LIOUVILLE THEOREMS AND SPECTRAL EDGE BEHAVIOR ON ABELIAN COVERINGS OF COMPACT MANIFOLDS

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ABSTRACT. The paper describes relations between Liouville type theorems for solutions of a periodic elliptic equation (or a system) on an abelian cover of a compact Riemannian manifold and the structure of the dispersion relation for this equation at the edges of the spectrum. Here one says that the Liouville theorem holds if the space of solutions of any given polynomial growth is finite dimensional. The necessary and sufficient condition for a Liouville type theorem to hold is that the real Fermi surface of the elliptic operator consists of finitely many points (modulo the reciprocal lattice). Thus, such a theorem generically is expected to hold at the edges of the spectrum. The precise description of the spaces of polynomially growing solutions depends upon a 'homogenized' constant coefficient operator determined by the analytic structure of the dispersion relation. In most cases, simple explicit formulas are found for the dimensions of the spaces of polynomially growing solutions in terms of the dispersion curves. The role of the base of the covering (in particular its dimension) is rather limited, while the deck group is of the most importance.

The results are also established for overdetermined elliptic systems, which in particular leads to Liouville theorems for polynomially growing holomorphic functions on abelian coverings of compact analytic manifolds.

Analogous theorems hold for abelian coverings of compact combinatorial or quantum graphs.

1. INTRODUCTION

The classical Liouville theorem claims that any harmonic function (i.e., a solution of the Laplace equation $\Delta u = 0$) in $\mathbb{R}^n$ that has a polynomial upper bound is in fact a (harmonic) polynomial. In particular, the space of all harmonic functions that grow not faster than

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$C(1+|x|)^N$, is of the finite dimension[1]

$$h_{n,N} := \begin{pmatrix} n+N \\ N \end{pmatrix} - \begin{pmatrix} n+N-2 \\ N-2 \end{pmatrix}. \tag{1.2}$$

The problem of extending this result to more general elliptic operators and to Laplace–Beltrami operators on general Riemannian manifolds of nonnegative Ricci curvature[2] has gained prominence since the work of S. T. Yau [87]. The questions asked concern the finite dimensionality of the spaces of solutions of a prescribed polynomial growth, estimates of (and in rare cases formulas for) their dimensions, and the structure of these solutions. One can find recent advances, reviews, and references in [24, 59, 60]. In particular, Yau's conjecture on the validity of the Liouville theorem for Riemannian manifolds of nonnegative Ricci curvature was proven in full generality by T. H. Colding and W. P. Minicozzi in [24] (see also references to previous partial solutions in [59, 60]).

In the flat situation, an amazing case was discovered by M. Avellaneda and F.-H. Lin [5] and later was also studied by J. Moser and M. Struwe [69]. In these papers the authors dealt with polynomially growing solutions of second-order divergence form elliptic equations

$$Lu = -\sum_{1\leq i,j\leq n} (a^{i,j}(x)u_{x_i})_{x_j} = 0 \tag{1.3}$$

with real coefficients that are periodic with respect to the lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$. They obtained a comprehensive answer for (1.3) (see also [24, 59] for related results and references). Using the formalism of homogenization theory, it was proved that the space of all solutions of the equation $Lu = 0$ of polynomial growth of order at most N has the same dimension $h_{n,N}$ (see (1.2)) as the space of harmonic polynomials in $\mathbb{R}^n$ of the same rate of growth. Moreover, any solution v of the equation $Lv = 0$ in $\mathbb{R}^n$ of polynomial growth is representable as a finite sum of the form

$$v(x) = \sum_{j=(j_1,\dots,j_n)\in\mathbb{Z}^n_+} x^j p_j(x), \tag{1.4}$$

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[1]We will also use the notation

$$q_{n,N} := \begin{pmatrix} n+N \\ N \end{pmatrix} \tag{1.1}$$

for the dimension of the space of all polynomials of degree at most N in n variables. Notice that $q_{n-1,N}$ also coincides with the dimension of the space of all homogeneous polynomials of degree N in n variables, so in particular, $h_{n,N} = q_{n,N} - q_{n,N-2} = q_{n-1,N-1} + q_{n-1,N}$.

[2]Without a condition on the curvature, the hyperbolic plane, where there is an infinite dimensional space of bounded harmonic functions, provides a counterexample.

where the functions $p_j(x)$ are periodic with respect to the group of periods of the equation and $x^j = \prod_i x_i^{j_i}$.

One can say that there has been no complete understanding of this result concerning periodic equations. In particular, one can ask the following natural questions:

(i) Is it important that the operator is of divergence form?
(ii) Can the results be generalized to higher order equations?
(iii) Is it possible to determine for a given periodic elliptic equation whether the Liouville theorem holds?
(iv) How crucial is the usage of homogenization theory tools (which automatically restricts the class of equations)?
(v) Same questions about elliptic systems.
(vi) Can these results be generalized for covering spaces of compact manifolds?

Some partial answers to these questions were obtained in [58, 61]. In [61], the results for the divergence type operators (1.3) in $\mathbb{R}^n$ were generalized to the case of second-order periodic operators without lower order terms. At the same time, [58] contains a necessary and sufficient condition for the validity of the Liouville theorem for a general periodic elliptic operator in $\mathbb{R}^n$, as well as a description (in most cases, implicit) of the dimensions of the corresponding spaces of solutions. In particular, an explicit formula was given for a general second-order periodic elliptic operator that admits a global positive solution.

Simultaneously, Liouville type theorems for holomorphic functions on complex analytic manifolds were studied (see [19, 20, 43, 62, 65, 66] and references therein). For instance, one asks whether Liouville theorems for holomorphic functions on coverings of compact analytic manifolds (or more generally, on coverings of manifolds with the Liouville property) hold true. One should mention the results of [62], where it was shown in particular that nilpotent coverings of compact complex analytic manifolds do have the Liouville property for *bounded* holomorphic functions (i.e., the space of such functions is finite-dimensional). It was not clear whether in these cases the spaces of holomorphic functions of a given polynomial growth are of finite dimensions as well. The exception was the result of [19], where such a Liouville theorem was proven for abelian coverings of compact Kähler manifolds (see also [20]). This result also follows from [24], while this relation with Liouville theorems for harmonic functions disappears for non-Kähler case.

One should also mention parallel studies concerning Liouville theorems for harmonic functions on graphs (e.g., [43, 68]).

The goal of this paper is to provide results of Liouville type that clarify this issue for abelian covers of compact manifolds. The results apply to elliptic equations and systems (including overdetermined ones) on abelian coverings of compact Riemannian manifolds, as well as to holomorphic functions on abelian coverings of compact complex manifolds, and to periodic equations on abelian coverings of combinatorial and quantum graphs. The crucial techniques used in the paper are different from the ones used in all the works cited above, with the exception of the authors' paper [58]. The ideas and the techniques come from the Floquet theory [51, 77] and are related to some spectral notions common in the solid state physics [7]. The reader can find all the necessary preliminary information in the next two sections. In comparison with [58], the present paper provides explicit formulas for the dimensions of the corresponding spaces, where [58], in general, contains only an implicit algorithm to calculate these numbers. Moreover, multiplicities at spectral edges, as well as overdetermined systems (in particular, $\overline{\partial}$-operators and Liouville theorems for holomorphic functions) are now allowed. Furthermore, operators on graphs are also considered. The general theorems are applied to a variety of specific examples of operators.

In order to outline the results of the paper, let us introduce some objects. Consider a normal abelian covering[3] of a compact d-dimensional Riemannian manifold M

$$X \overset{G}{\mapsto} M,$$

where G is the (abelian) deck group of the covering. Without loss of generality, one can assume that $G = \mathbb{Z}^n$. In fact, no harm will be done if the reader imagines for simplicity that $X = \mathbb{R}^n$, $G = \mathbb{Z}^n$, and M is the torus $\mathbb{R}^n/\mathbb{Z}^n$ (albeit in general, the dimension d of M does not have to be equal to n).

We will need to consider characters χ of G, i.e., homomorphisms of G into the multiplicative group $\mathbb{C}^*$ of nonzero complex numbers. Unitary characters map G into the group S^1 of complex numbers of absolute value 1. For any character χ, a function f on X will be called χ-*automorphic* if $f(gx) = \chi(g)f(x)$ for any $x \in X$, $g \in G$.

Let P be an elliptic G-periodic operator on X (in what follows, we use for shortness the word 'periodic' instead of 'G-periodic'). For any character χ, consider the space of χ-automorphic functions on X. It can also be interpreted as the space of sections of a linear bundle over M determined by χ. It is invariant with respect to P, so one can

[3]The word 'covering' in this paper always means 'normal covering'.

consider the restriction $P(\chi)$ of P to this space[4]. In the particular case of $\chi(g) \equiv 1$, $P(1)$ is just the elliptic operator P_M on M, whose lifting to X is P. In 'non-pathological' cases, the spectra of all operators $P(\chi)$ are discrete. The spectrum of $P(\chi)$, as a multiple-valued function of the character χ, is said to be the *dispersion curve* or *dispersion relation.*

Let $N \geq 0$. We say that *the Liouville theorem of order N for the equation $Pu = 0$* holds true if the space $\mathrm{V}_N(P)$ of solutions of the equation $Pu = 0$ on X that satisfy $|u(x)| \leq C(1+\rho(x))^N$ for all $x \in X$ is of finite dimension. Here $\rho(x)$ is the distance of $x \in X$ from a fixed point $x_0 \in X$.

We can now formulate a general (and somewhat vague at this point) statement that outlines our main results for the elliptic case contained in theorems 18, and 19. The results for the overdetermined, holomorphic, and graph cases can be found in Sections 6 and 7.

Main Theorem.

(1) *If the Liouville theorem of an order $N \geq 0$ for the equation $Pu = 0$ holds true, then it holds for any order.*
(2) *In order for the Liouville theorem to hold, it is necessary and sufficient that the number of unitary characters χ for which the equation $Pu = 0$ has a nonzero χ-automorphic solution is finite.*
(3) *If the Liouville theorem holds and the spectra of all the operators $P(\chi)$ are discrete, then under some genericity conditions on the operator P, the dimension of the space $\mathrm{V}_N(P)$ can be computed in terms of the dispersion relation for P.*
(4) *If the Liouville theorem holds, then all solutions that belong to $\mathrm{V}_N(P)$ are linear combinations of Floquet solutions of order N (see the definitions in the text below).*
(5) *Under the same conditions, one can describe a constant coefficient ('homogenized') linear differential operator $\Lambda(D)$ on $\mathbb{R}^n$, such that there is a one-to-one correspondence between the polynomial solutions of $\Lambda v = 0$ on $\mathbb{R}^n$ and the polynomially growing solutions of $Pu = 0$ on X.*

We will see that this result in particular means that one should naturally expect the Liouville theorem to hold only when zero is at an edge of the spectrum of P. This is true, for instance, for the operators considered in [5, 61, 69], when zero is the bottom of the spectrum.

It is interesting to notice that the dimension of X does not have to be equal to n, so the operators P and Λ might act on manifolds

[4]Exact definitions of function spaces and operators are provided in the next section.

of different dimensions. This happens since the Liouville property is of a 'homogenized' nature, i.e., it is something one sees by looking at the manifold X 'from afar'. Thus, the local details of the manifold are essentially lost and one sees the euclidian space $\mathbb{R}^n$ instead. In other words, the free rank of the deck group of the covering plays a more prominent role for Liouville theorems than the dimension of the manifold[5].

Similar results hold for elliptic systems, including overdetermined ones that are elliptic in the sense of being a part of an elliptic complex of operators. The reader can find basic notions and results concerning elliptic complexes in many books and articles (e.g., [41, Vol. III, Section 19.4], [78, Section 3.2.3], [84, Section IV.5]). For the particular case of the Cauchy–Riemann $\bar{\partial}$ operator, one obtains a Liouville theorem for holomorphic functions on abelian coverings (Theorem 30). Analogs for operators on combinatorial and quantum graphs are also straightforward to obtain.

The outline of the paper is as follows. The next section introduces necessary notations and preliminary results from the Floquet theory, in particular the definition and properties of the *Floquet–Gelfand transform*. The proofs are the same as for the case of periodic operators on $\mathbb{R}^d$ and are hence mostly omitted (see, e.g., how they can be worked out in parallel to the flat case in [48] and also in [21, 22, 83]). The crucial Section 3 is devoted to the detailed study of the so called *Floquet–Bloch solutions*. In Section 4 we derive Liouville type theorems for elliptic systems. Section 5 provides some examples of applications to specific periodic operators, and Section 6 treats overdetermined systems, including the case of analytic functions on complex manifolds. Graphs are briefly considered in Section 7. The last sections contain concluding remarks and acknowledgments.

2. Notations and preliminary results

Let us introduce first some standard notions of Floquet theory (see [29, 51, 77]), which we will adjust to the case of abelian covers (this does not require any change in the substance).

Let X be a noncompact smooth Riemannian manifold of dimension d equipped with an isometric, properly discontinuous, and free action of a **finitely generated abelian group** G. The action of an element $g \in G$ on $x \in X$ will be denoted by gx. Consider the orbit space $M = X/G$, which due to our conditions is a Riemannian manifold of

[5]This resonates with M. Gromov's notion of quasi-isometry, when the space of a covering might be indistinguishable from its deck group [34, 38].

its own. **We will assume that M is compact**. Hence, we are dealing with an abelian covering of a compact manifold

$$\pi : X \to M(= X/G). \tag{2.1}$$

Switching to a subcovering $X \to \tilde{M}$ $(\to M)$ with a compact $\tilde{M}$, one can eliminate the torsion part of G. In what follows, we could substitute $\tilde{M}$ for M and hence reduce the group G to $\mathbb{Z}^n$ with some $n \in \mathbb{N}$. **We will therefore assume from now on that $G = \mathbb{Z}^n$**. This will not reduce the generality of the results.

Let P be an elliptic operator of order m on X with smooth coefficients[6] that commutes with the action of G. Such an operator can be pushed down to an elliptic operator P_M on M (or conversely, P is the lifting of P_M to X). The ellipticity is understood in the sense of the nonvanishing of the principal symbol of the operator P on the cotangent bundle (with the zero section removed) $T^*X \setminus (X \times \{0\})$. The dual operator (the formal adjoint) P^* has similar properties (in particular, P^* is also G-periodic). Here the duality is provided by the bilinear rather than the sesquilinear[7] form

$$< g, f >= \int\limits_X f(x)g(x)\,\mathrm{d}x.$$

All the preparatory facts and main statements here hold for linear periodic matrix operators that are either standard elliptic (some times called elliptic in the Petrovsky sense) or elliptic in the Douglis–Nirenberg sense (e.g., [28], [41, Vol. III, Section 19.5], and [78, Section 3.1.2.1]). The only difference in the proofs between the scalar and system cases arises in the necessity of introducing appropriate spaces of vector-valued functions, exactly as it was done in [51, Section 3.4]. Doing this, however, would on one hand be very routine, and on the other hand would make reading the text more difficult. Bearing this in mind, we will provide detailed considerations for scalar linear elliptic operators only. They transfer with no effort to systems.

For any *quasimomentum* $k \in \mathbb{C}^n$ we denote by γ_k the character of $G = \mathbb{Z}^n$ defined as $\gamma_k(g) = \mathrm{e}^{\mathrm{i}k\cdot g}$. Here $g = (g_1, \dots, g_n) \in \mathbb{Z}^n$ and $k\cdot g = k_1g_1+\dots+k_ng_n$. We will also use the notation $|g| = |g_1|+\dots+|g_n|$ and for a multi-index $j = (j_1, \dots, j_n) \in \mathbb{Z}^n_+$ we denote $g^j = \prod_i g_i^{j_i}$. If $k \in \mathbb{R}^n$, the corresponding character is unitary. Due to the obvious 2π-periodicity of γ_k with respect to k, it is sufficient to restrict ourselves

[6]The smoothness condition can be significantly reduced (see the corresponding remarks in Section 8).

[7]This is not essential, but simplifies somewhat the calculations.

to the real vectors k in the *Brillouin zone* $\mathrm{B} = [-\pi, \pi]^n$, which is a fundamental domain of the reciprocal (dual) lattice $G^* = (2\pi\mathbb{Z})^n$. Periodizing B (i.e., considering the torus $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$), one obtains the dual group $\mathbb{T}^n$ to $G = \mathbb{Z}^n$.

For any $k \in \mathbb{C}^n$, we define the subspace $L^2_k(X)$ of $L^2_{\mathrm{loc}}(X)$ consisting of all functions $f(x)$ that are γ_k-automorphic, i.e., such that $f(gx) = \gamma_k(g)f(x) = \mathrm{e}^{\mathrm{i}k\cdot g}f(x)$ for a.e. $x \in X$. Alternatively, this space can be defined as follows. We can identify γ_k with a one-dimensional representation of G and consider the one-dimensional flat vector bundle E_k over M associated with this representation. Then elements of $L^2_k(X)$ can be naturally identified with L^2-sections of E_k. A similar construction works also for other classes of functions (e.g., from Sobolev spaces). We will identify G-periodic (i.e., γ_0-automorphic) functions on X with functions on M. Due to the periodicity of the operator P, it leaves the spaces L^2_k invariant, and so its restrictions to these subspaces define elliptic operators $P(k)$ on the spaces of sections of the bundles E_k over M [8]. If P is selfadjoint, then $P(k)$ is selfadjoint for any real quasimomentum k.

It is natural that Fourier transform with respect to the periodicity group G reduces the space $L^2(X)$, as well as the original operator P on X, to the direct integral of operators $P(k)$ on sections of E_k:

$$L^2(X) = \int_{\mathrm{B}}^{\oplus} L^2_k(X)\,\mathrm{d}k, \tag{2.2}$$

and

$$P = \int_{\mathrm{B}}^{\oplus} P(k)\,\mathrm{d}k. \tag{2.3}$$

The integral is understood with respect to the normalized Haar measure on the dual group $\mathbb{T}^n$, which boils down to the normalized Lebesgue measure $\mathrm{d}k$ on the Brillouin zone B. The isomorphism (in fact, an isometry) in (2.2) is provided by an analog of the Fourier transform (see [51, Section 2.2], [77, 83]), which we will call the *Floquet–Gelfand transform* $\mathcal{U}$:

$$f(x) \to \mathcal{U}f(k,x) = \sum_{g\in G} f(gx)\gamma_{-k}(g) \qquad k \in \mathbb{C}^n. \tag{2.4}$$

[8]In the case when $X = \mathbb{R}^n$ with the natural $\mathbb{Z}^n$ action, these operators can be identified with the "shifted" versions $P(x, D + k)$ of the operator P acting on the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ (see [51, 77]).

This transform is the main tool in the Floquet theory for PDEs (e.g., [51, 77, 81, 83]). It was introduced first in [32] in order to obtain expansions into Bloch generalized eigenfunctions for periodic selfadjoint elliptic operators.

It is not hard to describe the image of a Sobolev space $H^s(X)$ under the Floquet–Gelfand transform. In order to do so, let us consider a quasimomentum $k \in \mathbb{C}^n$ and denote by H^s_k the closed subspace of the space $H^s_{\text{loc}}(X)$ consisting of γ_k-automorphic functions. It is clear that H^s_k can be naturally equipped with the structure of a Hilbert space and that it can be identified with the space $H^s(E_k)$ of H^s-sections of the bundle E_k over M.

One can show[9] that

$$\mathcal{E}^s := \bigcup_{k\in\mathbb{C}^n} H^s_k \tag{2.5}$$

forms a holomorphic $2\pi\mathbb{Z}^n$-periodic Banach vector bundle. As any infinite dimensional analytic Hilbert bundle over a Stein domain, it is trivializable [23] (see also the survey [88] and theorems 1.3.2, 1.3.3, and 1.5.23 in [51]).

We collect now several statements from [77, Theorem XIII.97], [51, Theorem 2.2.2], and [58, 83], recasted into the abelian covering form:

Theorem 1. (1) *For any nonnegative integer m, the operator*

$$\mathcal{U} : H^m(X) \to L^2(\mathbb{T}^n, \mathcal{E}^m)$$

is an isometric isomorphism, where $L^2(\mathbb{T}^n, \mathcal{E}^m)$ is the space of square integrable sections over the torus (identified with the Brillouin zone B*) of the bundle $\mathcal{E}^m$, equipped with the natural topology of a Hilbert space.*

(2) *Let $K \Subset X$ be a domain in X such that $\bigcup_{g\in G} gK = X$. Let also the space*

$$\mathrm{C}^m(X) := \left\{ \phi \in H^m_{\text{loc}}(X) \mid \sup_{g\in G} ||\phi||_{H^m(gK)}(1+|g|)^N < \infty \ \ \forall N \right\}$$

be equipped with the natural Fréchet topology. Then

$$\mathcal{U} : \mathrm{C}^m(X) \to C^\infty(\mathbb{T}^n, \mathcal{E}^m)$$

is a topological isomorphism, where $C^\infty(\mathbb{T}^n, \mathcal{E}^m)$ is the space of C^∞ sections of the bundle $\mathcal{E}^m$ over the complex torus $\mathbb{T}^n$, equipped with the standard topology.

[9]See Theorem 2.2.1 in [51] for the case $X = \mathbb{R}^n$. The general case of abelian covers over compact manifolds is entirely parallel.

(3) *Let the elliptic operator P be of order m. Then under the transform $\mathcal{U}$ the operator*

$$P : \mathrm{C}^m(X) \to \mathrm{C}^0(X)$$

becomes the operator

$$C^\infty(\mathbb{T}^n, \mathcal{E}^m) \overset{P(k)}{\to} C^\infty(\mathbb{T}^n, \mathcal{E}^0)$$

of multiplication by the holomorphic Fredholm morphism $P(k)$ between the fiber bundles $\mathcal{E}^m$ and $\mathcal{E}^0$.

3. Floquet–Bloch solutions

We now need to introduce and study our main notions: Bloch and Floquet solutions of periodic differential equations.

Definition 2. Let $k \in \mathbb{C}^n$. A γ_k-automorphic function $u(x)$ on X is said to be a *Bloch function with quasimomentum k*. In other words, it is a function with the property $u(gx) = \gamma_k(g)u(x) = \mathrm{e}^{\mathrm{i}k\cdot g}u(x)$ for any $x \in X$, $g \in G$. To put it differently, $u(x)$ is transformed according to an irreducible representation of the group G with the character γ_k.

A *Bloch solution* of an equation is a solution that is a Bloch function.

Notice that every continuous Bloch function on X with a real quasimomentum (i.e., transformed according to an irreducible unitary representation) is bounded. Any such Bloch function u that belongs to $L^2_{\mathrm{loc}}(X)$ is bounded in the following integral sense: for any compact $K \Subset X$ we have $\sup_{g\in G} \|u\|_{L^2(gK)} < \infty$.

In the case when $X = \mathbb{R}^n$, $G = \mathbb{Z}^n$, and $M = \mathbb{T}^n$, any Bloch function with quasimomentum k has the form

$$u(x) = \mathrm{e}^{\mathrm{i}k\cdot x}p(x)$$

with a $\mathbb{Z}^n$-periodic function $p(x)$. In fact, a similar (albeit less natural) representation holds for Bloch functions on any abelian cover $X \overset{G}{\mapsto} M$. Indeed, let K be any fundamental domain of X with respect to the action of G and $f \in C_0^\infty(X)$ be a nonnegative function which is strictly positive on K. We define for any $j = 1, \ldots, n$

$$h_j(x) := \sum_{g=(g_1,\ldots,g_n)\in G=\mathbb{Z}^n} f(gx)\exp(-g_j).$$

Then $h_j(x)$ clearly is a positive function satisfying $h_j(gx) = \mathrm{e}^{g_j}h_j(x)$ for any $g = (g_1, \ldots, g_n) \in G$. It is an analog of e^{x_j} on $\mathbb{R}^n$. Thus, one can define analogs of powers $x^l = x_1^{l_1}\cdots x_n^{l_n}$ and of exponents $e^{ik\cdot x}$ as follows: for $l=(l_1,\dots,l_n)\in\mathbb{Z}_+^n$ let

$$[x]^l := \prod_{j=1}^{n} [\log h_j(x)]^{l_j}\,,$$

and for any quasimomentum $k\in\mathbb{C}^n$

$$e_k(x) := \exp(\mathrm{i}k_1\log h_1(x)+\cdots+\mathrm{i}k_n\log h_n(x)).$$

Notice that $e_k(x)$ is a nonvanishing Bloch function on X with quasimomentum k, which is positive for $\mathrm{i}k\in\mathbb{R}^n$. Thus, any Bloch function u on X with a quasimomentum k is given by

$$u(x) = e_k(x)p(x),$$

where $p(x)$ is G-periodic.

The construction of Bloch functions can be described in a more invariant way [2]. Consider a basis ω_j of the space of all closed differential 1-forms on M (modulo the exact ones) such that their lifts w_j to X are exact. According to De Rham's theorem, this basis is finite. One can now achieve the same goals as before defining $h_j(x)=\exp(\int\limits_o^x w_j)$ for a fixed reference point $o\in X$.

We can now define a more general class than Bloch functions.

Definition 3. A function $u(x)$ on X is said to be a *Floquet function with quasimomentum* $k\in\mathbb{C}^n$ if it can be represented in the form

$$u(x) = e_k(x)\left(\sum_{\substack{j=(j_1,\dots,j_n)\in\mathbb{Z}_+^n\\ |j|\le N}} [x]^j p_j(x)\right), \tag{3.1}$$

where the functions p_j are G-periodic.

The number N in this representation will be called *the order* of the Floquet function.

A *Floquet solution* of an equation is a solution that is a Floquet function.

This definition is modelled closely after the notion of Floquet solution that is common in $\mathbb{R}^n$ (e.g., [51]), where the formula is the same, one just replaces the 'powers' $[x]^j$ by the true powers x^j. However, unlike the notion of a Bloch function, it lacks invariance. To alleviate this, we briefly address now a different way to define Bloch and Floquet functions (solutions) on abelian coverings [63].

For any $g \in G$, let us define the first difference operator Δ_g acting on functions on X as follows:

$$\Delta_g u(x) = u(gx) - u(x). \tag{3.2}$$

Clearly, u is a periodic function (i.e., a Bloch function with zero quasimomentum) if an only if it is annihilated by Δ_g for any $g \in G$. In fact, it is sufficient to check this property for any set $\{g_j\}$ of generators of G. One wonders whether one can check in a similar way whether a function is a Bloch function with a nonzero quasimomentum and whether Floquet functions allow for similar tests. In order to get the answer, we need to introduce a twisted version of the first difference, that depends of the quasimomentum:

$$\Delta_{g;k} u(x) = \chi_{-k}(g)u(gx) - u(x) = \mathrm{e}^{-\mathrm{i}k\cdot g}u(gx) - u(x). \tag{3.3}$$

We also need to introduce iterated finite differences of order N with quasimomentum k as follows:

$$\Delta_{g_1,\dots,g_N;k} = \Delta_{g_1;k} \cdot \dots \cdot \Delta_{g_N;k}, \tag{3.4}$$

where $g_j \in G$ (it will always be sufficient to use only elements (maybe repeated) of a fixed set of generators of G).

We can now answer the question by proving the following

Lemma 4. *A function $u(x)$ on X is a Floquet function of order N with quasimomentum k if and only if it is annihilated by any difference of order $N+1$ with quasimomentum k:*

$$\Delta_{g_1,\dots,g_{N+1};k} u = 0 \qquad \forall g_1, \dots, g_{N+1} \in G,$$

(choosing g_j from a fixed set of generators is sufficient). In particular, it is a Floquet function of order N with quasimomentum 0 *if and only if*

$$\Delta_{g_1} \cdot \dots \cdot \Delta_{g_{N+1}} u = 0 \qquad \forall g_1, \dots, g_{N+1} \in G. \tag{3.5}$$

Proof: Let us provide the proof for the case $k = 0$ first. As it has already been mentioned, the necessity and sufficiency of the condition (3.5) checks out easily for $N = 0$, where it boils down to $\Delta_g u = 0$ for all $g \in G$, i.e., to the periodicity of u. Necessity for any N now follows easily by induction if one takes into account the representation (3.1). Indeed, one computes that (using the same standard basis $\{g_j\}$ of $\mathbb{Z}^n$ as before)

$$\Delta_{g_j}[x]^{(l_1,\dots,l_n)} = l_j[x]^{(l_1,\dots,l_j-1,\dots,l_n)} + \text{ lower order terms.} \tag{3.6}$$

Here 'lower order terms' contain linear combinations of $[x]^l$'s of strictly lower total degrees. Since the difference operators do not alter periodic functions, we obtain that any Δ_{g_j} reduces Floquet functions of order

$N+1$ to the ones of order N, which concludes the induction step of the proof of necessity for $k=0$.

Let us prove sufficiency which also follows by induction with respect to the order of the Floquet function. We have already checked it for $N=0$. Assume that this has been proven for orders up to N. Suppose that u satisfies $\Delta_{g_1}\cdot\ldots\cdot\Delta_{g_{N+2}}u=0$ for any $g_1,\ldots,g_{N+2}\in G$. Take a set of generators $\{g_j\}_{j=1}^n$ in G. Then the functions $f_j:=\Delta_{g_j}u$ satisfy the condition of the Lemma for the order N. According to the induction hypothesis, we conclude that

$$f_j(x)=\left(\sum_{\substack{l=(l_1,\ldots,l_n)\in\mathbb{Z}_+^n\\|l|\leq N}}[x]^l p_{l,j}(x)\right)$$

with periodic functions $p_{l,j}$.

We claim that for any Floquet function f of order N (with $k=0$) there exists a Floquet function v of order $N+1$ such that $\Delta_{g_j}v=f$. Without loss of generality we assume that $j=1$. Since the difference operator Δ_{g_1} does not change periodic coefficients, it is sufficient to check the statement for $f=[x]^l$, where $|l|\leq N$. Now induction with respect to l_1 finishes the job. Namely, (3.6) provides a linear system of a triangular structure for recursively determining the coefficients of v such that $\Delta_{g_j}v=f$.

Therefore, a Floquet function v_1 of order $N+1$ exists such that $\Delta_{g_1}v_1=\Delta_{g_1}u$. This means that the function $u_2=u-v_1$ is periodic with respect to the one-parameter subgroup generated by g_1. Since the condition of the Lemma is still satisfied for u_2, and since u_2 is g_1-periodic, we can continue this process and find a Floquet function v_2 of order $N+1$ that is g_1-periodic and such that $u_3=u-v_1-v_2$ is periodic with respect to both g_1 and g_2. Continuing this process, we get the conclusion of the Lemma for $k=0$.

Let us now prove the statement for an arbitrary quasimomentum k. According to the definition (3.1) of Floquet functions, any Floquet function u of order N with quasimomentum k has the form $u=e_k(x)v$, where v is a Floquet function of order N with quasimomentum $k=0$. A straightforward calculation shows that if any two functions u and v satisfy $u=e_k(x)v$, then one has $\Delta_{g;k}u=e_k(x)\Delta_g v$. Thus, the statement of the lemma for an arbitrary quasimomentum k follows from the one for $k=0$. □

We have provided several different ways to interpret the notion of Floquet solutions: using explicit formulas analogous to the ones in $\mathbb{R}^n$ (constructing analogs of coordinate functions and exponents by either explicit constructions, or by using some special differential forms on M), as well as in terms of some difference operators[10]. Another way to think of Floquet functions is to imagine the finite dimensional subspace E generated by the G-shifts of such a function as a Jordan block for the action of G [74]. This suggests a relation to indecomposable (non-unitary) representations of G (e.g., [9, Ch. 6, Sect. 3]). However, we do not pursue this approach, due to the known difficulties of classifying such representations [33].

Remark 5. Any continuous Floquet function $u(x)$ of order N with a real quasimomentum satisfies the growth estimates

$$|u(x)| \leq C(1+|g|)^N \qquad \forall g \in G \text{ and } x \in gK,$$

where C depends on K and u. In general, one needs to replace this growth estimate by an integral one, as it was done before for Bloch functions.

We now introduce a notion that plays a very important role in solid state physics, photonic crystal theory, as well as in the general theory of periodic PDEs [7, 51, 52, 71]. It will be also crucial for formulation of our main results.

Definition 6. The (complex) *Fermi surface* F_P of the operator P (at the zero energy level) consists of all quasimomenta $k \in \mathbb{C}^n$ such that the equation $Pu = 0$ on X has a nonzero Bloch solution with a quasimomentum k . The *real Fermi surface* $F_{P,\mathbb{R}}$ is $F_P \cap \mathbb{R}^n$.

Equivalently, $k \in F_P$ means the existence of a nonzero solution u of the equation $P(k)u = 0$. In the Euclidean case, such a u has a form $u(x) = \mathrm{e}^{\mathrm{i}k\cdot x}p(x)$, where $p(x)$ is a G-periodic function. Fermi surface plays in the periodic situation the role of the characteristic variety (the set of zeros of the symbol) of a constant coefficient differential operator.

Introducing a spectral parameter λ, one arrives at the notion of the *Bloch variety*:

Definition 7. The (complex) *Bloch variety* B_P of the operator P consists of all pairs $(k, \lambda) \in \mathbb{C}^{n+1}$ such that the equation $Pu = \lambda\, u$ has a nonzero Bloch solution u with a quasimomentum k. The *real Bloch variety* $B_{P,\mathbb{R}}$ is $B_P \cap \mathbb{R}^{n+1}$.

[10] Difference operators approach has been successfully used in related studies of periodic equations and Liouville type problems for holomorphic functions in [19, 62, 64, 69].

The Bloch variety B_P can be treated as the graph of a (multivalued) function $\lambda(k)$, which is called the *dispersion relation*. If the spectra of the operators $P(k)$ on M are discrete, we can single out continuous branches λ_j of this multivalued dispersion relation. They are called the *band functions* [77, 51]. The Fermi surface is obviously the zero level set of the dispersion relation.

In order to justify the notion of a band function, we need to guarantee the discreteness of the spectrum of the operators $P(k)$ on M for all $k \in \mathbb{C}^n$. In other words, we need to exclude the pathological (but possible) situation of the spectrum of P covering the complex plane. One of the reasons for such strange spectral behavior is the Fredholm index being not equal to zero. However, even when the index is equal to zero, such pathology can occur, as the example of the operator $\mathrm{e}^{\mathrm{i}\varphi}\mathrm{d}/\mathrm{d}\varphi$ on the unit circle shows. Self-adjointness of P is one of the conditions that would obviously guarantee discreteness. Another example is a second-order elliptic periodic operator in $\mathbb{R}^n$ of the form

$$L = -\sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^{n} b_i(x)\partial_i + c(x) \tag{3.7}$$

with real and smooth coefficients. More sufficient conditions can be found for example in [1].

Lemma 8. *The Fermi and Bloch varieties are the sets of all zeros of entire functions of a finite order in $\mathbb{C}^n$ and $\mathbb{C}^{n+1}$, respectively.*

This is proven in [51, theorems 3.1.7 and 4.4.2] for the flat case. The case of a general abelian covering does not require any change in the proof.

The lemma implies in particular, that the band functions $\lambda(k)$ are piecewise-analytic (e.g., when there is no level crossing, one has analyticity due to the standard perturbation theory [44]). This statement was originally proven in [85] for Schrödinger operators.

Another useful property of the Bloch and Floquet varieties is the relation between the corresponding varieties of the operators P and P^*:

Lemma 9. [51, Theorem 3.1.5] *A quasimomentum k belongs to F_{P^*} if and only if $-k \in F_P$. Analogously, $(k, \lambda) \in B_{P^*}$ if and only if $(-k, \lambda) \in B_P$. In other words, the dispersion relations $\lambda(k)$ and $\lambda^*(k)$ for the operators P and P^* are related as follows:*

$$\lambda^*(k) = \lambda(-k). \tag{3.8}$$

We will need to see how the structure of the functions of Floquet type (see Definition 3) and in particular, of Floquet solutions of our periodic equation reacts to the Floquet–Gelfand transform. For instance, in the constant coefficient case, where the role of the Floquet solutions is played by the exponential polynomials

$$u(x) = \mathrm{e}^{\mathrm{i}k\cdot x} \sum_{|j|\leq N} p_j x^j,$$

where $p_j \in \mathbb{C}$, such functions are Fourier transformed into distributions supported at the point $(-k)$. The next statement shows that under the Floquet–Gelfand transform, each Floquet type function (3.1) corresponds, in a similar way, to a (vector valued) distribution supported at the quasimomentum $(-k)$. This, and some other properties of Floquet solutions that play a crucial role in establishing our Liouville type theorems are collected in the next lemma.

Every Floquet type function u with a real quasimomentum is of polynomial growth, and thus determines a (continuous linear) functional on the previously defined space $\mathrm{C}^0(X)$ (see Theorem 1). If it satisfies the equation $Pu = 0$ for a periodic elliptic operator of order m, then as such a functional it is orthogonal to the range of the dual operator $P^* : \mathrm{C}^m(X) \to \mathrm{C}^0(X)$. According to Theorem 1, after the Floquet–Gelfand transform any such functional becomes a functional on $C^\infty(\mathbb{T}^n, \mathcal{E}^0)$ that is orthogonal to the range of the operator of multiplication by the Fredholm morphism $P^*(k) : \mathcal{E}^m \to \mathcal{E}^0$. The following auxiliary result (see [58]) describes all such functionals[11].

Lemma 10. (1) *A continuous linear functional u on $\mathrm{C}^0(X)$ is generated by a Floquet type function with a quasimomentum k_0 if and only if after the Floquet–Gelfand transform it corresponds to a functional on $C^\infty(\mathbb{T}^n, \mathcal{E}^0)$ which is a distribution ϕ that is supported at the point $-k_0$, i.e., has the form*

$$\langle \phi, f\rangle = \sum_{|j|\leq N} \left(q_j, \left.\frac{\partial^{|j|} f}{\partial k^j}\right|_{-k_0} \right)_{L^2(M)} \qquad f \in C^\infty\left(\mathbb{T}^n, \mathcal{E}^0\right), \tag{3.9}$$

where $q_j \in L^2(M)$. The orders N of the Floquet function and of the corresponding distribution ϕ are the same.

(2) *Let a_k be the dimension of the kernel of the operator*

$$P(k) : H^m_k \to L^2_k.$$

[11] Although the discreteness of the spectrum of $P(k)$ was assumed throughout the whole paper [58], it is in fact not needed for (and was not used in the proof of) this lemma.

Then the dimension of the space of Floquet solutions of the equation $Pu = 0$ of order at most N with a quasimomentum k is finite and does not exceed $a_k q_{n,N}$.

The estimate on the dimension given in the Lemma 10 is very crude and can often be improved. The next result (Theorem 13) provides in some cases exact values of these dimensions. This theorem is the crucial part of the proof of (Liouville) Theorem 18. In order to state and prove it, we need to introduce some notions.

First of all, the analytic Hilbert bundle $\mathcal{E}^m$ is locally trivial for any m. Since the previous Lemma shows that our interest in Floquet solutions is local with respect to the quasimomentum k, we can trivialize the bundles and hence assume that the analytic families of operators $P(k)$ and $P^*(k)$ act in a fixed Hilbert space. At this moment, we will need the additional condition that the spectra of these operators are discrete (see the corresponding discussion earlier in the text). Assume now that zero is an eigenvalue of the adjoint operator $P^*(-k_0) : H^m_{-k_0}(X) \to L^2_{-k_0}(X)$ (since under the conditions we imposed on the operator, its Fredholm index is zero, this means that the operator $P(k_0)$ has eigenvalue zero as well). Suppose further that the algebraic multiplicity of the null eigenvalue is equal to r. Consider a closed curve Υ in $\mathbb{C}$ separating 0 from the rest of the spectrum of $P^*(-k_0)$ and the corresponding (analytically depending on k in a neighborhood of k_0) r-dimensional spectral projector $\Pi(k)$ for $P^*(-k)$. Let $\{e_j\}_{j=1}^r$ be an orthonormal basis of the spectral subspace of $P^*(-k_0)$ corresponding to the point $0 \in \mathbb{C}$ (i.e., the range of $\Pi(k_0)$). Some of our main results will be expressed in terms of the following $r \times r$ matrix function

$$\lambda(k)_{ij} = \langle e_j, (P^*(k)\Pi(k)e_i\rangle. \tag{3.10}$$

This matrix function is analytic with respect to k in a neighborhood of k_0.

Remark 11. We would like to mention that in what follows, the results will be invariant with respect to a multiplication of the matrix λ from either side by an invertible matrix-function analytic in a neighborhood of k_0. This means, in particular, that if $r = 1$, then $\lambda(k)$ can equivalently be chosen to be equal to the analytic branch around k_0 of the eigenvalue of $P^*(-k)$ that vanishes at k_0.

Moreover, instead of $\{\Pi(k)e_i\}$ in (3.10) one can use any holomorphic basis $e_j(k)$ in the range of $\Pi(k)$ near $k = k_0$. One can also use instead of $\langle e_j, \bullet\rangle$ any family $\{f_j(k)\}_{j=1}^r$ of holomorphic with respect to k (in a neighborhood of k_0) functionals that is complete on the range of $\Pi(k)$.

Consider the Taylor expansion of $\lambda(k)$ around the point k_0 into homogeneous matrix-valued polynomials:

$$\lambda(k) = \sum_{l \geq 0} \lambda_l(k - k_0). \tag{3.11}$$

In this paper we will be mostly concerned with the first nonzero term λ_{l_0} of the expansion.

Definition 12. Let Q be a homogeneous polynomial in n variables with matrix coefficients of dimension $r \times r$, and let $Q(D)$ be the differential matrix operator with the symbol Q. A $\mathbb{C}^r$-valued polynomial $p(x)$ in $\mathbb{R}^n$ is called Q-*harmonic* if it satisfies the system of differential equations $Q(D)p = 0$.

Let $\mathcal{P}$ denote the vector space of all $\mathbb{C}^r$-valued polynomials in n variables, and let P_l be the subspace of all such homogeneous polynomials of degree l. So,

$$\mathcal{P} = \bigoplus_{l=0}^{\infty} \mathrm{P}_l \tag{3.12}$$

and

$$\mathcal{P}_N := \bigoplus_{l=0}^{N} \mathrm{P}_l \tag{3.13}$$

is the subspace of all such vector valued polynomials of degree at most N. If $Q(k)$ is a homogeneous polynomial of degree s with values in $r \times r$ matrices, then the matrix differential operator $Q(D)$ maps P_{l+s} to P_l. If the determinant $\det Q$ is not identically equal to zero, then this mapping is surjective for any l (this will follow from the proof of the theorem below). Hence, the mapping $Q(D) : \mathcal{P} \to \mathcal{P}$ has a (non-uniquely defined) linear right inverse R that preserves the homogeneity of polynomials.

Theorem 13. *Assume that zero is an isolated eigenvalue of algebraic multiplicity r of the operator $P^*(-k_0) : H^m_{-k_0}(X) \to L^2_{-k_0}(X)$,. Let $\lambda(k)$ be defined in a neighborhood of k_0 as in (3.10)*[12]*. Let also λ_{l_0} be the first nonzero term of the Taylor expansion (3.11). Then*

(1) *For any $N \geq 0$, the dimension of the space of Floquet solutions of the equation $Pu = 0$ in X with quasimomentum k_0 and of order at most N is finite and does not exceed $rq_{n,N}$.*

[12] Any analytic function in a neighborhood of k_0 that differs from $\lambda(k)$ by a left and right multiplication by analytic invertible matrix functions will produce the same results in what follows.

(2) *If* $\det \lambda_{l_0}$ *is not identically equal to zero (for instance, this is the case when the eigenvalue is simple, i.e.,* $r = 1$*), then for any* $N \geq 0$ *the dimension of the space of Floquet solutions of the equation* $Pu = 0$ *in* X *of order at most* N *and with quasi-momentum* k_0 *is equal to*

$$r\left[\left(\begin{array}{c} n+N \\ N \end{array}\right) - \left(\begin{array}{c} n+N-l_0 \\ N-l_0 \end{array}\right)\right]. \tag{3.14}$$

(3) *This dimension coincides with the dimension of the space of all* λ_{l_0}*-harmonic polynomials of degree of at most* N *with values in* $\mathbb{C}^r$*. Moreover, given a linear right inverse* R *of the mapping* $\lambda_{l_0}(D) : \mathcal{P} \to \mathcal{P}$ *that preserves homogeneity, one can construct an explicit isomorphism between the corresponding spaces.*

Proof: As before, we will assume that the bundles are analytically trivialized around the point k_0 and hence all operators act between fixed spaces that we will denote H^m and H^0. In order to simplify notations, we assume that $k_0 = 0$ (this can always be achieved by a change of variables). Let us denote by $N(k)$ the range of the projector $\Pi(k)$ and choose a closed complementary subspace M to $N(0)$ in H^m. The subspace M stays complementary to $N(k)$ in a neighborhood of 0 and so

$$H^m = M \oplus N(k).$$

Thus, $P^*(-k)$ has zero kernel on M for all k in a neighborhood of 0. This implies that the range of $P^*(-k)$ on M forms an analytic Banach vector bundle $R(k)$ in a neighborhood of 0 (e.g., Theorem 1.6.13 of [51]). Representing now the operator $P^*(-k)$ in the block form according to the decompositions

$$H^m = M \oplus N(k)$$

and

$$H^0 = R(k) \oplus N(k),$$

we get

$$P^*(-k) = \left(\begin{array}{cc} B(k) & 0 \\ 0 & \tilde{\lambda}(-k) \end{array}\right),$$

where $B(k)$ is an analytic invertible operator-function and the matrix analytic function $\tilde{\lambda}(-k)$ differs from $\lambda(-k)$ only by multiplying by an invertible analytic matrix function and thus for our purposes can be replaced by the latter one.

Let us now have a functional ϕ on $C^\infty(\mathbb{T}^n, \mathcal{E}_0)$ supported at 0, such that it is orthogonal to the range of the operator of multiplication by $P^*(-k)$. Then it must be equal to zero on all sections of the bundle $R(k)$ (since they are all in the range, due to invertibility of $B(k)$).

This means that the restriction of such functionals to the sections of the finite-dimensional bundle $N(k)$ is an one-to-one mapping. This reduces the problem to the following: find the dimension of the space of all distributions of order N supported at the origin such that they are orthogonal to the sub-module generated by the matrix $\lambda(-k)$ in the module of germs of analytic vector valued functions. One can change variables to eliminate the minus sign in front of k. Due to the finiteness of the order of the distribution, the problem further reduces to the following: find the dimension of the cokernel of the mapping

$$\Lambda_N : \mathcal{P}_N \to \mathcal{P}_N \,.$$

Here $\Lambda_N(p)$ for $p \in \mathcal{P}_N$ is the Taylor matrix-valued polynomial of order N at 0 of the product $\lambda(k)p(k)$. Let us write the block matrix Λ_{ij} of the operator Λ_N that corresponds to the decomposition $\mathcal{P}_N = \bigoplus_{l=0}^{N} \mathrm{P}_l$. Then $\Lambda_{ij} = 0$ for $i - j < l_0$. For $i - j \geq l_0$ the entry Λ_{ij} is the operator of multiplication by λ_{i-j} acting from P_j into P_i. Since $\det(\lambda_{l_0})$ is not identically equal to zero, it follows that for $i - j = l_0$ the operator Λ_{ij} of multiplication by λ_{l_0} has zero kernel.

In order to prove the theorem, we need to find the dimension of the cokernel of Λ_N, as well as to obtain the cokernel's description.

The first statement of the theorem is now obvious, since the dimension of the cokernel of Λ_N cannot exceed the dimension of the ambient space $\mathcal{P}_N$, which is equal to $rq_{n,N}$.

Let us now approach the second statement. Since Λ_N is a square matrix, $\dim \operatorname{Coker}\Lambda_N = \dim \operatorname{Ker}\Lambda_N$. The latter dimension, however, is easy to find, due to the triangular structure of the equation $\Lambda_N p = 0$, if it is written in the block matrix form according to the decomposition (3.13):

$$\begin{pmatrix} 0 & \dots & 0 & \lambda_{l_0} & \lambda_{l_0+1} & \dots & \lambda_N \\ 0 & \dots & 0 & 0 & \lambda_{l_0} & \dots & \lambda_{N-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \dots & & & & 0 & \lambda_{l_0} \\ 0 & \dots & & & & & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & & & & & 0 \end{pmatrix} \begin{pmatrix} p_N \\ p_{N-1} \\ \vdots \\ p_{N-l_0} \\ p_{N-l_0-1} \\ \vdots \\ p_0 \end{pmatrix} = 0. \tag{3.15}$$

Here $p = p_0 + \cdots + p_N$ is the expansion of $p \in \mathcal{P}_N$ into homogeneous terms. Since $\det\lambda_{l_0} \neq 0$, one concludes immediately from (3.15) that $p_0 = \ldots = p_{N-l_0} = 0$, while other components are arbitrary. This gives the dimension of the kernel (and hence of the cokernel) of Λ_N as

$$r\left[\left(\begin{array}{c} n+N \\ N \end{array}\right)-\left(\begin{array}{c} n+N-l_0 \\ N-l_0 \end{array}\right)\right],$$

which proves the second statement of the theorem.

In order to prove the last assertion of the theorem, we need describe the elements of the cokernel of Λ_N. Hence, we need to find the kernel of the adjoint matrix Λ_N^*. The adjoint matrix acts in the space $\bigoplus\limits_{l=0}^{N} \mathrm{P}_l^*$, where P_l^* can be naturally identified with the space of linear combinations with coefficients in $\mathbb{C}^r$ of the derivatives of order l of the Dirac's delta-function at the origin. Here we have $\Lambda_{ij}^* = 0$ for $j-i < l_0$, and for $j-i \geq l_0$ the entry Λ_{ij}^* is the dual to the operator of multiplication by λ_{j-i} acting from P_i into P_j. In particular, since for $j-i = l_0$ the latter operator is injective, we conclude that the operators Λ_{ij}^* are surjective. This enables one to describe the structure of the kernel of Λ_N^* (and hence of cokernel of Λ_N). Namely, let

$$\psi = (\psi_0, \dots, \psi_N) \in \bigoplus_{l=0}^{N} \mathrm{P}_l^*$$

be such that $\Lambda_N^* \psi = 0$. Due to the triangular structure of Λ_N^*, we can solve this system:

$$\sum_{j \geq i+l_0} \Lambda_{ij}^* \psi_j = 0 \qquad i = 0, \dots, N-l_0.$$

Taking the Fourier transform, we can rewrite this system in the form

$$\sum_{j \geq i+l_0} \lambda_{j-i}(D)\widehat{\psi_j} = 0 \qquad i = 0, \dots, N-l_0,$$

where $\widehat{\psi}$ denotes the Fourier transform of ψ. Therefore, $\widehat{\psi_j}$ is a homogeneous polynomial of degree j in $\mathbb{R}^n$. For $i = N-l_0$ we have

$$\lambda_{l_0}(D)\widehat{\psi_N} = 0.$$

This equality means that $\widehat{\psi_N}$ can be chosen as an arbitrary λ_{l_0}-harmonic homogeneous polynomial of order N (with values in $\mathbb{C}^r$). Moving to the previous equation, we analogously obtain

$$\lambda_{l_0}(D)\widehat{\psi_{N-1}} + \lambda_{l_0+1}(D)\widehat{\psi_N} = 0,$$

or

$$\lambda_{l_0}(D)\widehat{\psi_{N-1}} = -\lambda_{l_0+1}(D)\widehat{\psi_N}.$$

The right hand side is already determined, and the nonhomogeneous equation, as we concluded before, always has a solution, for instance

$$-R\left(\lambda_{l_0+1}(D)\widehat{\psi_N}\right).$$

This means that

$$\widehat{\psi_{N-1}} + R\left(\lambda_{l_0+1}(D)\widehat{\psi_N}\right)$$

is a λ_{l_0}-harmonic homogeneous polynomial of order $N-1$. We see that the solution $\widehat{\psi_{N-1}}$ exists and is determined up to an addition of any homogeneous λ_{l_0}-harmonic polynomial of degree $N-1$. Continuing this process until we reach $\widehat{\psi_0}$, we conclude that the mapping

$$\psi = (\psi_0, \ldots, \psi_N) \to \phi = (\phi_0, \ldots, \phi_N),$$

where

$$\phi_j = \widehat{\psi_j} + R\sum_{i>j} \lambda_{i-j+l_0}(D)\widehat{\psi_i}$$

establishes an isomorphism between the cokernel of the mapping Λ_N and the space of λ_{l_0}-harmonic polynomials of degree at most N. This proves the theorem. □

In the simplest possible cases, the theorem immediately implies the following:

Corollary 14. *Under the hypotheses of Theorem 13 one has:*

(1) *If k_0 is a noncritical point of a single band function $\lambda(k)$ such that $\lambda(0) = 0$, then the dimension of the space of Floquet solutions of the equation $Pu = 0$ on X of order at most N with a quasimomentum k_0 is equal to $q_{n-1,N}$, i.e., it is equal to the dimension of the space of all polynomials of degree at most N in $\mathbb{R}^{n-1}$.*

(2) *If the Taylor expansion at a point k_0 of a single band function $\lambda(k)$ such that $\lambda(0) = 0$ starts with a nonzero quadratic form*[13]*, then the dimension of the space of Floquet solutions of the equation $Pu = 0$ on X of order at most N with quasimomentum k_0 is equal to $h_{n,N}$, i.e., it is equal to the dimension of the space of harmonic (in the standard sense) polynomials of degree at most N in $\mathbb{R}^n$. In particular, this condition is satisfied at nondegenerate extrema of dispersion curves, i.e., the condition is satisfied at nondegenerate spectral edges.*

In both cases an isomorphism can be provided explicitly as in the previous theorem.

[13]This form can be degenerate, as opposed to our assumption in [58, Corollary 11].

4. Liouville type theorems for elliptic periodic systems

In this section we shall use the results of the previous sections to establish Liouville theorems for periodic equations. We will consider at the moment an arbitrary linear (square) matrix elliptic operator P on the space X of the abelian covering $X \to M$ with smooth G-periodic coefficients that satisfies the assumptions made in Section 2[14]. As before, without loss of generality we can limit the consideration to the case $G = \mathbb{Z}^n$. Any of the standard meanings of ellipticity of a system would do, e.g., ellipticity in Petrovsky or Douglis–Nirenberg sense [28].

Definition 15. Let $K \Subset X$ be a domain in X such that $\underset{g\in G}{\cup} gK = X$. For $N = 0, 1, \ldots$, define the space

$$\mathrm{V}_N(P) := \left\{ u \mid Pu = 0 \text{ in } X, \text{ and } \sup_{g\in G} \left[||u||_{L^2(gK)} (1+|g|)^{-N} \right] < \infty \right\}.$$

We say that *the Liouville theorem of order $N \geq 0$ holds true for the operator P* if $\dim \mathrm{V}_N(P) < \infty$.

We say that *the Liouville theorem holds true for the operator P* if it holds for any order $N \geq 0$.

Abusing notations, we will call solutions in $\mathrm{V}_0(P)$ *bounded solutions.* Since obviously $\mathrm{V}_{N_1}(P) \subset \mathrm{V}_{N_2}(P)$ for $N_1 < N_2$, Liouville theorem of higher order implies the lower order ones. It is not clear *a priori* that the converse holds. The next results show that this is in fact true, at least in our situation of abelian coverings. This observation apparently has failed to be made in previous studies, which sometimes lead to investigations of some individual Liouville theorems, e.g., for $N = 0$ without noticing their simultaneous validity for all N.

We will start with an auxiliary statement, which is an analog of the classical theorem on the structure of distributions supported at a single point. It is a generalization of Lemma 25 in [58], where Fredholm rather than semi-Fredholm property is assumed. For the results of this section Lemma 25 in [58] would be sufficient, while we need the full strength of the lemma below to treat overdetermined systems and holomorphic functions in Section 6.

Lemma 16. *Let T be a C^∞-manifold and $P : T \to L(H_1, H_2)$ be a C^∞-function with values in the space $L(H_1, H_2)$ of bounded linear operators*

[14] The results hold with the same proofs for periodic elliptic equations in sections of periodic vector bundles on X.

between the Hilbert spaces H_1 and H_2. Assume that for each $k \in T$ the operator $P(k)$ is right semi-Fredholm (e.g., [88]*), i.e., it has a closed range and a finite dimensional cokernel. Then*

(1) *If $P(k)$ is surjective for all points k in T, then the multiplication operator*

$$C^\infty(T, H_1) \stackrel{P(k)}{\rightarrow} C^\infty(T, H_2) \tag{4.1}$$

is surjective.

(2) *For any fixed $k_0 \in T$ the dimension of the space of functionals of the form*

$$\left[\sum_{j\leq N} D_{j,k}(< g_j, \phi >)\right]_{k_0} \tag{4.2}$$

that are orthogonal to the range of the multiplication operator (4.1) is finite. Here g_j are continuous linear functionals on H_2, the pairing $< g_j, \phi >$ denotes the duality between H_2^* and H_2, $D_{j,k}$ are linear differential operators with respect to k on T, and $N \geq 0$.

(3) *If $P(k)$ is surjective for all points k except of a finite subset $F \subset T$, then any continuous linear functional g on the space of smooth vector functions $C^\infty(T, H_2)$ that annihilates the range of the multiplication operator*

$$C^\infty(T, H_1) \stackrel{P(k)}{\rightarrow} C^\infty(T, H_2) \tag{4.3}$$

has the form

$$< g, \phi >= \sum_{k_l\in F} \left[\sum_{j\leq N} D_{j,k}(< g_j, \phi >)\right]_{k_l}, \tag{4.4}$$

in the notations of the previous statement of the lemma.

Proof: Let us establish the validity of the first statement of the lemma. Due to the existence of partitions of unity, the statement is local. Locally, following for instance the proof of Theorem 2.7 in [88], one can construct a smooth one-sided (right) inverse $Q(k)$ to the operator function $P(k)$. Now multiplication by $Q(k)$ provides a right inverse to (4.1), which proves the surjectivity.

To prove the second statement, let us consider a closed subspace $M \subset H_1$ complementary to the kernel of $P(k_0)$. Then the operator $P(k_0) : M \rightarrow H_2$ is injective and Fredholm. This injectivity property is preserved in a neighborhood U of k_0. In particular, the subspace

$M(k) = P(k)(M) \subset H_2$ of finite codimension forms a smooth subbundle $\mathcal{M}$ in $U \times H_2 \to U$. Now, any smooth H_2-valued function $f(k)$ such that $f(k) \in M(k)$, belongs to the range of the operator (4.1). Hence, functionals orthogonal to the range can be pushed down to the smooth sections of the finite dimensional bundle $\mathcal{C} = \bigcup_{k \in U} H_2/M(k)$ over U. It is clear that the functional one gets on this bundle preserves the structure (4.2). Such functionals on a finite dimensional bundle, however, form a finite dimensional space (for any fixed N).

The third statement can be proven analogously to the similar statement in [51, Corollary 1.7.2]. Namely, under the conditions of the statement, and taking into account the first claim of the lemma, any functional annihilating the range of the operator of multiplication by $P(k)$ must be supported at the finite set F over which $P(k)$ is not surjective. We can reduce the consideration to a neighborhood U of a single point $k_0 \in F$. Now, the proof of the second statement reduces the functional to one supported at the point k_0 and defined on smooth sections of a finite-dimensional bundle $\mathcal{C}$ over U. Thus, the standard representation of distributions supported at a point implies (4.4). □

The next theorem shows that the existence of a polynomially growing solution implies the existence of a nonzero bounded Bloch solution (i.e., a solution automorphic with respect to a unitary character of G).

Theorem 17. *The equation $Pu = 0$ has a nonzero polynomially growing solution if and only if it has a nonzero Bloch solution with a real quasimomentum (such a solution is automatically bounded), i.e., if and only if the real Fermi surface $F_{P,\mathbb{R}} = F_P \cap \mathbb{R}^n$ is not empty.*

Proof: Assume that $F_{P,\mathbb{R}} = \emptyset$. Then $P(k)$ is surjective for all $k \in \mathbb{T}^n$. Indeed, if this were not the case, we could find a nonzero functional of the type (3.9) with $N = 0$ and some k. According to the first statement of Lemma 10, this would mean the existence of a Bloch solution. Now, the first statement of Lemma 16 guarantees the surjectivity of the mapping

$$C^\infty(\mathbb{T}^n, \mathcal{E}^m) \overset{P(k)}{\to} C^\infty(\mathbb{T}^n, \mathcal{E}^0)$$

and hence the absence of any nontrivial functionals on $C^\infty(\mathbb{T}^n, \mathcal{E}^0)$ that annihilate the image of this mapping. Since under the Floquet–Gelfand transform $\mathcal{U}$, any polynomially growing solution $u(x)$ of $Pu = 0$ is mapped to such a functional, we conclude that $u = 0$. □

The next result provides necessary and sufficient conditions under which Liouville theorems hold for equations of the type we consider. It also establishes that the validity of Liouville type theorems does not depend on the order of polynomial growth.

Theorem 18. *Under the conditions we have imposed on the covering X and the operator P, the following statements are equivalent:*

(1) *The number of points in the real Fermi surface $F_{P,\mathbb{R}}$ is finite (i.e., Bloch (or automorphic) solutions exist for only finitely many unitary characters γ_k).*
(2) *There exists $N \geq 0$ such that the Liouville theorem of order N holds true.*
(3) *The Liouville theorem holds (i.e., it holds for any order N).*

Proof: (2) $\Rightarrow$ (1). Any Bloch solution with a real quasimomentum k (i.e., corresponding to a unitary character) is bounded and hence belongs to the space $\mathrm{V}_N(P)$ for any N. Since such solutions with different characters are linearly independent, the validity of the Liouville theorem for some value of N implies that the number of the corresponding characters is finite. Since the characters of all Bloch solutions with real quasimomenta constitute the real Fermi variety, $F_{P,\mathbb{R}}$ is finite.

The implication (3) $\Rightarrow$ (2) is obvious.

Let us now prove (1) $\Rightarrow$ (3). Let u be a nonzero polynomially growing solution. It can be interpreted as a continuous functional on C^0 annihilating the range of the dual operator $P^* : \mathrm{C}^m \to \mathrm{C}^0$. After the Floquet–Gelfand transform $\mathcal{U}$, one obtains a functional on $C^\infty(\mathbb{T}^n, \mathcal{E}^0)$ orthogonal to the range of the operator

$$C^\infty(\mathbb{T}^n, \mathcal{E}^m) \overset{P(k)}{\to} C^\infty(\mathbb{T}^n, \mathcal{E}^0).$$

By our assumption, $F_{P,\mathbb{R}}$ is finite, thus the second and the third statements of Lemma 16 and Theorem 13 finish the proof of statement (3). $\square$

While this theorem establishes conditions under which Liouville theorem holds, it does not tell much about the dimensions of the spaces $\mathrm{V}_N(P)$ of polynomially growing solutions, besides those being finite. It also does not address the structure of these solutions. The next result provides some estimates and even explicit formulas for the dimensions, as well a representation for the solutions.

Theorem 19. *Suppose that the Liouville theorem holds for an elliptic operator P and let $d_N := \dim \mathrm{V}_N(P)$. Then the following statements hold:*

(1) *Each solution $u \in \mathrm{V}_N(P)$ can be represented as a finite sum of Floquet solutions:*

$$u(x) = \sum_{k \in F_{P,\mathbb{R}}} \sum_j u_{k,j}(x), \tag{4.5}$$

where each $u_{k,j}$ is a Floquet solution with a quasimomentum k, and $F_{P,\mathbb{R}} = F_P \cap \mathbb{R}^n$.

(2) *For all $N \in \mathbb{N}$, we have*

$$d_N \leq d_0 q_{n,N} < \infty ,$$

where $q_{n,N}$ is the dimension of the space of all polynomials of degree at most N in n variables.

(3) *Assume that the spectra of the operators $P(k)$ are discrete for all k and that for each real quasimomentum $k \in F_{P,\mathbb{R}}$ the conditions of Theorem 13 are satisfied. Then for each $N \geq 0$ the dimension d_N of the space $\mathrm{V}_N(P)$ is equal to*

$$\sum_{k \in F_{P,\mathbb{R}}} r_k \left[\begin{pmatrix} n+N \\ N \end{pmatrix} - \begin{pmatrix} n+N-l_0(k) \\ N-l_0(k) \end{pmatrix} \right]. \tag{4.6}$$

Here r_k and $l_0(k)$ are respectively the multiplicity of the zero eigenvalue and the order of the first nonzero Taylor term of the dispersion relation at the point $k \in F_{P,\mathbb{R}}$ (see these notions explained before Theorem 13). The terms in this sum are the dimensions of the spaces of $\lambda_{l_0(k)}(D)$-harmonic polynomials, and polynomially growing solutions can be described in terms of these polynomials analogously to Theorem 13.

Proof: All these statements follow immediately from Lemma 10, Theorem 13, Lemma 16, and Theorem 18. □

5. Examples of Liouville theorems for specific operators

Let us recall that the L^2-spectrum of the operator P is the union over $k \in \mathrm{B}$ of the spectra of $P(k)$ (e.g., [51, Theorem 4.5.1] and [77, Theorem XIII.85]). In other words, the spectrum of P coincides with the range of the dispersion relation over the Brillouin zone B. We have also discussed that the real Fermi surface for P is just the zero level set for the dispersion relation over the Brillouin zone. Theorem 18 of the previous section shows that the Liouville theorem holds if and only if this Fermi surface is finite[15]. One expects this to happen normally at 'extrema' of the dispersion relation (albeit this is neither necessary, nor sufficient). In other words, speaking for instance about the *selfadjoint case*, one should expect Liouville theorem to hold mainly when zero is at the edge of a spectral gap, although it is possible in principle to have interior points of the spectrum where such a thing

[15] In particular, outside the spectrum the Liouville theorem holds vacuously, according to Theorem 18 and Theorem 4.5.1 in [51].

could occur as well[16]. One notices that the cases considered in [5, 61, 69] all correspond to zero being at the bottom of the spectrum (in the second-order nonselfadjoint case we mean by the bottom of the spectrum the generalized principal eigenvalue, see [2, 58, 64] and Section 5.4 below). This explains why the homogenization techniques employed in these papers could be successful. Indeed, homogenization works exactly at the bottom of the spectrum. The results of this work show that one should also consider internal spectral edges, where the standard homogenization does not apply.

So, let us now try to look at some examples where one can apply the results of the previous sections. Theorem 18 is applicable to any elliptic periodic equation or system of equations on any abelian covering of a compact manifold. According to this theorem, one only needs to establish the finiteness of the real Fermi surface. On the other hand, Theorem 19, which provides formulas for the dimensions of the spaces of polynomially growing solutions, as well as some representations of such solutions, is more demanding. It requires that one guarantees discreteness of the spectra of the 'cell' operators $P(k)$, and most of all, requires understanding of the analytic structure of the dispersion curve near the Fermi surface. Concerning this structure, it is expected that the following is true:

Conjecture 20. *Let P be a 'generic' selfadjoint second-order elliptic operator with periodic coefficients, and let (λ_-, λ_+) be a nontrivial gap in its spectrum. Then each of the gap's endpoints is a unique (modulo the dual lattice) and nondegenerate extremum of a single band function $\lambda_j(k)$.*

This conjecture is crucial in many problems of mathematical physics, spectral theory, and homogenization (see its further discussion in Section 8) and is widely believed to hold. Unfortunately, the only known theorem of this kind is the recent result of [46], which states that generically a gap edge is an extremum of a single band function. A similar conjecture probably holds for equations on abelian coverings of compact manifolds. Theorem 18 shows that validity of Conjecture 20 would imply the following statement:

Theorem 21. *If Conjecture 20 holds true, then 'generically', at the spectral edges of a selfadjoint elliptic operator of second-order with periodic coefficients on $\mathbb{R}^n$, the Liouville theorem holds, and for each*

[16]Two spectral zones with touching edges could provide such an example.

$N \geq 0$ the dimension of the space V_N is equal to $h_{n,N}$, the dimension of the space of all harmonic polynomials of order at most N in n variables.

In the similar situation on abelian coverings of compact manifolds, the rank of the deck group would appear in the answer rather than the dimension of the manifold (see Section 8).

At the bottom of the spectrum, however, much more is known [15, 16, 18, 31, 45, 76, 80]. Even this limited (in terms of spectral location) information together with theorems 18 and 19 provides one with many specific examples that go far beyond the equations considered in [5, 61, 69].

The first trivial remark is that if zero is outside the spectrum of the operator P, then according to Theorem 18 the Liouville property holds vacuously. Indeed, in this case the real Fermi surface is empty, and hence the equation $Pu = 0$ has no polynomially growing solutions. Let us look now at some less trivial examples.

5.1. **Schrödinger operators.** Let $X \to M$ be, as before, a noncompact abelian covering of a d-dimensional compact manifold and $H = -\Delta + V(x)$ be a Schrödinger operator on X with a periodic real valued potential $V \in L^{r/2}_{\mathrm{loc}}(X)$, $r > d$. Then the result of [45] for $X = \mathbb{R}^d$ and of [48] in the general case states that the lowest band function $\lambda_1(k)$ has a unique nondegenerate minimum Λ_0 at $k = 0$. All other band functions are strictly greater than Λ_0. In particular, the bottom of the spectrum of the operator H is at Λ_0. Let us assume that $\Lambda_0 = 0$ (or replace the operator by $H = -\Delta + V(x) - \Lambda_0$). Then theorems 18 and 19 become applicable, since the real Fermi surface consists of a single point $k = 0$ and the band function has a simple nondegenerate minimum at this point (i.e., $r = 1$ and $l_0 = 2$ in the notations of Theorem 19). Thus, one obtains

Theorem 22.
(1) *The Liouville theorem holds for the operator $H - \Lambda_0$.*
(2) *The dimension of the space $\mathrm{V}_N(H - \Lambda_0)$ equals $h_{n,N}$ (where n is the free rank of G).*
(3) *Every solution $u \in \mathrm{V}_N(H - \Lambda_0)$ of the equation $Hu - \Lambda_0 u = 0$ is a Floquet solution of order N with quasimomentum $k = 0$, i.e., it can be represented as*

$$u(x) = \sum_{|j| \leq N} [x]^j p_j(x)$$

with G-periodic functions p_j.

5.2. **Magnetic Schrödinger operators.** Although the first result of this subsection holds also in the general situation of abelian coverings, we will present it for simplicity in the flat case.

Consider the following selfadjoint magnetic Schrödinger operator on $\mathbb{R}^n$

$$H = [\mathrm{i}\nabla + A(x)]^2 + V(x) \tag{5.1}$$

with periodic electric and magnetic potentials V and A, respectively[17]. The introduction of a periodic magnetic potential into the operator is known to change properties of the Fermi surface significantly (e.g., [31, 80]). However, small magnetic potentials do not destroy the properties of our current interest. Indeed, let V be as in the previous statement, then there exists $\varepsilon > 0$ (depending on V) such that for any periodic real valued magnetic potential A such that

$$||A||_{L^r(\mathbb{T}^n)} < \varepsilon$$

and

$$\int\limits_{\mathbb{T}^n} A(x)\,\mathrm{d}x = 0 \tag{5.2}$$

the following holds true: The lowest band function $\lambda_1(k)$ of H attains a unique nondegenerate minimum Λ_{k_0} at a point k_0 (albeit, not necessarily at $k = 0$). All other band functions are strictly greater than Λ_{k_0}. Indeed, when both V and A are sufficiently small, this is proven in [31]. It is not hard, though, to allow for arbitrary electric and small magnetic potential. Indeed, the case when $A = 0$ is covered by the result of [45] (see the previous subsection). Now, the statement of Lemma 8 (see also [51, Theorem 4.4.2]) can be easily extended without any change in the proof to include analyticity with respect to the potentials (e.g., [31]). Namely, there exists an entire function $f(k, \lambda, A, V)$ of all its arguments such that $f(k, \lambda, A, V) = 0$ is equivalent to

$$(k, \lambda) \in B_{(\mathrm{i}\nabla+A)^2+V}\ ,$$

where as before B_H is the Bloch variety of the operator H. This, together with the just mentioned result of [45] for $A = 0$, imply that the lowest band function $\lambda_1(k)$ of H attains a unique nondegenerate minimum Λ_{k_0} at a point k_0 for sufficiently small magnetic potentials[18].

[17] One can see that the considerations and the result of this subsection immediately extend to more general abelian coverings. We avoid doing so, in order not to introduce any new notions required for defining the magnetic operators in such a setting.

[18] The degree of smallness of A depends on the electric potential V.

Now one uses theorems 18 and 19 again to obtain

Theorem 23. *Assume that V, A, and k_0 satisfy the above assumptions, then*

(1) *The Liouville theorem holds for $u \in \mathrm{V}_N(H - \Lambda_{k_0})$.*
(2) *Any solution $u \in \mathrm{V}_N(H - \Lambda_{k_0})$ is representable in the Floquet form*

$$v(x) = \mathrm{e}^{\mathrm{i}k_0 \cdot x} \sum_{|j| \leq N} x^j p_j(x)$$

with periodic functions $p_j(x)$.
(3) *The dimension of the space $\mathrm{V}_N(H - \Lambda_{k_0})$ is equal to $h_{n,N}$.*

Note that the normalization (5.2) can be always achieved by a gauge transformation, which does not affect the spectrum and the Liouville property.

Besides magnetic Schrödinger operators with small magnetic potentials, another very special subclass of periodic magnetic Schrödinger operators is formed by the so called *Pauli operators*. Consider the following Pauli operators in $\mathbb{R}^2$:

$$P_{\pm} = (\mathrm{i}\nabla + A)^2 \pm (\partial_{x_1} A_2 - \partial_{x_2} A_1), \tag{5.3}$$

where $A(x) = (A_1(x), A_2(x))$ is periodic. The structure of the dispersion curves at the bottom of the spectrum of such an operator was studied in [15]. It was shown that the dispersion relation for $P_{\pm}$ attains at the point $k = 0$ its single nondegenerate minimum with the value $\Lambda_0 = 0$. This implies that the result for the Pauli operators holds exactly like for the Schrödinger operator in the previous subsection, and in fact in a more precise form, since the minimal value Λ_0 of the band function is known. Namely, every solution $u \in \mathrm{V}_N(P_{\pm})$ of the equation $P_{\pm} u = 0$ on $\mathbb{R}^2$ is representable in the form (1.4). The dimension of the space $\mathrm{V}_N(P_{\pm})$ is equal to $h_{2,N} = 2N + 1$, which is the dimension of the space of all harmonic polynomials of order at most N in two variables.

The cases above of small magnetic potentials and of Pauli operators form a rather special subclass of periodic magnetic Schrödinger operators, in the sense that one still finds a single nondegenerate minimum of band functions at the bottom of the spectrum. However, this is not true anymore for the whole class of periodic magnetic Schrödinger operators, which should influence significantly our Liouville theorems. Although there is not much known here yet, the results of [80] provide the first glimpse at the possibilities. In that paper the following operator in $\mathbb{R}^2$ is considered:

$$M_t = (\mathrm{i}\nabla + tA(x))^2 - t^2,$$

where $t \in \mathbb{R}$ and the magnetic potential $A(x)$ is $(0, a(x_1))$ with the 1-periodic function $a(x_1)$ that is equal to $\operatorname{sign}(x_1)$ on $[-0.5, 0.5]$. The main result of [80] is that for $|t| < 2\sqrt{3}$ the bottom of the spectrum of M_t is 0 and it is attained at the quasimomentum $k = 0$, where the band function has a simple nondegenerate minimum. This implies immediately the same Liouville theorem as for the Pauli operators. The situation changes for $|t| = 2\sqrt{3}$, when this minimum (which is still 0 and is attained at $k = 0$) becomes degenerate. Namely, it is shown in [80] that the lowest eigenvalue $\lambda_1(k)$ of the operators $M_t(k)$ for $|t| = 2\sqrt{3}$ has the following Taylor expansion at $k = 0$:

$$\lambda_1(k) = k_1^2 + \frac{1}{42}k_2^4 - \frac{1}{10}k_1^2k_2^2 + O(|k|^6).$$

This shows the degeneration of the Hessian. On the other hand, since the first nonzero homogeneous term of the expansion is of order two, Theorem 19 still says that the Liouville theorem sounds the same as before (i.e., $\dim \mathrm{V}_N(M_{\pm 2\sqrt{3}}) = 2N + 1$). This changes, though, for a little bit larger values of $|t|$, when the bottom of the spectrum shifts to the negative half-axis and two points of extremum appear with a nonvanishing Hessian at these quasimomenta. This implies that the Liouville theorem will apply after shifting to the bottom of the spectrum (i.e., after adding an appropriate positive scalar term to M_t), but the dimensions of the spaces V_N of polynomially growing solutions will increase above $h_{2,N}$.

Theorem 24. (1) *The Liouville theorem holds for the operators $P_\pm$ and M_t for $|t| \le 2\sqrt{3}$.*

(2) *The dimensions of the corresponding spaces V_N are equal to $h_{2,N} = 2N + 1$.*

(3) *Every solution $u \in \mathrm{V}_N$ can be represented as*

$$u(x) = \sum_{|j| \le N} x^j p_j(x)$$

with G-periodic functions p_j.

(4) *For a sufficiently small $\varepsilon > 0$ and the values $|t| \in (2\sqrt{3}, 2\sqrt{3} + \varepsilon)$, the Liouville theorem holds at the bottom of the spectrum of the operator M_t, while the dimension of the space $\mathrm{V}_N(M_t)$ is equal to $2h_{2,N}$.*

It is also conjectured in [80] that for some values of t, the operator M_t with the magnetic potential equal to $A(x) = (\operatorname{sign}(x_2), \operatorname{sign}(x_1))$ on the cube $[-0.5, 0.5]^2$ and $\mathbb{Z}^2$-periodic, will exhibit a complete degeneration of the bottom of the spectrum in the sense of complete vanishing of the

Hessian. If this happens to be true, then according to Theorem 19, it would lead to an increase in the dimensions of the spaces V_N.

5.3. **Operators admitting regular factorization.** A very general and wide class of selfadjoint second-order periodic operators H on $\mathbb{R}^n$, for which one can study thoroughly the dispersion curves at the bottom of the spectrum, was introduced in [16, 17]. It consists of operators that allow for what the authors of [16, 17] call *regular factorization*:

$$H = \overline{f(x)}b(D)^*G(x)b(D)f(x). \tag{5.4}$$

Here $b(D) = \sum b_j D_{x_j}$ is a constant coefficients first-order linear differential operator, the periodic function $f(x)$ is such that $f, f^{-1} \in L^\infty(\mathbb{R}^n)$, and the periodic matrix-function $G(x)$ satisfies $c_0 I \leq G(x) \leq c_1 I$ for some $0 < c_0 \leq c_1$. Then, it was shown in [17] that the bottom of the spectrum of H is 0 and it is attained at the quasimomentum $k = 0$, where the band function has a simple nondegenerate minimum. As in the previous subsections, this implies

Theorem 25. (1) *The Liouville theorem holds for the operator H (5.4).*
(2) $\dim \mathrm{V}_N(H) = h_{n,N}$.
(3) *Every solution $u \in \mathrm{V}_N(H)$ can be represented as*

$$u(x) = \sum_{|j| \leq N} x^j p_j(x)$$

with G-periodic functions p_j.

5.4. **Nonselfadjoint second-order operators.** Throughout this subsection, we again consider the flat case $X = \mathbb{R}^n$ only. However, the result still holds on general abelian coverings, which does not require any significant change in the proof.

In all the examples that were discussed so far in this section, we considered only selfadjoint operators. There is, however, a class of (in general nonselfadjoint) second-order operators in $\mathbb{R}^n$, which can be studied thoroughly and which plays an important role in probability theory.

Consider second-order uniformly elliptic operators on $\mathbb{R}^n$ of the following form (3.7):

$$L = -\sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^{n} b_i(x)\partial_i + c(x) \qquad x \in \mathbb{R}^n \tag{5.5}$$

with coefficients that are *real* and periodic.

For an operator of this type, an important function $\Lambda(\xi) : \mathbb{R}^n \to \mathbb{R}$ can be introduced[19], whose properties were studied in detail in [2, 64, 76]. It is defined by the condition that the equation

$$Lu = \Lambda(\xi)u$$

has a *positive* Bloch solution of the form

$$u_\xi(x) = \mathrm{e}^{\xi\cdot x} p_\xi(x), \tag{5.6}$$

where $p_\xi(x)$ is G-periodic.

We also need to define the following number:

$$\Lambda_0 = \max_{\xi\in\mathbb{R}^n} \Lambda(\xi). \tag{5.7}$$

It follows from [2, 64] that Λ_0 can also be described as follows:

$$\Lambda_0 = \sup\{\lambda \in \mathbb{R} \mid \exists u > 0 \text{ such that } (L-\lambda)u = 0 \text{ in } \mathbb{R}^n\}. \tag{5.8}$$

In the selfadjoint case, Λ_0 is the bottom of the spectrum of the operator L. The common name for Λ_0 is *the generalized principal eigenvalue* of the operator L in $\mathbb{R}^n$.

We assemble the information we need about the function $\Lambda(\xi)$ and the generalized principal eigenvalue Λ_0 in the following lemma. The reader can find proofs of these statements in [2, 64, 76] and more detailed references in [58].

Lemma 26.
1. *The value $\Lambda(\xi)$ is uniquely determined for any $\xi \in \mathbb{R}^n$.*
2. *The function $\Lambda(\xi)$ is bounded from above, strictly concave, analytic, and has a nonzero gradient at all points except at its maximum point.*
3. *Consider the operator*

   $$L(\xi) = \mathrm{e}^{-\xi\cdot x} L \mathrm{e}^{\xi\cdot x} = L(x, D - i\xi)$$

   on the torus $\mathbb{T}^n$. Then $\Lambda(\xi)$ is the principal eigenvalue of $L(\xi)$ with a positive eigenfunction p_ξ. Moreover, $\Lambda(\xi)$ is algebraically simple.
4. *The Hessian of $\Lambda(\xi)$ is nondegenerate at all points.*
5. *$\Lambda_0 \geq 0$ if and only if the operator L admits a positive (super-) solution. This condition is satisfied in particular when $c(x) \geq 0$.*
6. *$\Lambda_0 \geq 0$ if and only if the operator L admits a positive solution of the form (5.6).*

[19] The notation Λ has been used before in this paper to denote a somewhat different -albeit related - object. However, these two notations do not collide in this subsection, so the reader should not get confused.

(7) $\Lambda_0 = 0$ *if and only if the equation* $Lu = 0$ *admits exactly one normalized positive solution in* $\mathbb{R}^n$.

(8) *If* $c(x) = 0$, *then* $\Lambda_0 = 0$ *if and only if* $\int\limits_{\mathbb{T}^n} b(x)\psi(x)\,\mathrm{d}x = 0$, *where* ψ *is the principal eigenfunction of* L^* *on* $\mathbb{T}^n$, *and* $b(x) = (b_1(x), \dots, b_n(x))$. *In particular, divergence form operators satisfy this condition.*

(9) *Let* $\xi \in \mathbb{R}^n$, *and assume that* $u_\xi(x) = \mathrm{e}^{\xi\cdot x}p_\xi(x)$ *and* $u^*_{-\xi}$ *are positive Bloch solutions of the equations* $Lu = 0$ *and* $L^*u = 0$, *respectively. Denote by* ψ *the periodic function* $u_\xi u^*_{-\xi}$. *Consider the function*

$$\tilde{b}_i(x) = b_i(x) - 2\sum_{j=1}^{n} a_{ij}(x)\left\{\xi_j + [p_\xi(x)]^{-1}\,\partial_j p_\xi(x)\right\}.$$

Then $\Lambda_0 = 0$ *if and only if*

$$\int\limits_{\mathbb{T}^n} \tilde{b}_i(x)\psi(x)\,\mathrm{d}x = 0 \qquad i = 1, \dots, n.$$

Now one can describe Liouville type theorems for an operator of the form (5.5) assuming that $\Lambda_0 \geq 0$. This assumption implies that the operator admits a positive supersolution.

It is shown in [58] that if $\Lambda(0) \geq 0$, then the Fermi surface F_L can touch the real space only at the origin (modulo the reciprocal lattice $G^* = (2\pi\mathbb{Z})^n$) and in this case $\Lambda(0) = 0$.

In fact, we have the following stronger result which extends Lemma 15 in [58] to nonreal λ. In addition, the proof of the statement below is more elementary than in that lemma.

Lemma 27. *Let* $k = \gamma - i\xi \in \mathbb{C}^n$. *If* $Re\,\lambda < \Lambda(\xi)$, *then* (k, λ) *does not belong to the Bloch variety* B_L *of the operator* L. *Moreover, if* $Re\,\lambda = \Lambda(\xi)$, *then* $(k, \lambda) \in B_L$ *if and only if* $\gamma \in G^*$, ξ *belongs to the zero level set* Ξ *of* $\Lambda(\xi)$, *and* $Im\,\lambda = 0$.

Proof: Without loss of generality, we may assume that $\xi = 0$, $\Lambda(\xi) = 0$, and $L\mathbf{1} = 0$. Assume that $(k, \lambda) \in B_L$, and let $u(x) = \mathrm{e}^{\mathrm{i}\gamma\cdot x}p(x)$ be a Floquet solution with a quasimomentum k of the equation

$$Lu = \lambda u \qquad \text{in } \mathbb{R}^n,$$

where $Re\,\lambda \leq 0$. Take the complex conjugate

$$L\bar{u} = \bar{\lambda}\bar{u} \qquad \text{in } \mathbb{R}^n,$$

and compute

$$L(|u|^2) = \bar{u}Lu + uL\bar{u} - 2\sum_{i,j=1}^{n} a_{ij}u_{x_i}\bar{u}_{x_j} = 2Re\,\lambda|u|^2 - 2\sum_{i,j=1}^{n} a_{ij}u_{x_i}\bar{u}_{x_j}.$$

Notice that for each $\zeta \in \mathbb{C}^n$, we have

$$\sum_{i,j=1}^{n} a_{ij}\zeta_i\bar{\zeta}_j = \sum_{i,j=1}^{n} a_{ij}Re\,\zeta_i Re\,\zeta_j + \sum_{i,j=1}^{n} a_{ij}Im\,\zeta_i Im\,\zeta_j \geq 0.$$

Therefore,

$$L(|u|^2) \leq 0. \tag{5.9}$$

Thus, $|u|^2 = |p(x)|^2$ is a periodic nonnegative subsolution of the equation $Lu = 0$ in $\mathbb{R}^n$. By the strong maximum principle $|u(x)|^2 =$ constant. In particular, $L(|u|^2) = 0$. Since we have equality in (5.9), it follows that $Re\,\lambda = 0$ and

$$\sum_{i,j=1}^{n} a_{ij}u_{x_i}\bar{u}_{x_j} = \sum_{i,j=1}^{n} a_{ij}Re\,u_{x_i}Re\,u_{x_j} + \sum_{i,j=1}^{n} a_{ij}Im\,u_{x_i}Im\,u_{x_j} = 0.$$

It follows that $u =$ constant and therefore, $\mathrm{e}^{\mathrm{i}\gamma\cdot x}$ is a periodic function. Consequently, $\gamma \in G^*$. Since $L\mathbf{1} = 0$, it follows that $Im\,\lambda = 0$. $\square$

Lemma 27 and Theorem 18 imply that if $\Lambda(0) > 0$, then the Liouville theorem holds vacuously, and the equation $Lu = 0$ does not admit any nontrivial polynomially growing solution.

Suppose now that $\Lambda(0) = 0$ (i.e., $0 \in F_{L,\mathbb{R}}$). Then by Lemma 27 $F_{L,\mathbb{R}} = \{0\}$. Moreover, Lemma 26 implies that if $\Lambda_0 > 0$, then the point $k = 0$ is a noncritical point of the dispersion relation, and if $\Lambda_0 = 0$ then $k = 0$ is a nondegenerate extremum. In terms of Theorem 19, we have in the first case $l_0 = 1$, while in the second $l_0 = 2$. Moreover, in both cases $r = 1$. Theorems 18 and 19 imply now that the Liouville theorem holds, and every solution $u \in \mathrm{V}_N(L)$ is representable in the form (1.4). The dimension of the space $\mathrm{V}_N(L)$ is equal to $h_{n,N}$ in the case when $\Lambda_0 = 0$, and to $q_{n-1,N}$ when $\Lambda_0 > 0$.

We summarize these results in the following statement:

Theorem 28. *Let L be a periodic operator of the form (5.5) such that $\Lambda_0 \geq 0$. Then*

(1) *The Liouville theorem holds vacuously if $\Lambda(0) > 0$, i.e., the equation $Lu = 0$ does not admit any nontrivial polynomially growing solution.*

(2) *If* $\Lambda(0) = 0$ *and* $\Lambda_0 > 0$, *then the Liouville theorem holds for* L,

$$\dim \mathrm{V}_N(L) = q_{n-1,N} = \begin{pmatrix} n+N-1 \\ N \end{pmatrix},$$

and every solution $u \in \mathrm{V}_N(L)$ *of the equation* $Lu = 0$ *can be represented as*

$$u(x) = \sum_{|j| \le N} x^j p_j(x)$$

with G*-periodic functions* p_j.

(3) *If* $\Lambda(0) = 0$ *and* $\Lambda_0 = 0$, *then the Liouville theorem holds for* L,

$$\dim \mathrm{V}_N(L) = h_{n,N} = \begin{pmatrix} n+N \\ N \end{pmatrix} - \begin{pmatrix} n+N-2 \\ N-2 \end{pmatrix},$$

and every solution $u \in \mathrm{V}_N(L)$ *of the equation* $Lu = 0$ *can be represented as*

$$u(x) = \sum_{|j| \le N} x^j p_j(x)$$

with G*-periodic functions* p_j.

It is interesting to notice that for selfadjoint operators, one never faces the case $l = 1$ at the spectral edges, as it does happen above when $\Lambda(0) = 0$ and $\Lambda_0 > 0$.

6. Liouville theorems for overdetermined elliptic periodic systems and for holomorphic functions

We will show here how the techniques and results of the preceding sections can be applied to some overdetermined elliptic systems, in particular to the Cauchy–Riemann operators on abelian covers of compact complex manifolds. The main construction stays the same, so we will be brief in its description.

We consider as before an abelian covering $X \overset{G}{\mapsto} M$ of a compact Riemannian manifold. Let also L_j $(j = 0, 1, \dots)$ be finite dimensional smooth vector bundles on M equipped with a Hermitian metric and $\mathcal{L}_j$ be the sheaves of their smooth sections. Suppose that we have an elliptic complex of differential operators

$$0 \to \mathcal{L}_0 \overset{P}{\to} \mathcal{L}_1 \overset{P_1}{\to} \mathcal{L}_2 \overset{P_2}{\to} \dots.$$

We can lift this complex to an elliptic complex on X for which we will use the same notations:

$$0 \to \mathcal{L}_0 \overset{P}{\to} \mathcal{L}_1 \overset{P_1}{\to} \mathcal{L}_2 \overset{P_2}{\to} \dots. \tag{6.1}$$

Notice that the lifted bundles have a natural G-action with respect to which the operators are periodic.

We will be interested in the spaces $\mathrm{V}_N(P)$ of the global solutions $u(x)$ of the equation $Pu = 0$ on X that have polynomial growth in the sense that for any compact $K \Subset X$

$$\|u\|_{L^2(gK,L_0)} \leq C(1+|g|)^N \qquad \forall g \in G \tag{6.2}$$

for some C and N. As before, without loss of generality we can assume that $G = \mathbb{Z}^n$. The Floquet–Gelfand transform reduces the elliptic complex (6.1) and its dual to the direct integrals with respect to the quasimomentum k of elliptic complexes on M

$$0 \to \mathcal{E}^1 \overset{P(k)}{\to} \mathcal{E}^2 \overset{P_1(k)}{\to} \dots, \tag{6.3}$$

and

$$0 \leftarrow \mathcal{F}^1 \overset{P^*(k)}{\leftarrow} \mathcal{F}^2 \overset{P_1^*(k)}{\leftarrow} \dots, \tag{6.4}$$

where $\mathcal{E}^j, \mathcal{F}^j$ are appropriately defined analytic Banach vector bundles over $\mathbb{C}^n$ (a more detailed setup of the relevant spaces can be found for instance in [74].). As previously, solutions of polynomial growth of $Pu = 0$ after applying the Floquet–Gelfand transform become functionals on $C^\infty(\mathbb{T}^n, \mathcal{E}^1)$ orthogonal to the range of the operator

$$C^\infty(\mathbb{T}^n, \mathcal{F}^2) \overset{P^*(k)}{\to} C^\infty(\mathbb{T}^n, \mathcal{F}^1). \tag{6.5}$$

Notice that due to the ellipticity of the complex and the compactness of the base, it is Fredholm as a complex of operators between the appropriate Sobolev spaces (e.g., [41, Vol. III, Sections 19.4 and 19.5], [78, Section 3.2., in particular Theorem 13], [84, Section IV.5]). Hence, the operators $P^*(k)$ on M belong to the set Φ_r of the right semi-Fredholm operators between the corresponding Sobolev spaces of sections. This means that they have closed ranges of finite codimension, while having infinite dimensional kernels. It is also clear that as in the elliptic case before, $P(k)$ depends analytically (in fact, polynomially) on k. The Fermi surface F_P of P is introduced as the set of all values of k for which the equation $P(k)u = 0$ has a nonzero solution[20]. As before, it coincides with the set where the dual operator $P^*(k)$ has a nonzero cokernel. This enables us to carry over the considerations of the previous section to prove the following result:

Theorem 29. *The following conditions are equivalent:*

(1) *The (real) Fermi surface $F_{P,\mathbb{R}}$ is finite.*

[20]It can be shown similarly to the elliptic case that the Fermi surface, as in the elliptic case, is an analytic subset of $\mathbb{C}^n$.

(2) *For some $N \geq 0$ the dimension d_N of the space $\mathrm{V}_N(P)$ of solutions of polynomial growth of order N is finite (i.e., the Liouville theorem of order N holds).*
(3) *The dimension of $\mathrm{V}_N(P)$ is finite for any $N \geq 0$ (i.e., the Liouville theorem holds) and $d_n \leq d_0 q_{n,N}$.*

If one of the above conditions holds, then any solution of polynomial growth is a linear combination of Floquet solutions.

Proof: Since any point $k \in F_{P,\mathbb{R}}$ provides a bounded Bloch solution and those for different values of k (modulo the dual lattice) are linearly independent, the implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are obvious. Let us now assume (1) and prove (3). According to Lemma 16, all functionals of order N orthogonal to the range of (6.5) can be expressed as linear combinations over $k \in F_{P,\mathbb{R}}$ of the form (4.4). The same Lemma together with Lemma 10 and the finiteness of $F_{P,\mathbb{R}}$ show the finite dimensionality of this space and the Floquet representation for all solutions of this type. $\square$

We apply the above theorem to the following interesting case.

Theorem 30. *Let $X \xrightarrow{G} M$ be an abelian cover of a compact (complex) analytic manifold M. Let X be equipped with the lift of an arbitrary Riemannian metric from M and ρ be the corresponding distance. For $N = 0, 1, \ldots$, let $A_N(X)$ be the space of all analytic functions on X growing not faster than $C(1 + \rho(x))^N$. Then:*

(1) *The real Fermi surface of the $\bar{\partial}$-operator defined on functions on X contains only the origin: $F_{\bar{\partial},\mathbb{R}} = \{0\}$.*
(2) *$A_0(X)$ consists of constants.*
(3) *For any $N \geq 0$, $\dim A_N(X) \leq q_{n,N} < \infty$. All elements of the space $A_N(X)$ are holomorphic Floquet–Bloch functions with quasimomentum $k = 0$.*
(4) *There exists a finite family of holomorphic functions $\{f_j\}$ on X such that all elements of $A_N(X)$ are polynomials of $\{f_j\}$.*

Proof: Let $k \in \mathbb{R}^n$ be in $F_{\bar{\partial},\mathbb{R}}$. This means that there exists a nonzero analytic function $u(x)$ of Bloch type, i.e., automorphic with a unitary character γ_k of the group G. The absolute value of $u(x)$ is then G-periodic and thus can be pushed down to M. Due to the compactness of M, it must attain its maximum. Hence, the function $u(x)$ attains somewhere on X the maximum of its absolute value. The standard maximum principle now implies that $u =$ constant. This means in particular that $k = 0$. Hence, (1) holds.

Representation (4.4) of Lemma 16 implies now that $A_0(X)$ consists of constants. Indeed, this space is comprised of analytic functions on X

automorphic with respect to unitary characters of G. As we have just shown, every such automorphic function is automatically constant.

Let us extend the $\bar\partial$ operator on X to the elliptic Dolbeault complex (e.g., [84, Chapter IV, Examples 2.6 and 5.5]):

$$0 \to \mathcal{A}^{0,0} \overset{\bar\partial}{\to} \mathcal{A}^{0,1} \overset{\bar\partial}{\to} \mathcal{A}^{0,2} \overset{\bar\partial}{\to} \dots, \tag{6.6}$$

where $\mathcal{A}^{p,q}$ is the sheaf of germs of smooth (p,q)-forms on X. The conditions of Theorem 29 are now satisfied, which proves the statement (3).

Finally, the implication (3)⇒(4) was established in [19]. □

7. Liouville theorems on combinatorial and quantum graphs

Liouville type theorems have also been studied intensively on discrete objects (graphs, discrete groups) (e.g., [43, 68]). Our technique is insensitive to the local structure of the base of the covering, and hence it can also be applied to abelian co-compact coverings of graphs.

Let Γ be a countable noncompact graph with a free co-compact action of an abelian group G of a free rank n. Denote by $\tilde\Gamma$ the finite graph Γ/G. Consider any periodic finite order difference operator P on Γ. One can define all the basic notions: solutions of polynomial growth, Floquet solutions, quasimomenta, Fermi variety, etc., exactly the way they were defined before. The same procedure as before leads to the following result that we state without a proof:

Theorem 31. (1) *If the Liouville theorem of order $N \geq 0$ for the equation $Pu = 0$ on Γ holds true, then it holds to any order.*

(2) *In order for the Liouville theorem to hold, it is necessary and sufficient that the real Fermi surface $F_{P,\mathbb{R}}$ (i.e., the set of unitary characters χ for which the equation $Pu = 0$ has a nonzero χ-automorphic solution) is finite.*

(3) *If the Liouville theorem holds true, then:*
   (a) *Each solution $u \in \mathrm{V}_N(P)$ can be represented as a finite sum of Floquet solutions of order up to N.*
   (b) *If $d_N = dim\, \mathrm{V}_N(P)$, then for all $N \in \mathbb{N}$, we have*

$$d_N \leq d_0 q_{n,N} < \infty\,,$$

   where $q_{n,N}$ is the dimension of the space of all polynomials of degree at most N in n variables.
   (c) *Assume that for each real quasimomentum $k \in F_{P,\mathbb{R}}$ the conditions of Theorem 13 are satisfied. Then for each $N \geq$*

0 the dimension d_N of the space $\mathrm{V}_N(P)$ is equal to

$$\sum_{k \in F_{P,\mathbb{R}}} r_k \left[\begin{pmatrix} n+N \\ N \end{pmatrix} - \begin{pmatrix} n+N-l_0(k) \\ N-l_0(k) \end{pmatrix} \right].$$

Here r_k and $l_0(k)$ are respectively the multiplicity of the zero eigenvalue and the order of the first nonzero Taylor term of the dispersion relation at the point $k \in F_{P,\mathbb{R}}$. The terms in this sum are the dimensions of the spaces of $\lambda_{l_0(k)}$-harmonic polynomials, and polynomially growing solutions can be described in terms of these polynomials analogously to Theorem 13.

The same can also be done for the so called *quantum graphs*, where a differential (rather than difference) equation is considered on a graph that is treated as a one-dimensional singular variety rather than a purely combinatorial object (see the details in [52]-[55]). A quantum graph Γ is a graph with two additional structures. First of all, the graph must be *metric*, i.e., each edge e must be supplied with a (finite in our case) length $l_e > 0$ and correspondingly with an 'arc length' coordinate. This allows one to do differentiations and integrations and to define differential operators on Γ. The simplest ones are $-\mathrm{d}^2x/\mathrm{d}x^2 + V(x)$, where differentiation is done with respect to the edge coordinates, and $V(x)$ is a sufficiently nice (e.g., measurable and bounded) potential. In order to define such an operator, one needs to impose some boundary conditions at vertices. All such 'nice' conditions have been described (e.g., [39, 49, 55]). The simplest example is the so called Neumann (or Kirchhoff) condition that requires that the functions are continuous at the vertices and that the sum of outgoing derivatives at each vertex is equal to zero. A *quantum graph* is a metric graph equipped with a selfadjoint operator P of the described kind.

Let now Γ be a noncompact quantum graph with a free co-compact isometric action of an abelian group G of a free rank n, and let $\tilde{\Gamma}$ be the compact quantum graph Γ/G. We assume that the Hamiltonian P of the quantum graph is G-periodic. The basic notions (solutions of polynomial growth, Floquet solutions, quasimomenta, Fermi variety, etc.) can again be defined for this situation. In particular, we consider the operators $P(k)$, and assume that for each $k \in \mathbb{C}^n$ the spectrum of $P(k)$ is discrete.

Theorem 32. *The statements of Theorem 31 hold for the quantum graph Γ under the conditions just described.*

8. Further remarks

(1) In most parts of the paper we assumed that all the coefficients of the operators P and P^* are C^∞-smooth. This assumption was made for simplicity only, and in fact one does not need such a strong restriction (e.g., see the discussion in [58, Remark 6.1] and [51, Section 3.4.D]). For example, for the results of Theorem 28 to hold, it is sufficient (but not truly necessary) that the coefficients of L and L^* are Hölder continuous. It is clear that the conditions on the coefficients could be significantly relaxed if the operators were considered in the weak sense, or by means of their quadratic forms. This does not change the general techniques of the proofs, which rely on the analytic dependence with respect to the quasimomenta, rather than on the particular regularity properties of the coefficients. One only needs to guarantee the compactness of the resolvents of the operators $P(\chi)$. More precisely, we need that both the operator P_M and its dual P_M^* define Fredholm mappings between the Sobolev space $H^m(M)$ and $L_2(M)$, and this condition can be weakened further.

(2) A stronger statement than Theorem 17 holds: the existence of a subexponentially growing solution also implies the existence of a Bloch one. This is an analog of the Shnol's theorem (see [27, 37, 79]). It is provided in Theorem 4.3.1 of [51] for the case of periodic equations in $\mathbb{R}^n$, however carrying it over to the case of more general abelian coverings does not present any difficulty. An analogous theorem holds for periodic combinatorial and quantum graphs [56].

(3) The interesting feature of the main results of the present paper is that they relate the Liouville property to the local behavior of the dispersion relations near the 'edges' of the spectrum of the operator. This threshold behavior is responsible for many phenomena, in particular homogenization [3, 15, 16, 10, 25, 26, 42], structure of the impurity spectrum arising when a periodic medium is locally perturbed [12, 13], Anderson localization, and others. There are less explored issues of this kind, for instance the behavior of the Green's function [8, 57, 70, 86], integral representations of solutions of different classes, and finally Liouville theorems. The last two were also looked upon in our previous paper [58]. The results of the papers quoted above, as well as of this paper, deduce properties of solutions from an assumed spectral edge behavior of the dispersion relation.

However, there are only extremely few results concerning such precise spectral behavior for specific (or even generic) operators. The notable exception is the bottom of the spectrum, where much is known (e.g., [45, 76, 48] and recent advances in [15, 17, 80]). An initial study of the internal edges was conducted in the interesting paper [46], however the knowledge in this case remains far from being satisfactory. It is also appropriate to mention that the study of analytic properties of the Fermi and Bloch varieties, even for 'generic' operators, happens to be a very hard problem (e.g., [35, 36, 47]), and there are several unproven conjectures about the generic behavior of these varieties (see for example, [6, 71]).

(4) As it was mentioned above, the behavior at the bottom of the spectrum is responsible for homogenization. Our results on Liouville theorems indicate that in some sense some kind of "homogenization" exists also at the interior edges of the spectrum. A recent progress has been made towards precise understanding of the meaning of this 'homogenization' in [11] for ODEs and [16] for PDEs.

(5) An unexpected and counter-intuitive result of our study of Liouville theorems is that they depend on the **order** l_0 of the leading term λ_{l_0} of the dispersion relation only, rather than on its full structure. For instance, if $l_0 = 2$, the dimensions are the same as for the Laplace operator, even if λ_2, the Hessian of the dispersion relation, is degenerate (e.g., the case of the operator M_t with $t = 2\sqrt{3}$ in Section 5). Normally in homogenization theory, the whole term λ_{l_0} is important.

(6) Although our results are complete for the case of non-multiple spectral edge ($r = 1$ in terms of Theorem 13 and 19), in the multiplicity case when $r > 1$ we require that the determinant of the leading term λ_{l_0} of the Taylor expansion is not identically equal to zero. Simple examples show that without this non-degeneracy condition, the proof of (4.6) (and probably the result as well) does not work anymore. Indeed, suppose that we are dealing with a single point k of the Fermi surface and that the matrix $\lambda(k)$ is diagonal, with each diagonal entry $\lambda_{jj}(k)$ having its own leading term $\lambda_{l_j}(k)$. In this case, it is not hard to check that the summand in (4.6) needs to be replaced by

$$\sum_j \left[\begin{pmatrix} n+N \\ N \end{pmatrix} - \begin{pmatrix} n+N-l_j \\ N-l_j \end{pmatrix} \right], \tag{8.1}$$

in order for the result to hold. One can also see that in the non-selfadjoint case the non-degeneracy condition in its present form essentially forces the algebraic and geometric multiplicities of the zero eigenvalue to coincide. These deficiencies can be mollified by allowing more general ways of choosing the leading term λ_{l_0}, e.g., in the 'Douglis–Nirenberg sense' analogous to the corresponding definition of ellipticity [28]. However, at the present moment it is not clear to the authors how to avoid having any non-degeneracy conditions in the case of a multiple eigenvalue. Fortunately, in most known to us cases that are interesting for applications, one expects multiplicity to be absent.

(7) The result of Theorem 30 for the case of Kähler manifold can be extracted from the elliptic case, by switching the consideration to harmonic functions. Such result was obtained in [19]. This, however, does not work in general and the technique presented in this text is needed.

(8) It is interesting to mention that the dimension (and in fact the geometry) of the underlying manifold X does not explicitly enter in our main results. It definitely influences the geometry of the dispersion curves and hence the Liouville theorems. However, on the surface it looks like the group $G = \mathbb{Z}^n$ matters more than the manifold. In particular, one can easily cook up a **two dimensional** covering $X \xrightarrow{\mathbb{Z}^n} M$ with an arbitrarily large n (take for example the standard 2-dimensional jungle gym JG^2 in $\mathbb{R}^n$, see [75]). Then in the results that concern the dimensions of the spaces of polynomially growing solutions, one sees mostly the influence of n. In particular, one can get large dimensions of these spaces in such a manner. The reason is that, as it has been mentioned before, our Liouville theorems are of 'homogenized' nature, and being seen from afar, the covering manifold X looks like $\mathbb{R}^n$ [34, 38].

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