PERTURBATIONS OF ORTHOGONAL POLYNOMIALS WITH PERIODIC RECURSION COEFFICIENTS

DAVID DAMANIK[1], ROWAN KILLIP[2], AND BARRY SIMON[3]

Abstract. We extend the results of Denisov–Rakhmanov, Szegő–Shohat–Nevai, and Killip–Simon from asymptotically constant orthogonal polynomials on the real line (OPRL) and unit circle (OPUC) to asymptotically periodic OPRL and OPUC. The key tool is a characterization of the isospectral torus that is well adapted to the study of perturbations.

1. Introduction

This is a paper about the spectral theory of orthogonal polynomials on the real line (OPRL) and orthogonal polynomials on the unit circle (OPUC), that is, the connection of the underlying (spectral) measure and the recursion coefficients.

Specifically, given a probability measure, $d\rho$, on $\mathbb{R}$ with bounded but infinite support, the orthonormal polynomials, $p_n(x)$, obey a recursion relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_{n+1}p_n(x) + a_n p_{n-1}(x) \tag{1.1}$$

where the Jacobi parameters $\{a_n, b_n\}_{n=1}^\infty$ obey $b_j \in \mathbb{R}$, $a_j \in (0,\infty)$. As is well known (see, e.g., [98, Section 1.3]), (1.1) sets up a one-one correspondence between uniformly bounded $\{a_n, b_n\}_{n=1}^\infty$ and such measures, $d\rho$ (this is sometimes called Favard's theorem).

Similarly, probability measures, $d\mu$, on $\partial\mathbb{D}$ which are nontrivial (i.e., their support is not a finite set of points) are in one-one correspondence with sequences $\{\alpha_n\}_{n=0}^\infty$ of Verblunsky coefficients in $\mathbb{D} \equiv \{z : |z| < 1\}$ via the recursion relation of the orthonormal polynomials $\varphi_n(z)$, namely,

$$z\varphi_n(z) = \rho_n \varphi_{n+1}(z) + \bar{\alpha}_n \varphi_n^*(z) \tag{1.2}$$

where

$$\varphi_n^*(z) = z^n\,\overline{\varphi_n(1/\bar{z})} \qquad \rho_n = (1 - |\alpha_n|^2)^{1/2} \tag{1.3}$$

Underlying the association of measures and recursion coefficients are matrix representations. For OPRL, we take the matrix for multiplication by x in the

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[1] Department of Mathematics, Rice University, Houston, TX 77005. E-mail: damanik@rice.edu. Supported in part by NSF grant DMS-0500910.

[2] Department of Mathematics, UCLA, Los Angeles, CA 90095. E-mail: killip@math.ucla.edu. Supported in part by NSF grant DMS-0401277 and a Sloan Foundation Fellowship.

[3] Mathematics 253-37, California Institute of Technology, Pasadena, CA 91125. E-mail: bsimon@caltech.edu. Supported in part by NSF grant DMS-0140592 and U.S.–Israel Binational Science Foundation (BSF) Grant No. 2002068.

$\{p_n\}_{n=0}^\infty$ basis of $L^2(\mathbb{R}, d\rho)$, which is the tridiagonal Jacobi matrix

$$J = \begin{pmatrix} b_1 & a_1 & 0 & \\ a_1 & b_2 & a_2 & \ddots \\ 0 & a_2 & \ddots & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix} \tag{1.4}$$

For OPUC, one takes the matrix, $\mathcal{C}$, for multiplication by z in the basis obtained by orthonormalizing $\{1, z, z^{-1}, z^2, z^{-2}, \dots\}$ in $L^2(\partial\mathbb{D}, d\mu)$. This CMV matrix (see [98, Section 4.2]) has the form

$$\mathcal{C} = \mathcal{L}\mathcal{M} \tag{1.5}$$

$$\mathcal{L} = \Theta(\alpha_0) \oplus \Theta(\alpha_2) \oplus \cdots \tag{1.6}$$

$$\mathcal{M} = \mathbf{1}_{1\times 1} \oplus \Theta(\alpha_1) \oplus \Theta(\alpha_3) \oplus \cdots \tag{1.7}$$

$$\Theta(\alpha) = \begin{pmatrix} \bar\alpha & \rho \\ \rho & -\alpha \end{pmatrix} \tag{1.8}$$

Note that $\mathcal{C}$ is unitary, while J is self-adjoint.

As a model for what we wish to prove, let us briefly survey some of the main results relating to (slowly decaying) perturbations of the free case, that is, $a_n \equiv 1$, $b_n \equiv 0$ for OPRL and $\alpha_n \equiv 0$ for OPUC.

(1) *Weyl's Theorem* [111, 13, 1, 43, 49]. If $a_n \to 1$, $b_n \to 0$, then $\sigma_{\text{ess}}(d\rho) = [-2, 2]$ and if $\alpha_n \to 0$, then $\sigma_{\text{ess}}(d\mu) = \partial\mathbb{D}$. Here $\sigma_{\text{ess}}(d\rho)$ is the support of the measure, $d\rho$, with all isolated points removed.

(2) *Denisov–Rakhmanov Theorem* [86, 87, 74, 30, 82]. If $\sigma_{\text{ess}}(d\rho) = \Sigma_{\text{ac}}(d\rho) = [-2, 2]$, then $a_n \to 1$ and $b_n \to 0$. If $\sigma_{\text{ess}}(d\mu) = \Sigma_{\text{ac}}(d\mu) = \partial\mathbb{D}$, then $\alpha_n \to 0$. Here $\Sigma_{\text{ac}}(d\rho)$ is defined by: If $d\rho = f(x)\,dx + d\rho_{\text{s}}$ (in the OPUC case, dx is $d\theta/2\pi$) with $d\rho_{\text{s}}$ singular, then $\Sigma_{\text{ac}}(d\rho) = \{x : f(x) \neq 0\}$ as a measure class, that is, a set modulo sets of Lebesgue measure zero.

(3) *Szegő's Theorem* [105, 106, 94, 79, 59]. In the OPUC case, define

$$Z(d\mu) \equiv -\int \log\biggl(\frac{d\mu_{\text{ac}}}{d\theta}\biggr)\frac{d\theta}{2\pi} \tag{1.9}$$

Then $Z(d\mu) < \infty$ if and only if

$$\sum_{j=0}^\infty |\alpha_j(d\mu)|^2 < \infty \tag{1.10}$$

In the OPRL case, define

$$Z(d\rho) \equiv -\int_{-2}^2 \log\biggl(2\pi(4-E^2)^{1/2}\,\frac{d\rho_{\text{ac}}}{dE}\biggr)\frac{dE}{2\pi(4-E^2)^{1/2}} \tag{1.11}$$

Then, if we assume $\operatorname{supp}(d\rho) \subset [-2, 2]$, we have

$$Z(d\rho) < \infty \iff \limsup \sum_{j=1}^N \log(a_j) > -\infty \tag{1.12}$$

and if that holds, then

$$\sum_{n=1}^{\infty} (a_n - 1)^2 + b_n^2 < \infty \tag{1.13}$$

and

$$\sum (a_n - 1) \quad \text{and} \quad \sum b_n \tag{1.14}$$

are conditionally convergent to finite numbers.

(4) *Killip–Simon Theorem* [59]. For OPRL, define

$$Q(d\rho) = -\int_{-2}^{2} \log\biggl(\pi(4-E^2)^{-1/2}\, \frac{d\rho_{\text{ac}}}{dE}\biggr) \frac{(4-E^2)^{1/2}\, dE}{\pi} \tag{1.15}$$

and let $\{E_j^\pm\}_{j=1}^{N_\pm}$ be the point masses of $d\rho$ (eigenvalues of J) in $(2,\infty)\cup(-\infty,-2)$. Then (1.13) holds if and only if $\sigma_{\text{ess}}(d\rho) = [-2,2]$, $Q(d\rho) < \infty$, and $\sum_{j=1}^{N_\pm} (|E_j^\pm| - 2)^{3/2} < \infty$.

(5) *Nevai's Conjecture* [80, 59]. For OPRL, if $\sum_{n=1}^\infty |a_n - 1| + |b_n| < \infty$, then $Z(d\rho) < \infty$ (Z given by (1.11)).

The five results listed above capture different aspects of the philosophy that the measure is close to the free case if and only if the coefficients are asymptotic to the free ones. In this paper, we study extensions of all these results to perturbations of a periodic sequence of Jacobi or Verblunsky coefficients, that is,

$$a_{n+p}^{(0)} = a_n^{(0)} \qquad b_{n+p}^{(0)} = b_n^{(0)} \qquad n \geq 1 \tag{1.16}$$

or

$$\alpha_{n+p}^{(0)} = \alpha_n^{(0)} \qquad n \geq 0 \tag{1.17}$$

and some fixed $p \geq 1$. Note that $p = 1$ is the perturbation of the free case considered above. For simplicity in the OPUC case, we will normally suppose p is even—indeed, the shape of a CMV matrix repeats itself only after shifting by two rows/columns. As explained in Section 15, the situation when p is odd can be reduced to this using sieving. For OPRL, p is arbitrary.

The philosophy described above becomes more subtle when we move to the periodic setting; rather than having a single 'free operator' we have a manifold of them (the isospectral torus). Nevertheless—and this is the main thrust of the paper—spectral measures that are close to those of the isospectral torus correspond to coefficients that approach the isospectral torus. One of the key obstructions here is that a sequence of coefficients may approach the isospectral torus without converging to any particular point therein.

In order to make these heuristics precise, we need to make a few definitions. To keep the presentation as coherent as possible, we will focus our attention on the OPRL/Jacobi case for the remainder of the introduction.

To any pair of p-periodic sequences, $\{a_n^{(0)}, b_n^{(0)}\}_{n\in\mathbb{Z}}$, we can associate a two-sided Jacobi matrix J_0. Two such pairs of sequences are termed *isospectral* if the corresponding Jacobi matrices have the same spectrum. We write $\mathcal{T}_{J_0}$ for the set of p-periodic sequences that are isospectral to J_0. Topologically, this is a torus as explained in Subsection 2.14 below.

Given two bounded sequences $\{a_n, b_n\}_{n=1}^\infty$ and $\{a_n', b_n'\}_{n=1}^\infty$, we define

$$d_m((a,b),(a',b')) = \sum_{k=0}^{\infty} e^{-k} [|a_{m+k} - a_{m+k}'| + |b_{m+k} - b_{m+k}'|] \tag{1.18}$$

which is a metric for the product topology on $\times_m^\infty\big((0,R]\times[-R,R]\big)$. The OPUC analog is

$$d_m((\alpha),(\alpha'))=\sum_{k=0}^\infty e^{-k}|\alpha_{m+k}-\alpha'_{m+k}| \tag{1.19}$$

a metric for $\times_m^\infty\mathbb{D}$. The distance from a point to a set is defined in the usual way:

$$d_m((a,b),\mathcal{T})=\inf\{d_m((a,b),(a',b')):(a',b')\in\mathcal{T}\} \tag{1.20}$$

and similarly in the OPUC case.

We begin with the periodic analog of Weyl's Theorem.

Theorem 1.1 (Last–Simon [67])**.** *Let J_0 be a two-sided periodic Jacobi matrix and J a one-sided Jacobi matrix with Jacobi parameters $\{a_m,b_m\}_{m=1}^\infty$. If*

$$d_m((a,b),\mathcal{T}_{J_0})\to 0 \tag{1.21}$$

then

$$\sigma_{\text{ess}}(J)=\sigma(J_0) \tag{1.22}$$

As indicated, this result first appeared in [67]. It is derived from a theorem that had earlier been proven with different methods by others [41, 69, 85]. The inclusion $\sigma_{\text{ess}}(J)\supset\sigma(J_0)$ follows easily using trial vectors; the reverse seems to be more sophisticated. In Section 8, we prove this using the methods of this paper. The OPUC version appears here as Theorem 15.1; it was also proved in [67].

Note that (1.21) does not imply that there is a sequence $\{(a'_n,b'_n)\}\in\mathcal{T}_{J_0}$ such that

$$d_m((a,b),(a',b'))\to 0$$

It is much weaker. Equality of essential spectra under this stronger hypothesis follows immediately from Weyl's original theorem on compact perturbations.

Our first major new result is an analog of the Denisov–Rakhmanov Theorem.

Theorem 1.2. *Let J_0 be a two-sided periodic Jacobi matrix and J a one-sided Jacobi matrix with Jacobi parameters $\{a_m,b_m\}_{m=1}^\infty$. If $\sigma_{\text{ess}}(J)=\sigma(J_0)$ and*

$$\Sigma_{\text{ac}}(J)=\sigma(J_0) \tag{1.23}$$

then $d_m((a,b),\mathcal{T}_{J_0})\to 0$.

Remark. Using Theorem 1.4 below, we will show that the hypotheses of this theorem can hold while (a,b) only approaches $\mathcal{T}_{J_0}$ without actually having a limit.

A two-sided p-periodic Jacobi matrix is said to have all gaps open if the spectrum has exactly p connected components—the largest number possible. As explained in Section 2, this is holds generically (indeed, on a dense open set).

Our next new result is

Theorem 1.3. *Let J_0 be a two-sided periodic Jacobi matrix with all gaps open and parameters $\{a_n^{(0)},b_n^{(0)}\}_{n=-\infty}^\infty$. Also, let J be a one-sided Jacobi matrix with parameters $\{a_n,b_n\}_{n=1}^\infty$ and spectral measure $d\rho$. We assume that $\sigma_{\text{ess}}(J)=\sigma(J_0)$ and*

$$\sum_{j=1}^N \operatorname{dist}(E_j,\sigma_{\text{ess}}(J))^{1/2}<\infty \tag{1.24}$$

where $\{E_m\}_{j=1}^N$ *enumerates the eigenvalues of* J *outside* $\sigma(J_0)$. *Then*

$$-\int_{\sigma(J_0)} \log\biggl(\frac{d\rho_{\mathrm{ac}}}{dx}\biggr) \operatorname{dist}(x,\mathbb{R}\setminus\sigma(J_0))^{-1/2}\,dx < \infty \tag{1.25}$$

implies

$$\lim_{N\to\infty} \sum_{j=1}^{pN} \log\biggl(\frac{a_j}{a_j^{(0)}}\biggr) \tag{1.26}$$

exists and lies in $(-\infty,\infty)$. *Conversely,* (1.25) *holds so long as*

$$\limsup_{N\to\infty} \sum_{j=1}^{N} \log\biggl(\frac{a_j}{a_j^{(0)}}\biggr) > -\infty \tag{1.27}$$

and in this case, the limit in (1.26) *exists and lies in* $(-\infty,\infty)$.

Lastly, if (1.25) *or* (1.27) *holds, then*

$$\sum_{m=0}^\infty d_m((a,b),\mathcal{T}_{J_0})^2 < \infty \tag{1.28}$$

and there is $J_1\in\mathcal{T}_{J_0}$, *so*

$$d_m(J,J_1)\to 0 \tag{1.29}$$

Remarks. 1. Thus, when (1.24)–(1.27) hold, J has a limit J_1 in $\mathcal{T}_{J_0}$. In the normal direction to $\mathcal{T}_{J_0}$, the convergence is ℓ^2 (in the sense of (1.28)). But in the tangential direction, we only prove it has a limit. It would be interesting to know what can be said about how slowly (1.29) can occur and to know if there are examples where (1.24)–(1.28) hold but

$$\sum_{m=0}^\infty d_m(J,J_1)^2 = \infty \tag{1.30}$$

2. Notice that (1.27) will only fail if the partial sums converge to $-\infty$.

3. The final statement that there is $J_1\in\mathcal{T}_{J_0}$ with (1.29) is not our result but one of Peherstorfer–Yuditskii [84]. With our methods, we can sometimes prove that the a_m's approach a periodic limit which are the a's of a point in $\mathcal{T}_{J_0}$ but we have not been able to control the b's.

4. By (2.23), all $a_j^{(0)}$ in (1.26) and (1.27) can be replaced by $\mathrm{Cap}(\sigma(J_0))$, the logarithmic capacity of the spectrum of J_0.

Our third new result is

Theorem 1.4. *Let* J_0 *be a two-sided Jacobi matrix with all gaps open and parameters* $\{a_n^{(0)},b_n^{(0)}\}_{n\in\mathbb{Z}}$. *Let* J *be a Jacobi matrix with parameters* $\{a_n,b_n\}_{n=1}^\infty$. *Then*

$$\sum_{m=0}^\infty d_m((a,b),\mathcal{T}_{J_0})^2 < \infty \tag{1.31}$$

if and only if

(i) $\sigma_{\mathrm{ess}}(J)=\sigma(J_0)$,

(ii) $\displaystyle\sum_{j=1}^N \operatorname{dist}(E_j,\sigma_{\mathrm{ess}}(J))^{3/2}<\infty$, *and*

(iii) $\displaystyle -\int_{\sigma(J_0)} \log\biggl(\frac{d\rho_{\text{ac}}}{dx}\biggr) \operatorname{dist}(x, \mathbb{R}\setminus\sigma(J_0))^{1/2}\,dx < \infty.$

Here $\{E_m\}_{m=1}^N$ *enumerates the (discrete) spectrum of* J *outside* $\sigma(J_0)$.

Remarks. 1. Since (i)–(iii) are equivalent to (1.31), one may easily construct examples were (i)–(iii) hold, but there is no J_1 with (1.29). This provides the examples promised in the remark after Theorem 1.2. It also shows a stark difference between (1.24)–(1.25) and (ii)–(iii). In terms of the spectral measure, this difference is reflected only in the behavior near the band edges.

2. As we will see (Section 12), there are results even if all gaps are not open, but for Theorem 1.4 they are not so easy to express directly in terms of the a's and b's.

3. A special case of part of Theorem 1.4 is known, namely, Killip [57] proved that $\Sigma_{\text{ac}}(J) = \sigma(J_0)$ for $\{a_n, b_n\}_{n=0}^\infty$ obeying $\sum_{n=1}^\infty |a_n - a_n^{(0)}|^2 + |b_n - b_n^{(0)}|^2 < \infty$ (which is a strictly stronger hypothesis than (1.31)).

Theorem 1.5. *Let* J_0 *be a two-sided Jacobi matrix and* J *a Jacobi matrix with Jacobi parameters* $\{a_n, b_n\}_{n=1}^\infty$. *Suppose*

$$\sum_{n=1}^\infty |a_n - a_n^{(0)}| + |b_n - b_n^{(0)}| < \infty \tag{1.32}$$

Then (1.25) *holds.*

Remarks. 1. Condition (1.32) can be replaced by

$$\sum_{n=1}^\infty d_n((a,b), \mathcal{T}_{J_0}) < \infty \tag{1.33}$$

Indeed, if (1.33) holds, then (1.32) holds with $\{a_n^{(0)}, b_n^{(0)}\}$ replaced by some fixed sequence in $\mathcal{T}_{J_0}$.

2. As we will show (see Proposition 3.5), the theorems above continue to hold if d_m is replaced by

$$\tilde d_m((a,b),(a',b')) = \sum_{k=0}^{p-1} (|a_{m+k} - a'_{m+k}| + |b_{m+k} - b'_{m+k}|) \tag{1.34}$$

For OPUC, we need to sum k from 0 to p in order to get an equivalence; see the discussion at the end of Section 4.

In Section 15, we prove an OPUC analog of each of these theorems. We need to replace the all-gaps-open hypothesis with a stronger one (that holds generically). The deficiency is not so much with our method, but rather that an independent question, which is known in the Jacobi case (independence of the Toda Hamiltonians), is currently unresolved in the CMV case. Our results confirm Conjectures 12.2.3 and 12.2.4 of [99] as well as Conjectures 12.2.5 and 12.2.6 in the (generic) special case that all gaps are open.

For the case of OPUC with a single gap, the analog of Theorem 1.2 is known and motivated Simon's conjectures in [99]. In that case, the isospectral tori are labelled by $a \in (0,1)$ and consist of $\{\alpha^{(\lambda)} : \lambda \in \partial\mathbb{D}\}$ where $\alpha_n^{(\lambda)} = \lambda a$. Then $d_m(\alpha, \mathcal{T}) \to 0$ is equivalent to $|\alpha_n| \to a$ and $\alpha_{n+1}/\alpha_n \to 1$. This is often called the López condition. Bello–López [10] proved the OPUC analog of Theorem 1.2 for this case if $\sigma_{\text{ess}}(J) = \sigma(J_0)$ is strengthened to $\sigma(J) = \sigma(J_0)$ (the analog of Rakhmanov's result). The full analog for this special case appears in Simon [99], Alfaro et al. [2], and Barrios et al. [9].

Associated to each two-sided p-periodic Jacobi matrix, J_0, is a polynomial, Δ_{J_0}, of degree p, known as the discriminant. This is a classical object described in detail in the next section. It is usually defined as the trace of the one-period transfer matrix. It is also the unique polynomial (with positive leading coefficient) such that

$$\sigma(J_0) = \{x : \Delta_{J_0}(x) \in [-2, 2]\}$$

In particular, two sequences of coefficients are isospectral if and only if they give rise to the same discriminant.

The key to the proofs of our results is what we call the magic formula. Let J be a *two-sided* Jacobi matrix, then

$$\Delta_{J_0}(J) = S^p + S^{-p} \tag{1.35}$$

if and only if $J \in \mathcal{T}_{J_0}$. Here S is the right shift (cf. (2.31)). In particular, (1.35) already implies that J is periodic! In the OPUC case, Δ is a polynomial in z and z^{-1}. It turns out that $\Delta(\mathcal{C})$ is always self-adjoint; moreover,

$$\Delta_{\mathcal{C}_0}(\mathcal{C}) = S^p + S^{-p} \tag{1.36}$$

if and only if $\mathcal{C} \in \mathcal{T}_{\mathcal{C}_0}$.

It has been previously noted that for periodic J_0, one has

$$\Delta_{J_0}(J_0) = S^p + S^{-p}$$

That this holds for some polynomial in J_0 is in Naĭman [76, 77]. That the polynomial is the discriminant was found by Sebbar–Falliero [93]. After learning of our results, L. Golinskii has kindly pointed out to us that Naĭman [77] also has a theorem which implies any J obeying (1.35) is periodic, the core of proving the converse. We will discuss this further in Section 3.

Nonetheless, the two facts that make (1.35) magical to us—that it characterizes the isospectral torus and that it is ideal for the study of perturbations—seem to have escaped prior notice.

While J may be tridiagonal and $\mathcal{C}$ five-diagonal, both $\Delta(J)$ and $\Delta(\mathcal{C})$ are $2p+1$-diagonal, that is, vanishing except for the main diagonal and p diagonals above and below. Thus, both will be tridiagonal if written as $p \times p$ blocks. The key to our proofs will be to extend results from the $a_n \equiv 1$, $b_n \equiv 0$ case to block tridiagonal matrices, and then use (1.35) or (1.36) to study perturbations of the periodic case.

The magic formula is very powerful and opens up many other avenues for study:

(a) Szegő and Jost asymptotics for periodic perturbations and, in particular, the analogs of Damanik–Simon [21].
(b) Periodic analogs of the results of Nevai–Totik [81] and its various recent extensions [22, 97, 101].
(c) Analogs of the Strong Szegő Theorem for the periodic case following Ryckman's paper [91] for the Jacobi case.

We should point out a major limitation of our results. If B is a disjoint finite union of closed intervals in $\mathbb{R}$ (or $\partial\mathbb{D}$), one can construct an isospectral torus of Jacobi (or CMV) matrices whose recursion coefficients are almost periodic. As discussed in Section 2, these are strictly periodic if and only if the harmonic measure of each interval is rational. There are obvious potential extensions of Theorems 1.1–1.5 to this setting, but except for Theorem 1.1 (where the method of [67] applies) and Theorem 1.2 (where Section 9 has some extensions), we do not know how to prove them (or even if they are true). There is no analog of Δ in the almost periodic case, so our method does not work directly.

Here is the plan of this paper. Section 2 reviews the theory of the (unperturbed) periodic problem. In Section 3, we prove the magic formula for OPRL, and in Section 4, the magic formula for OPUC. While we will not discuss Schrödinger operators in detail here, we discuss the magic formula for such operators in Section 5. As we have mentioned, the magic formula brings block Jacobi matrices into play, so Section 6 discusses matrix-valued OPRL and OPUC—mainly setting up notation. Section 7 uses known results on Rakhmanov's theorem for matrix-valued orthogonal polynomials to prove a Denisov-type extension which we use in Section 8 to prove Theorem 1.2; the section also proves half of Theorem 1.1. Section 9 provides two results that go beyond the periodic case to prove Denisov–Rakhmanov-type theorems for special almost periodic situations. Section 10, following [96], proves the P_2 sum rule of Killip–Simon [59] and the C_1 sum rule for matrix-valued measures, and Section 11 uses these results to prove Theorems 1.3 and 1.4. Section 12 explores what we can say if gaps are closed. Section 13 proves analogs to the Lieb–Thirring bounds of Hundertmark–Simon [53] as preparation for proving Theorem 1.5 in Section 14. Finally, Section 15 discusses the OPUC results.

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2. Review of the Periodic Problem

In this section, we collect some of the major elements in the strictly periodic case. As this is textbook material, we forgo proofs and historical discussion. Full details can be found, for example, in Chapter 11 of [99] or Chapter 5 of [102] and the references therein.

To discuss the strictly periodic case, we need to extend our operators to be two-sided, that is, to act on $\ell^2(\mathbb{Z})$. In the Jacobi/OPRL case, we simply continue the tridiagonal pattern with parameters $\{a_n, b_n\}_{n\in\mathbb{Z}}$. Two-sided (or extended) CMV matrices are formed as $\mathcal{C} = \mathcal{L}\mathcal{M}$, where $\mathcal{L}$ and $\mathcal{M}$ are doubly infinite direct sums

$$\mathcal{L} = \cdots \oplus \Theta_{-2}(\alpha_{-2}) \oplus \Theta_0(\alpha_0) \oplus \Theta_2(\alpha_2) \oplus \cdots \tag{2.1}$$

$$\mathcal{M} = \qquad \cdots \oplus \Theta_{-1}(\alpha_{-1}) \oplus \Theta_1(\alpha_1) \oplus \cdots \tag{2.2}$$

that are misaligned by one row/column, just as in (1.5)–(1.8).

We adopt the convention of indexing the elements of matrices so that

$$J_{00} = b_1 \qquad \mathcal{L}_{00} = \bar\alpha_0 \qquad \mathcal{M}_{00} = -\alpha_{-1} \tag{2.3}$$

except $\mathcal{M}_{00} = 1$ in the one-sided case.

2.1. **Transfer Matrices.**

Let J be a two-sided Jacobi matrix. A sequence $\{u_n\}$ obeys $(J - x)u \equiv 0$ if and only if

$$a_{n+1}u_{n+1} + (b_{n+1} - x)u_n + a_n u_{n-1} = 0 \tag{2.4}$$

or, what is equivalent,

$$\begin{pmatrix} u_{n+1} \\ a_{n+1}u_n \end{pmatrix} = A_n \begin{pmatrix} u_n \\ a_n u_{n-1} \end{pmatrix} \tag{2.5}$$

with

$$A_n(x) = \frac{1}{a_{n+1}} \begin{pmatrix} x - b_{n+1} & -1 \\ a_{n+1}^2 & 0 \end{pmatrix} \tag{2.6}$$

Note that the desire to have A_n depend only on one pair (a_{n+1}, b_{n+1}) and have determinant equal to one resulted in the factors a_{n+1} and a_n appearing in (2.5). (The same price is usually paid when writing Sturm–Liouville equations as first-order systems.) The choice is not the most common one (although it is used in Pastur–Figotin [83]), but we feel it is the 'right' one since, in particular, $\det(A_n(x)) = 1$.

In the OPUC case we define

$$A_n(z) = \frac{1}{\rho_n} \begin{pmatrix} z & -\bar{\alpha}_n \\ -\alpha_n z & 1 \end{pmatrix} \tag{2.7}$$

which encodes the recurrence relation (1.2):

$$\begin{pmatrix} \varphi_{n+1}(z) \\ \varphi_{n+1}^*(z) \end{pmatrix} = A_n \begin{pmatrix} \varphi_n(z) \\ \varphi_n^*(z) \end{pmatrix} \tag{2.8}$$

We will now explain the link to formal (i.e., not necessarily ℓ^2) eigenvectors of a two-sided CMV matrix, $\mathcal{C}$. It is not as simple as (2.5). Suppose $(\mathcal{C} - z)u = 0$ with $z \neq 0$ and let $v = \mathcal{Z}^{-1}\mathcal{M}u$ where $\mathcal{Z}$ denotes the diagonal matrix with entries

$$\mathcal{Z}_{jj} = \begin{cases} z & : j \text{ odd} \\ 1 & : j \text{ even} \end{cases}$$

and $\mathcal{M}$ is as in (2.2). Then

$$z \begin{pmatrix} u_{2n+2} \\ v_{2n+2} \end{pmatrix} = z A_{2n+1}(z) \begin{pmatrix} v_{2n+1} \\ u_{2n+1} \end{pmatrix} = A_{2k+1}(z) A_{2k}(z) \begin{pmatrix} u_{2n} \\ v_{2n} \end{pmatrix} \tag{2.9}$$

The key observation used to verify (2.9) is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A_n(z) \begin{pmatrix} y \\ x \end{pmatrix} \iff \begin{pmatrix} zy \\ y' \end{pmatrix} = \Theta(\alpha_n) \begin{pmatrix} x \\ x' \end{pmatrix} \tag{2.10}$$

2.2. **The Discriminant.** As in Sturm–Liouville theory, the discriminant is defined as the trace of the one-period transfer matrix:

$$\Delta(z) = \operatorname{Tr}(T(z)) \tag{2.11}$$

where

$$T(z) = \begin{cases} A_{p-1}(z) \cdots A_1(z) A_0(z) & \text{(OPRL)} \\ z^{-p/2} A_{p-1}(z) \cdots A_1(z) A_0(z) & \text{(OPUC)} \end{cases} \tag{2.12}$$

In the OPUC case, p is even. Also, the factor $z^{-p/2}$ is there to cancel the extra factor of z on the left-hand side of (2.9). From a strictly OPUC point of view, it is more natural to omit this factor (as in [98, 99]); however, as the magic formula is an operator identity, we have elected to use the definition best adapted to this perspective. The only negative side effect of this choice is that our Lyapunov exponent (defined below) differs by $-\frac{1}{2}\log|z|$ from that in [98, 99].

For OPRL, the discriminant is a real polynomial of degree p with leading behavior

$$\Delta(x) = (a_1 \dots a_p)^{-1} \left[\prod_{j=1}^{p} (x - b_j^{(0)}) + O(x^{p-2}) \right] \tag{2.13}$$

$$= (a_1 \cdots a_p)^{-1} \biggl[x^p - \biggl(\sum_{j=1}^{p} b_j^{(0)} \biggr) x^{p-1} + O(x^{p-2}) \biggr] \tag{2.14}$$

For OPUC, it is a Laurent polynomial of total degree p with

$$\Delta(\bar{z}) = \overline{\Delta(1/z)} \tag{2.15}$$

so Δ is real-valued on $\partial\mathbb{D}$. Moreover,

$$\Delta(z) = (\rho_0 \rho_1 \cdots \rho_{p-1})^{-1} (z^{p/2} + \cdots + z^{-p/2}) \tag{2.16}$$

2.3. **The Lyapunov Exponent.** On an exponential scale, the behavior of formal eigenfunctions is determined by the Lyapunov exponent

$$\gamma(z) = \lim_{n\to\infty} \frac{1}{np} \log \|T^n(z)\| \tag{2.17}$$

$$= \frac{1}{p} \log (\text{spectral radius of } T(z))$$

$$= \frac{1}{p} \log |\lambda_+(z)| \tag{2.18}$$

where $\lambda_\pm$ are the eigenvalues of $T(z)$ with the convention $|\lambda_+| \geq |\lambda_-|$.

As $\det(T(z)) \equiv 1$, these eigenvalues are the roots of

$$\lambda^2 - \Delta(z)\lambda + 1 = 0 \tag{2.19}$$

which implies

$$\lambda_\pm(z) = \frac{\Delta(z)}{2} \pm \frac{\sqrt{\Delta^2(z) - 4}}{2} \tag{2.20}$$

and so

$$\gamma(z) = \tfrac{1}{p} \log \Bigl| \tfrac{1}{2}\Delta(z) + \tfrac{1}{2}\sqrt{\Delta(z)^2 - 4} \Bigr| \tag{2.21}$$

2.4. **Gaps and Bands.** Our recurrence relations admit bounded solutions for a given z if and only if $\Delta(z) \in [-2, 2]$. In the Jacobi/OPRL case, this is a collection of intervals in $\mathbb{R}$. For CMV/OPUC, it is a collection of arcs in $\partial\mathbb{D}$. In either case, one may partition this set into p bands. These are the closures of the (disjoint) regions where $\Delta(z) \in (-2, 2)$. These can only intersect at the 'band edges', $\Delta^{-1}(\{-2, 2\})$.

The open gaps are the intervals/arcs that are complementary to the bands—excluding the two semi-infinite intervals in the OPRL case. When two bands touch, we refer to the common band edge as a closed gap.

$\Delta^2 - 4$ has simple zeros at the edges of the open gaps and double zeros at the closed gaps; indeed, this is a complete list of its zeros. It is possible to distinguish whether these zeros correspond to $\Delta(z) = \pm 2$ from the fact that there must be two zeros of $\Delta \pm 2$ between consecutive zeros of $\Delta \mp 2$ and the fact that Δ has positive leading coefficient.

2.5. **Spectrum.** In both cases, the spectrum of the two-sided operator (acting on $\ell^2(\mathbb{Z})$) is the union of the bands: $\sigma = \Delta^{-1}([-2, 2])$. It is purely absolutely continuous and of multiplicity two.

The spectrum of a two-sided p-periodic operator uniquely determines its discriminant; see Lemma 3.3. One consequence of this was noted already in the introduction: isospectral tori are the classes of p-periodic recurrence coefficients that lead to the same discriminant.

In the case of a one-sided operator, the essential spectrum remains $\Delta^{-1}([-2,2])$; it is absolutely continuous with multiplicity one. In addition, up to one eigenvalue may appear in each open gap.

2.6. **Potential Theory.** From the way they are defined, one can see that $\gamma(z)$ vanishes on the bands and is both positive and harmonic in the complement (in the OPUC case one must also exclude $z=0$). This leads to the solution of the Dirichlet problem for a charge at infinity,

$$g_{\mathbb{C}\setminus\sigma}(z;\infty) = \begin{cases} \frac{1}{p}\log\left|\frac{\Delta}{2}+\sqrt{\frac{\Delta^2}{4}-1}\right| & \text{(OPRL)} \\ \frac{1}{2}\log|z| + \frac{1}{p}\log\left|\frac{\Delta}{2}+\sqrt{\frac{\Delta^2}{4}-1}\right| & \text{(OPUC)} \end{cases} \tag{2.22}$$

and so to the logarithmic capacity of the spectrum,

$$\operatorname{Cap}(\sigma) = \begin{cases} \left(\prod_{j=1}^{p} a_j\right)^{1/p} & \text{(OPRL)} \\ \left(\prod_{j=0}^{p-1} \rho_j\right)^{1/p} & \text{(OPUC)} \end{cases} \tag{2.23}$$

2.7. **Harmonic Measure.** Taking normal derivatives in (2.22) leads to a formula for harmonic measure on σ (aka equilibrium measure for the logarithmic potential),

$$d\nu = \begin{cases} \frac{2}{p}\,\frac{|\Delta'(x)|}{\sqrt{4-\Delta^2(x)}}\,\frac{dx}{2\pi} & \text{(OPRL)} \\ \frac{2}{p}\,\frac{|\Delta'(e^{i\theta})|}{\sqrt{4-\Delta^2(e^{i\theta})}}\,\frac{d\theta}{2\pi} & \text{(OPUC)} \end{cases} \tag{2.24}$$

where $\operatorname{supp}(d\nu) = \sigma = \{z : |\Delta(z)| \le 2\}$. (*Note*: [99] has $1/p$ rather than $2/p$, but that is an error.)

Recognizing

$$\frac{\Delta'(x)}{\sqrt{4-\Delta^2(x)}} = \frac{d}{dx}\arccos\left(\frac{\Delta(x)}{2}\right) \tag{2.25}$$

we see that the harmonic measure of each band is exactly $1/p$. The connected components of the union of the components all have rational harmonic measure. This gives strong restrictions on sets that can be bands. In the OPRL case, rational harmonic measure of connected components is also sufficient for a set to be the spectrum of a periodic Jacobi matrix. In the OPUC case, there is an additional condition needed: after breaking the bands into arcs of harmonic measure $1/p$, the harmonic midpoints $\{\zeta_j\}_{j=1}^p$ of these intervals must obey $\prod_{j=1}^p \zeta_j = 1$. Clearly, the condition on the harmonic midpoints can be achieved by simply rotating σ. Discarding this condition gives rise to Verblunsky coefficients that are p-automorphic, $\alpha_{n+p} = e^{i\phi}\alpha_n$, rather than p-periodic.

2.8. **Thouless Formula.** Harmonic measure appears naturally in the theory in several other ways. It is the density of states measure:

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N} f(J)_{nn} = \int f(x)\,d\nu(x) \tag{2.26}$$

for every polynomial (or continuous function) f. The same formula holds with $\mathcal{C}$ replacing J. This connection, or more precisely the resulting expression for $\gamma(z)$ in terms of $\operatorname{Cap}(\sigma)$ and the logarithmic potential of $d\nu$, is known as the Thouless formula.

Two further characterizations of $d\nu$ involve the orthogonal polynomials. $d\nu$ is the weak limit of $\frac{1}{N}\sum_{n=0}^{N-1} p_n^2(x)\,d\rho(x)$ (resp. $\frac{1}{N}\sum_{n=0}^{N-1}|\varphi_n(e^{i\theta})|^2 d\mu(\theta)$). It is also the limiting density of zeros, that is, the weak limit of the probability measures, $d\nu_n$, which give weight $1/n$ to each of the zeros of p_n (resp. φ_n). These two characterizations are closely linked to (2.26); however, in the OPUC case one should keep in mind that for each n, the zeros of ϕ_n lie strictly inside $\mathbb{D}$.

2.9. **Floquet Theory.** Looking at the eigenvalues of T, one sees that when $\lambda_+ \neq \lambda_-$, there is a basis of formal (i.e., non-ℓ^2) eigenfunctions obeying

$$u_{m+kp} = \lambda_\pm^k u_m \tag{2.27}$$

If $\lambda_+ = \lambda_-$, which happens precisely at the band edges, then both are ± 1. If the edge abuts an open gap, there is only one eigenfunction obeying (2.27) since T has a Jordan block structure. At closed gaps, $T = \pm\mathbf{1}$ and so all solutions are p-periodic.

Solutions obeying

$$u_{m+kp} = e^{ik\theta} u_m \tag{2.28}$$

are called Floquet solutions and $e^{i\theta}$ is called the Floquet index; they have much the same role as plane waves in Fourier analysis. Since $\lambda_- = \lambda_+^{-1}$, if (2.28) has a solution, then $e^{-i\theta}$ is also a Floquet index.

In the OPRL case,

$$(2.28) \text{ holds} \iff \Delta(x) = 2\cos\theta \tag{2.29}$$

Thus, by the discussion above, for each $\theta \in (0,\pi)$, (2.28) or (2.29) holds for exactly p values of x: $x_1(\theta) < x_2(\theta) < \cdots < x_p(\theta)$. These $x_j(\theta)$ are known as the band functions.

The changes in the OPUC case are purely notational:

$$(2.28) \text{ holds} \iff \Delta(z) = 2\cos\theta \tag{2.30}$$

For $\theta \in (0,\pi)$, this has p solutions all of which lie in $\partial\mathbb{D}$.

2.10. **Direct Integrals.** Let S denote the right shift,

$$(Su)_n = u_{n-1} \tag{2.31}$$

If the sequences of coefficients are p-periodic, then J (or $\mathcal{C}$) commutes with S^p, which means that the two operators can be 'simultaneously diagonalized'. We elaborate this point in the OPRL case; the OPUC is almost identical.

Let us write

$$\mathcal{H}_p := \int^\oplus \ell_\theta^2 \, \frac{d\theta}{2\pi} = L^2\big([0,2\pi), \tfrac{d\theta}{2\pi}; \mathbb{C}^p\big)$$

where ℓ_θ^2 is the p-dimensional Hilbert space

$$\ell_\theta^2 = \{u | u_{n+p} = e^{i\theta} u_n\} \qquad \langle u|v\rangle_\theta = \sum_{n=1}^p \bar{u}_n v_n$$

From Fourier analysis, there is a unitary operator $\mathcal{F}\colon \ell^2(\mathbb{Z}) \to \mathcal{H}_p$ so that $\mathcal{F}S^p\mathcal{F}^{-1}$ is multiplication by $e^{i\theta}\mathbf{1}$ and $\mathcal{F}J\mathcal{F}^{-1}$ acts fiber-wise (i.e., on each ℓ_θ^2) as a $p\times p$ matrix, $J(\theta)$. In particular, the eigenvalues of $J(\theta)$ are the solutions of (2.29), that is, they are the band functions $x_j(\theta)$.

2.11. **Hyperelliptic Riemann Surfaces.** As $\sqrt{\Delta^2 - 4}$ appears repeatedly in the theory, it is natural that the associated Riemann surface should enter the analysis. $\Delta^2 - 4$ has simple zeros at the edges of open gaps and at $\inf \sigma(J)$ and $\sup \sigma(J)$. It has double zeros at the closed gaps. Let ℓ denote the number of open gaps, then $\sqrt{\Delta^2 - 4}$ has square root singularities at $2(\ell + 1)$ points, and so its natural analyticity domain is the genus ℓ Riemann surface, $\mathcal{S}$, obtained by taking two copies of $\mathbb{C} \setminus \sigma(J)$, gluing at the bands and adding points at ∞. There is a natural projection $\pi \colon \mathcal{S} \to \mathbb{C} \cup \{\infty\}$ which is 2 to 1 except at the branch points of $\sqrt{\Delta^2 - 4}$. A similar analysis works for OPUC, but now there are ℓ gaps and the genus is $\ell - 1$.

2.12. **Minimal Herglotz and Carathéodory Functions.** For a half-line periodic Jacobi matrix, the m function is defined by

$$m(z) = \langle \delta_0, (J - z)^{-1} \delta_0 \rangle \qquad \forall z \in \mathbb{C} \setminus \sigma(J) \tag{2.32}$$

This can be shown to obey a quadratic equation with polynomial coefficients

$$A(z) m(z)^2 + B(z) m(z) + C(z) = 0 \tag{2.33}$$

Moreover, these coefficients can be chosen to obey

$$B^2 - 4AC = \Delta^2 - 4 \tag{2.34}$$

This implies that $m(z)$ has meromorphic continuation to $\mathcal{S}$. Indeed, m has minimal degree (i.e., degree $\ell + 1$ in the OPRL case and ℓ in the OPUC case) among all meromorphic functions on $\mathcal{S}$ that are not of the form $g \circ \pi$ with g meromorphic on the Riemann sphere. It can be shown that there is a one-one correspondence between minimal meromorphic functions obeying $\operatorname{Im} m(z) > 0$ if $\operatorname{Im} z > 0$ and $m(z) = -z^{-1} + O(z^{-2})$ on the top sheet of $\mathcal{S}$ and all periodic Jacobi parameters with the same Δ. (We call these minimal Herglotz functions.)

There is a similar description for OPUC, but now one uses

$$F(z) = \langle \delta_0, (\mathcal{C} + z)(\mathcal{C} - z)^{-1} \delta_0 \rangle \tag{2.35}$$

which obeys $\operatorname{Re} F(z) > 0$ if $|z| < 1$ and $F(0) = 1$. Again F obeys a quadratic equation, showing that F has a meromorphic continuation to $\mathcal{S}$ of minimal degree, and again there is a one-one correspondence between all $\{\alpha_n\}_{n=0}^{p-1}$ with the same Δ and all minimal Carathéodory functions.

2.13. **Dirichlet Data.** One can describe the set of minimal Herglotz functions in terms of their poles. For each open gap, $\{G_j\}_{j=1}^{\ell}$, $\pi^{-1}(\overline{G}_j) \equiv T_j$ is a circle since π is 2 to 1 on G_j and one-one on $\overline{G}_j \setminus G_j$. A meromorphic Herglotz function has $\ell + 1$ simple poles, one at ∞ on the second sheet and the other ℓ, one in each T_j. Thus, the set of meromorphic Herglotz functions is homeomorphic to $\times_{j=1}^{\ell} T_j$ under the bijective map from such functions to its poles. A similar analysis holds for OPUC but now there is no pole at infinity, there are ℓ gaps, and $\times_{j=1}^{\ell} T_j$ describes the possible poles. The difference is that for OPRL, the dimension of the torus is ℓ, and for OPUC it is $\ell - 1$.

2.14. **Isospectral Tori.** By combining the bijective maps from periodic OPRL to minimal Herglotz functions and of such functions to Dirichlet data, we see for a Δ of period p with ℓ gaps,

$$\{(a_n, b_n)_{n=1}^{p} : \text{the discriminant is } \Delta\}$$

is an ℓ-dimensional torus in $\mathbb{R}^{2p}$. Generically, $\ell = p - 1$. In the OPUC case, generically $\ell = p$ and the torus is naturally embedded in $\mathbb{C}^p$. This torus is the isospectral torus which we will denote by $\mathcal{T}$ or $\mathcal{T}_{J_0}$ if a given periodic J_0 underlies our construction. For clarity of exposition, we will typically blur the distinction between p-tuples $(a_n, b_n)_{n=1}^p$ and the corresponding infinite sequences $\{a_n, b_n\}_{n\in\mathbb{Z}}$ of period p.

2.15. **Isospectral Flows.** The fact that spaces of p-periodic coefficients foliate into tori suggests that there is some kind of completely integrable system in the background. That is true: it is the Toda flow in the OPRL case and the defocusing Ablowitz–Ladik flow in the OPUC case. Since we will not need these below, we say no more about them, but see Chapter 6 of [102] for the OPRL case and Section 11.11 of [99] for OPUC.

3. The Magic Formula for Jacobi Matrices

Our goal in this section is to prove

Theorem 3.1. *Let J_0 be a two-sided p-periodic Jacobi matrix with discriminant $\Delta_{J_0}(x)$ and isospectral torus $\mathcal{T}_{J_0}$. Let J be a two-sided (not a priori periodic) Jacobi matrix. Then*
$$\Delta_{J_0}(J) = S^p + S^{-p} \iff J \in \mathcal{T}_{J_0} \tag{3.1}$$
where S is the right shift, (2.31)*, on $\ell^2(\mathbb{Z})$.*

We provide two proofs of the 'harder' direction $\Rightarrow$ or rather of
$$\Delta_{J_0}(J) = S^p + S^{-p} \Rightarrow J \text{ is periodic} \tag{3.2}$$
which is the key step. Our first proof is immediately below; the second, suggested to us by L. Golinskii, appears after Lemma 3.4.

Lemma 3.2. *Let $\ell = 1, 2, \dots$. Then*
$$(J^\ell)_{m,m+k} = \begin{cases} 0 & k > \ell \\ a_m a_{m+1} \cdots a_{m+k-1} & k = \ell \\ a_m a_{m+1} \cdots a_{m+\ell-2}(b_m + b_{m+1} + \cdots + b_{m+\ell-1}) & k = \ell - 1 \end{cases} \tag{3.3}$$

Proof. Writing
$$(J^\ell)_{m,m+k} = \sum_{i_1,\dots,i_{\ell-1}} J_{m,i_1} J_{i_1,i_2} \cdots J_{i_{\ell-1},m+k} \tag{3.4}$$
We see that since J is tridiagonal, all terms are zero if $k > \ell$, that we must have (with $i_0 \equiv m$, $i_\ell = m + k$) that $i_q - i_{q-1} = 1$ for $q = 1, \dots, \ell$ if $k = \ell$, and that if $k = \ell - 1$, $i_q - i_{q-1} = 1$ for all but one $q \in \{1, \dots, \ell\}$ and it is zero for that q. $\square$

Lemma 3.3. *If J and J_0 are periodic, then $\sigma(J) = \sigma(J_0)$ if and only if $\Delta_J = \Delta_{J_0}$.*

Remark. This lemma says that the spectrum determines the discriminant and vice versa. That the discriminant determines the spectrum is elementary: $\sigma = \{x : \Delta(x) \in [-2, 2]\}$. Therefore we only prove the other direction—indeed, we give two proofs.

First Proof. Harmonic measure $d\nu$ is intrinsic to the set σ; it is the solution of an electrostatic problem there. But then $d\nu$ determines Δ via (2.24). $\square$

Second Proof. σ determines the gaps—even closed gaps—via harmonic measure. The gap edges determine the zeros of $\Delta - 2$ and so $\Delta - 2$ up to a constant. The zeros of $\Delta + 2$ then determine the constant. $\square$

Proof of Theorem 3.1. For all $\theta \in [0, 2\pi)$, $J(\theta)$ is self-adjoint and so diagonalizable. Moreover, the eigenvalues of $J_0(\theta)$ are precisely the roots of $\Delta(x) = 2\cos(\theta)$. Thus

$$\Delta_{J_0}(J_0(\theta)) = (2\cos\theta)\mathbf{1} \tag{3.5}$$

But then $\Delta_{J_0}(J_0)$ and $S^p + S^{-p}$ both have direct integral decomposition with fibers $(2\cos\theta)\mathbf{1}$, so

$$\Delta_{J_0}(J_0) = S^p + S^{-p} \tag{3.6}$$

Since $J \in \mathcal{T}_{J_0} \Rightarrow \Delta_J = \Delta_{J_0}$, this proves $\Leftarrow$ in (3.1).

Now suppose LHS of (3.1) holds. By (2.14), (3.3), and

$$(S^p + S^{-p})_{m,m+p} = 1 \qquad (S^p + S^{-p})_{m,m+p-1} = 0 \tag{3.7}$$

this implies

$$a_m \cdots a_{m+p-1} = a_1^{(0)} \cdots a_p^{(0)} \tag{3.8}$$

and

$$\sum_{j=0}^{p-1} b_{m+j} = \sum_{j=0}^{p-1} b_{j+1}^{(0)} \tag{3.9}$$

In particular,

$$a_m \cdots a_{m+p-1} = a_{m+1} \cdots a_{m+p} \qquad \sum_{j=0}^{p-1} (b_{m+j+1} - b_{m+j}) = 0$$

which lead to

$$a_m = a_{m+p} \qquad b_m = b_{m+p} \tag{3.10}$$

so J is periodic.

Since J is periodic, $\Delta_J(J) = S^p + S^{-p}$; moreover, $\Delta_{J_0}(J) = S^p + S^{-p}$ by hypothesis. Thus we learn that applying the polynomial $\Delta_J - \Delta_{J_0}$ to J gives zero. By the $k = \ell$ case of (3.3), it must therefore be the zero polynomial, that is, $\Delta_J = \Delta_{J_0}$. Lemma 3.3 now completes the proof. $\square$

Remarks. 1. Showing that J was periodic only required equality in $\Delta_{J_0}(\mathcal{T}) = S^p + S^{-p}$, for the two most extreme upper (or lower) diagonals. Nevertheless, $J \in \mathcal{T}_{J_0}$ requires equality everywhere.

2. We need not suppose a priori that each $a_n > 0$ and can allow some $a_n = 0$ (J can still be defined on $\ell^2(\mathbb{Z})$), for (3.8) implies that if LHS of (3.1) holds, then each $a_n > 0$.

We now turn to our second proof of (3.2).

Lemma 3.4 (Naĭman [77])**.** *Let A be a two-sided* (*bounded*) *infinite matrix of finite width* (*i.e., for some w, we have that $|k - \ell| > w \Rightarrow A_{k\ell} = 0$*)*. Suppose*

$$[A, S^p + S^{-p}] = 0 \tag{3.11}$$

for some p, then

$$[A, S^p] = 0 \tag{3.12}$$

Remarks. 1. This is Lemma 2 in [77]; no proof is given.

2. $[A, B] \equiv AB - BA$

3. (3.12) has an equivalent form:

$$[A, S^p] = 0 \iff A_{k+p,\ell+p} = A_{k,\ell} \qquad \text{for all } k, \ell \tag{3.13}$$

4. As (3.13) shows, $[J, S^p] = 0$ for a Jacobi matrix if and only if a_k and b_k are p-periodic.

Proof. Since A has finite width, we can find diagonal matrices $D_{k_1}, D_{k_1+1}, \dots, D_{k_2}$ with $D_{k_1} \neq 0 \neq D_{k_2}$, so that

$$A = \sum_{j=k_1}^{k_2} D_j S^j \tag{3.14}$$

Since D_j is diagonal, so is $S^p D_j S^{-p}$. Thus

$$(S^p + S^{-p})A = \sum_{j=k_1}^{k_2} (S^p D_j S^{-p}) S^{j+p} + \sum_{j=k_1}^{k_2} (S^{-p} D_j S^p) S^{j-p}$$

$$A(S^p + S^{-p}) = \sum_{j=k_1}^{k_2} D_j S^{j+p} + \sum_{j=k_1}^{k_2} D_j S^{j-p}$$

Since the composition (3.14) uniquely determines each D_j, (3.11) implies

$$S^p D_{k_2} S^{-p} = D_{k_2}$$

that is, D_{k_2} is periodic. Thus, $D_{k_2} S^p$ commutes with $S^p + S^{-p}$, so we can remove it from (3.14) without losing (3.11). This shows inductively that each D_j is periodic. □

Second proof of (3.2). J commutes with $\Delta(J)$, so (3.12) holds. □

Our next goal is to compare $\tilde{d}_m((a, b), \mathcal{T})$ given by (1.34) and $d_m((a, b), \mathcal{T})$ given by (1.18). As well as satisfying natural curiosity, this relation also plays an important role (via Theorem 11.12) in the proofs of Theorems 1.3 and 1.4.

To capture the essence of what follows, let us pause to ponder the following: suppose $\tilde{d}_m((a, b), \mathcal{T}) = 0$ for all m, does this mean that $(a, b) \in \mathcal{T}$? The hypothesis tells us that each length-p block belongs to the isospectral torus; it does not a priori guarantee that the coefficients are periodic. Example 4.5 shows that this can occur in the OPUC case; however it does not occur for OPRL. The reason is simple: within the isospectral torus, $a_1, \dots, a_{p-1}$ determines a_p and $b_1, \dots, b_{p-1}$ determines b_p.

Proposition 3.5. *Given a p-periodic Jacobi matrix J_0, $1 \leq q \leq \infty$, and $\varepsilon > 0$, there is a constant C so that*

$$e^{1-p} \big\| \tilde{d}_m\big((a, b), \mathcal{T}_{J_0}\big) \big\|_{\ell^q} \leq \big\| d_m\big((a, b), \mathcal{T}_{J_0}\big) \big\|_{\ell^q} \leq C \big\| \tilde{d}_m\big((a, b), \mathcal{T}_{J_0}\big) \big\|_{\ell^q} \tag{3.15}$$

for all sequences $\{(a_n, b_n)\}$ obeying $\varepsilon^{-1} > a_n > \varepsilon > 0$. All ℓ^q norms are over $m \in \{1, 2, 3, \dots\}$.

The key input is

Lemma 3.6. *Given $\{(a_n, b_n)\}$ obeying $\varepsilon^{-1} > a_n > \varepsilon > 0$,*

$$\big| a_n - a_n^{(0)} \big| + \big| b_n - b_n^{(0)} \big| \leq \tilde{d}_m\big((a, b), (a^{(0)}, b^{(0)})\big) + C \sum_{r=m}^{n-p+1} \tilde{d}_r\big((a, b), \mathcal{T}_{J_0}\big)$$

for all $n \geq m$. The constant C depends only on ϵ.

Proof. The proof is by induction on n. For $m \leq n \leq m+p$, the sum disappears and the result is immediate from the definition of $\tilde{d}_m$.

For $n > m+p$, we consider the functions

$$f(a_1, \dots, a_p) := \sum_{j=1}^{p} \big[\log(a_j) - \log(a_j^{(0)})\big] \qquad g(b_1, \dots, b_p) := \sum_{j=1}^{p} \big[b_j - b_j^{(0)}\big]$$

These vanish on $\mathcal{T}_{J_0}$, as explained in the proof of Theorem 3.1.

As g is Lipschitz (with constant 1),

$$\begin{aligned}
\big|b_n - b_{n-p}\big| &= \big|g(b_n, \dots, b_{n-p+1}) - g(b_{n-1}, \dots, b_{n-p})\big| \\
&\leq \big|g(b_n, \dots, b_{n-p+1})\big| + \big|g(b_{n-1}, \dots, b_{n-p})\big| \\
&\leq \tilde{d}_{n-p+1}\big((a,b), \mathcal{T}_{J_0}\big) + \tilde{d}_{n-p}\big((a,b), \mathcal{T}_{J_0}\big)
\end{aligned}$$

In a similar way,

$$\big|\log[a_n] - \log[a_{n-p}]\big| = \big|f(a_n, \dots, a_{n-p+1}) - f(a_{n-1}, \dots, a_{n-p})\big|$$

leads to

$$\big|a_n - a_{n-p}\big| \leq C_\epsilon \big[\tilde{d}_{n-p+1}\big((a,b), \mathcal{T}_{J_0}\big) + \tilde{d}_{n-p}\big((a,b), \mathcal{T}_{J_0}\big)\big]$$

Combining these two inequalities gives

$$\begin{aligned}
\big|a_n - a_n^{(0)}\big| + \big|b_n - b_n^{(0)}\big| \leq \big|a_{n-p} - a_{n-p}^{(0)}\big| + \big|b_n - b_{n-p}\big| \\
+ (1 + C_\varepsilon)\big[\tilde{d}_{n-p+1}\big((a,b), \mathcal{T}_{J_0}\big) + \tilde{d}_{n-p}\big((a,b), \mathcal{T}_{J_0}\big)\big]
\end{aligned}$$

which completes the proof of the inductive step. $\square$

Proof of Proposition 3.5. The left-hand inequality in (3.15) follows immediately from the definitions of d_m and $\tilde{d}_m$; we focus on the second inequality.

Choose $(a^{(0)}, b^{(0)})$ minimizing $d_m((a,b), \mathcal{T}_{J_0})$; strictly, this amounts to a (inconsequential) change in J_0. Applying Lemma 3.6 in the definition of d_m gives

$$\begin{aligned}
d_m\big((a,b), \mathcal{T}_{J_0}\big) &\leq \tfrac{e}{e-1}\, \tilde{d}_m\big((a,b), \mathcal{T}_{J_0}\big) + C \sum_{k=0}^{\infty} \sum_{r=m}^{m+k} e^{-k}\, \tilde{d}_r\big((a,b), \mathcal{T}_{J_0}\big) \\
&\leq C' \sum_{j=0}^{\infty} e^{-j}\, \tilde{d}_{m+j}\big((a,b), \mathcal{T}_{J_0}\big)
\end{aligned}$$

The proposition follows because convolution with $e^{-j}\chi_{[0,\infty)}(j)$ is a bounded operator on all ℓ^q spaces. $\square$

4. The Magic Formula for CMV Matrices

Our goal in this section is to prove

Theorem 4.1. *Let p be even and let $\mathcal{C}_0$ be a two-sided p-periodic CMV matrix with discriminant $\Delta_{\mathcal{C}_0}(z)$ and isospectral torus $\mathcal{T}_{\mathcal{C}_0}$. Given a two-sided (not a priori periodic) CMV matrix, $\mathcal{C}$,*

$$\Delta_{\mathcal{C}_0}(\mathcal{C}) = S^p + S^{-p} \iff \mathcal{C} \in \mathcal{T}_{\mathcal{C}_0} \tag{4.1}$$

Remarks. 1. Notice that since $\mathcal{C}$ is unitary and $\Delta(e^{i\theta})$ is real, $\Delta_{\mathcal{C}_0}(\mathcal{C})$ is self-adjoint.

2. By (2.16) and the fact that $\mathcal{C}$ is five-diagonal, $\Delta_{\mathcal{C}_0}(\mathcal{C})$ has $2(p/2)$ diagonals above/below the main diagonal.

3. As in Section 3, we will first present our initial proof that

$$\Delta_{\mathcal{C}_0}(\mathcal{C}) = S^p + S^{-p} \Rightarrow \{\alpha_n\} \text{ is periodic} \tag{4.2}$$

and then a proof based on Golinskii's suggestion.

Lemma 4.2. *We have:*

$$(\mathcal{C}^\ell)_{m,m+k} = (\mathcal{C}^{-\ell})_{m,m+k} = 0 \qquad \text{if } k > 2\ell \tag{4.3}$$

$$(\mathcal{C}^\ell)_{2m,2m+2\ell} = \rho_{2m}\rho_{2m+1}\cdots\rho_{2m+2\ell-1} \tag{4.4}$$

$$(\mathcal{C}^\ell)_{2m+1,2m+2\ell+1} = 0 \tag{4.5}$$

$$(\mathcal{C}^{-\ell})_{2m,2m+2\ell} = 0 \tag{4.6}$$

$$(\mathcal{C}^{-\ell})_{2m+1,2m+\ell+1} = \rho_{2m+1}\rho_{2m+2}\cdots\rho_{2m+2\ell} \tag{4.7}$$

$$(\mathcal{C}^\ell)_{2m,2m+2\ell-1} = \rho_{2m}\rho_{2m+1}\cdots\rho_{2m+2\ell-2}\bar\alpha_{2m+2\ell-1} \tag{4.8}$$

$$(\mathcal{C}^\ell)_{2m+1,2m+2\ell} = -\alpha_{2m}\rho_{2m+1}\cdots\rho_{2m+2\ell-1} \tag{4.9}$$

$$(\mathcal{C}^{-\ell})_{2m,2m+2\ell-1} = -\bar\alpha_{2m-1}\rho_{2m}\rho_{2m+1}\cdots\rho_{2m+2\ell-2} \tag{4.10}$$

$$(\mathcal{C}^{-\ell})_{2m+1,2m+2\ell} = \rho_{2m+1}\cdots\rho_{2m+2\ell-1}\alpha_{2m+2\ell} \tag{4.11}$$

Proof. As $\mathcal{L}$ and $\mathcal{M}$ are tridagonal, $\mathcal{C}^\ell$ is a product of 2ℓ tridiagonal matrices, so (4.3) is immediate.

We will prove the results for $\mathcal{C}^\ell$. The results for $\mathcal{C}^{-\ell}$ are similar if we note

$$\Theta(\alpha)^{-1} = \Theta(\bar\alpha) \tag{4.12}$$

since Θ is unitary and symmetric.

Equation (4.4) follows from

$$\mathcal{L}_{2m,2m+1} = \rho_{2m} \qquad \mathcal{M}_{2m+1,2m+2} = \rho_{2m+1} \tag{4.13}$$

and (4.5) from

$$\mathcal{L}_{2m+1,2m+2} = 0 \tag{4.14}$$

Because of (4.14), the only way for $\mathcal{C}^\ell$ to get from $2m$ to $2m+2\ell-1$ is to increase index in the first $2\ell-1$ factors, which leads to (4.8). For the same reason, to get from $2m+1$ to $2m+2\ell$, the last $2\ell-1$ factor must increase index, leading to (4.9). ☐

Lemma 4.3. *If $\mathcal{C}$ and $\mathcal{C}_0$ are p-periodic, then $\sigma(\mathcal{C}) = \sigma(\mathcal{C}_0)$ if and only if $\Delta_{\mathcal{C}} = \Delta_{\mathcal{C}_0}$.*

Proof. Either proof of Lemma 3.3 carries over with no change. ☐

Proof of Theorem 4.1. The proof that

$$\Delta_{\mathcal{C}_0}(\mathcal{C}_0) = S^p + S^{-p} \tag{4.15}$$

is identical to the proof of (3.5).

For the converse, suppose

$$\Delta_{\mathcal{C}_0}(\mathcal{C}) = S^p + S^{-p} \tag{4.16}$$

In particular,

$$\Delta_{\mathcal{C}_0}(\mathcal{C})_{2m,2m+p-1} = 0 \tag{4.17}$$

By (2.16) and Lemma 4.2, this implies (recall p is even)

$$(\rho_0^{(0)} \dots \rho_{p-1}^{(0)})^{-1}(\rho_{2m}\rho_{m+1} \dots \rho_{2m+p-2})(\bar{\alpha}_{2m+p-1} - \bar{\alpha}_{2m-1}) = 0$$

so

$$\alpha_{2m+p-1} = \alpha_{2m-1} \tag{4.18}$$

Similarly, since

$$\Delta_{\mathcal{C}_0}(\mathcal{C})_{2m+1,2m+p} = 0$$

we get

$$(\rho_0^{(0)} \dots \rho_{p-1}^{(0)})^{-1}(\rho_{2m+1} \dots \rho_{2m+p-1})(\alpha_{2m+2} - \alpha_{2m+p-1}) = 0$$

which leads to

$$\alpha_{2m+p} = \alpha_{2m} \tag{4.19}$$

Thus, α has period p. That $\mathcal{C} \in \mathcal{T}_{\mathcal{C}_0}$ follows from Lemma 4.3 and the same argument used in the OPRL case. □

Next, we give a proof using Naĭman's lemma. We will need

Lemma 4.4. *Let $\mathcal{C}$ be the extended CMV matrix associated to $\{\alpha_n\}_{n=-\infty}^{\infty}$. Let p be even. If $S^p\mathcal{C} = \mathcal{C}S^p$, then*

$$\alpha_{n+p} = \alpha_n \tag{4.20}$$

for all n.

Proof. We have that $\mathcal{C}_{2j\ 2j+1}^2 + \mathcal{C}_{2j\ 2j+2}^2 = \rho_{2j}^2$ (see (4.2.14) of [98]), so ρ_{2j} is periodic. Thus, $\mathcal{C}_{2j\ 2j+2}/\rho_{2j} = \rho_{2j+1}$ is also periodic. So $\bar{\alpha}_{2j+1} = \mathcal{C}_{2j\ 2j+1}/\rho_{2j}$ is periodic as is $\alpha_{2j} = \mathcal{C}_{2j+1\ 2j+2}/(-\rho_{2j+1})$. □

Second proof that (4.2) *holds.* $\mathcal{C}$ commutes with $S^p + S^{-p}$, so by Naĭman's lemma (Lemma 3.4), which did not require that A be self-adjoint, $S^p\mathcal{C} = \mathcal{C}S^p$, which implies α is periodic by Lemma 4.4. □

We now turn to the OPUC version of Proposition 3.5. As noted in the introduction, it is not sufficient to sum over exactly one period:

Example 4.5. $(0, \frac{1}{2}, 0, \frac{1}{2}, 0, \dots)$ and $(0, -\frac{1}{2}, 0, -\frac{1}{2}, 0, \dots)$ are in the same isospectral torus, namely, the one with $p = 2$ and

$$\Delta(z) = \sqrt{\tfrac{4}{3}}\,(z + z^{-1})$$

Now consider $\alpha = (0, \frac{1}{2}, 0, -\frac{1}{2}, 0, \frac{1}{2}, 0, -\frac{1}{2}, \dots)$. If $\tilde{d}_m(\alpha, \mathcal{T}_{\mathcal{C}_0})$ were defined as sum from $k = 0$ to $p - 1$, it would be zero for all m, but $d_m(\alpha, \mathcal{T}_{\mathcal{C}_0})$ is not small. □

The problem, as this example shows, is that for sequences in $\mathcal{T}_{J_0}$, $(\alpha_0, \dots, \alpha_{p-2})$ does not determine α_{p-1}. But by periodicity, $\alpha_0, \dots, \alpha_{p-1}$ determines α_p. Thus, if we define

$$\tilde{d}_m(\alpha, \alpha') := \sum_{k=0}^{p} |\alpha_{m+k} - \alpha'_{m+k}| \tag{4.21}$$

then

$$|\alpha_m + p - \alpha_m| \le \tilde{d}_m(\alpha, \mathcal{T}_{J_0})$$

Plugging this into the proofs of Lemma 3.6 and Proposition 3.5 leads quickly to

Proposition 4.6. *Let $\mathcal{C}_0$ be a fixed periodic CMV matrix, then*

$$e^{-p}\big\|\tilde{d}_m(\alpha, \mathcal{T}_{\mathcal{C}_0})\big\|_{\ell^q} \le \big\|d_m(\alpha, \mathcal{T}_{J_0})\big\|_{\ell^q} \le C\big\|\tilde{d}_m(\alpha, \mathcal{T}_{\mathcal{C}_0})\big\|_{\ell^q} \tag{4.22}$$

for any sequence of Verblunsky coefficients $\{\alpha_n\}$.

5. The Magic Formula for Schrödinger Operators

In this section, we want to illustrate the potential applicability of our central idea to the spectral theory of one-dimensional Schrödinger operators,

$$H = -\frac{d^2}{dx} + V(x) \tag{5.1}$$

However, we will not pursue the applications in this paper but defer them to a future publication.

We will suppose $V \in L^1_{\text{unif}}$, that is, $\sup_x \int_{x-1}^{x+1} |V(y)|\, dy < \infty$. In that case, V is a form bounded perturbation of $-\frac{d^2}{dx^2}$ on $L^2(\mathbb{R}, dx)$ with relative bound zero, so H is a self-adjoint operator. Its form domain is the Sobolev space $H^1(\mathbb{R})$.

We need to say something about periodic Schrödinger operators. Suppose V_0 has period L, that is,

$$V_0(x+L) = V_0(x) \tag{5.2}$$

For arbitrary V in L^1_{loc} and $E \in \mathbb{C}$, let $u_D(x,E;V)$ and $u_N(x,E;V)$ (we will often drop the V if it is clear which V is intended) be the solutions of

$$-u'' + Vu = Eu \tag{5.3}$$

obeying the boundary conditions

$$u_D(0) = 0 \qquad u_D'(0) = 1 \qquad u_N(0) = 1 \qquad u_N'(0) = 0 \tag{5.4}$$

There is a unique solution of (5.3) in distributional sense which is absolutely continuous.

The transfer matrix that updates solutions of (5.3) (with data written as $\binom{u}{u'}$) is

$$T(x,E;V) = \begin{pmatrix} u_N(x,E) & u_D(x,E) \\ u_N'(x,E) & u_D'(x,E) \end{pmatrix} \tag{5.5}$$

$\det(T) = 1$ by constancy of the Wronskian. For periodic V_0, we define the discriminant by

$$\begin{aligned} \Delta_{V_0}(E) &= \operatorname{Tr}(T(L,E;V_0)) \\ &= u_N(L,E) + u_D'(L,E) \end{aligned} \tag{5.6}$$

As in the OPRL and OPUC cases, it is easy to see for the whole-line operator that

$$\sigma\bigl(-\tfrac{d^2}{dx^2} + V_0\bigr) = \Delta_{V_0}^{-1}([-2,2]) \tag{5.7}$$

and is purely absolutely continuous. Moreover (see, e.g., [89]), if

$$(S_y u)(x) = u(x-y) \tag{5.8}$$

then $H = -\frac{d^2}{dx^2} + V_0$ commutes with S_L and so has a direct integral decomposition,

$$H = \int^{\oplus} H(\theta)\, \frac{d\theta}{2\pi} \tag{5.9}$$

whose fibers, $H(\theta)$, are the operator (5.1) on $[0,L]$ with

$$u(L) = e^{i\theta} u(0) \qquad u'(L) = e^{i\theta} u'(0) \tag{5.10}$$

boundary conditions. $H(\theta)$ has purely discrete spectrum (i.e., $(H(\theta)+i)^{-1}$ is compact); the eigenvalues are precisely the solutions of

$$\Delta(E) = 2\cos(\theta) \tag{5.11}$$

Two periodic potentials of period L are called isospectral if and only if they have the same Δ. As in the Jacobi and CMV cases, the spectrum determines Δ, but this is more difficult to prove in the Schrödinger case. It is also known ([75] for nice V_0's; [15, 39, 54, 88]) that the set of V's isospectral to V_0 is a torus of dimension equal to the number of gaps which is typically infinite, so we will refer to an isospectral torus, $\mathcal{T}_{V_0}$. We can now state the main result in this section:

Theorem 5.1. *Let V_0 be periodic obeying* (5.2) *and let Δ_{V_0} be its discriminant and $\mathcal{T}_{V_0}$ its isospectral torus. Let V be in $L^1_{\text{loc,unif}}$ on $\mathbb{R}$ and $H = -\frac{d^2}{dx^2} + V$. Then*

$$\Delta_{V_0}(H) = S_L + S_{-L} \iff V \in \mathcal{T}_{V_0} \tag{5.12}$$

Here $S_{\pm L}$ denotes the shift operator, as in (5.8).

Remarks. 1. $\Delta_{V_0}(H)$ is defined by the functional calculus.

2. As in the last two sections, we will provide our initial proof that

$$\Delta_{V_0}(H) = S_L + S_{-L} \Rightarrow V \text{ periodic} \tag{5.13}$$

and then a simpler proof using an analog of Naĭman's lemma. This argument does not require Theorems 5.2 and 5.3 and the considerable machinery their proofs entail. That said, to show $\Delta_{V_0}(H) = S_L + S_{-L}$ plus V periodic implies $V \in \mathcal{T}_{V_0}$ *does* require Theorem 5.3, but it should be noted that one can prove Theorem 5.3 fairly easily without needing transformation formulae of Delsarte, Levitan, Gel'fand, Marchenko type.

We need two preliminaries whose proofs we defer to later in the section. We first make a definition:

Definition. For any $y > 0$, $\mathcal{R}_y$ consists of operators on $L^2(\mathbb{R})$ of the form

$$(Af)(x) = \tfrac{1}{2}\, f(x+y) + \tfrac{1}{2}\, f(x-y) + \int_{x-y}^{x+y} K(x,z) f(z)\, dz \tag{5.14}$$

where K is continuous and uniformly bounded on $\{(x,z) : |x-z| \le y\}$.

Note. It can happen that $K(x, x \pm y) \neq 0$, so if we think of K as an integral kernel on $\mathbb{R} \times \mathbb{R}$, it can be discontinuous at $|x-z| = y$.

Theorem 5.2. *If V_0 is L-periodic and V in $L^1_{\text{loc,unif}}$, then $\frac{1}{2}\Delta(H) \in \mathcal{R}_L$ and*

$$K(x, x+L) = -\tfrac{1}{4} \int_x^{x+L} (V(z) - V_0(z))\, dz \tag{5.15}$$

Note that (5.15) describes the 'matrix elements' of $\Delta_{V_0}(H) - (S_L + S_{-L})$ that are farthest from the diagonal. Indeed, just as in the other cases, one does not need the full statement $\Delta_{V_0}(H) = S_L + S_{-L}$ to see that V is periodic, only that $\langle f, (\Delta(H) - S_L - S_{-L})g\rangle = 0$ for f supported near x_0 and g near $x_0 + L$ (for all x_0).

Theorem 5.3. *$\Delta(E)$ is an entire function which obeys*

$$\text{(i)} \qquad |\Delta(E)| \le C \exp\bigl(L\sqrt{|E|}\bigr) \tag{5.16}$$

$$\text{(ii)} \qquad \lim_{\substack{E \to -\infty \\ E \text{ real}}} \frac{\Delta(E)}{\exp(L\sqrt{|E|})} = 1 \tag{5.17}$$

Proof of Theorem 5.1. If $V \in \mathcal{T}_{V_0}$, then $\Delta_V = \Delta_{V_0}$, so for the $\Leftarrow$ direction we need only prove

$$\Delta_{V_0}\big(\tfrac{d^2}{dx^2} + V_0\big) = S_L + S_{-L} \tag{5.18}$$

As before, this is equivalent to $\Delta_{V_0}(H(\theta)) = 2\cos\theta$ which follows from (5.11).

Conversely, if $\Delta_{V_0}(H) = S_L + S_{-L}$, then from Theorem 5.2 and the periodicity of V_0, we see

$$\int_x^{x+L} V(z)\,dz = \text{constant} \tag{5.19}$$

This implies that $V(x+L) - V(x) = 0$ for a.e. x, that is, V is periodic.

If $H(\theta)$ are the fibers of H in the direct integral decomposition, $\Delta_{V_0}(H) = S_L + S_{-L}$ implies

$$\Delta_{V_0}(H(\theta)) = 2\cos\theta \tag{5.20}$$

so, if Δ is the discriminant for V, we have $\Delta(z) = \pm 2 \Rightarrow \Delta_{V_0}(z) = \pm 2$. Moreover, (5.20) implies $\sigma(H) \subseteq \sigma(-\frac{d^2}{dx^2} + V_0)$, so any double zero of $\Delta \pm 2$ is a double zero of $\Delta_{V_0} \pm 2$. It follows that

$$g(z) = \frac{\Delta_{V_0}^2(z) - 4}{\Delta^2(z) - 4} \tag{5.21}$$

is analytic.

Since Δ_{V_0} and Δ are entire functions of order $\frac{1}{2}$ (by Theorem 5.3), $g(z)$ is of the form

$$g(z) = C\prod_{j=1}^{J}\bigg(1 - \frac{z}{z_j}\bigg) \tag{5.22}$$

where $z_1 < z_2 < \cdots$ are bounded from below. By (5.17), $\lim_{E\to-\infty} g(E) = 1$, which implies $g \equiv 1$, that is, $\Delta = \Delta_{V_0}$. $\square$

The argument used at the end of the proof to conclude that missing zeros cannot occur is reminiscent of ideas connected with the Hochstadt–Lieberman [52] and related theorems [45, 46].

We now turn to the proofs of Theorems 5.2 and 5.3. A critical role is played by the wave equation and the transformation operator formalism of Gel'fand–Levitan, further important work is due to Delsarte, Levin, and Marchenko; see the book of Marchenko [73] for references and history.

Define for $s > 0$,

$$C_s(z) = \cos\big(s\sqrt{z}\big) \qquad S_s(z) = z^{-1/2}\sin\big(s\sqrt{z}\big) \tag{5.23}$$

which are entire functions of z bounded on (a,∞) for any $a \in \mathbb{R}$. Thus $C_s(H)$ and $S_s(H)$ are bounded operators for any H that is bounded from below. We will need to study the form of $C_s(-\frac{d^2}{dx^2} + V)$. For bounded continuous V, this is discussed in Marchenko [73]. While his proofs extend to the L^1_{loc} case, it seems simpler to sketch the ideas:

Proposition 5.4. *$C_s^0 := C_s(-\frac{d^2}{dx^2}) \in \mathcal{R}_s$; indeed,*

$$(C_s^0 f)(x) = \tfrac{1}{2}\,[f(x+s) + f(x-s)] \tag{5.24}$$

If $S_s^0 := S_s(-\frac{d^2}{dx^2})$, then

$$(S_s^0 f)(x) = \tfrac{1}{2}\int_{x-s}^{x+s} f(y)\,dy \tag{5.25}$$

Remark. If $w(x,s) := (C_s^0 f)(x) + (S_s g)(x)$, then w obeys the wave equation $(\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial x^2})w = 0$ with initial data $w(x,0) = f$ and $\partial_s w(x,0) = g(x)$. Thus the proposition basically encodes d'Alembert's solution of the wave equation. From this point of view, Theorem 5.2 is connected to finite propagation speed for the wave equation.

Proof. Since cos is even,

$$\cos(s|k|) = \cos(sk) = \tfrac{1}{2}\left(e^{iks} + e^{-iks}\right) \tag{5.26}$$

(5.24) is just the Fourier transform of this. (5.25) follows from

$$S_s(z) = \int_0^s C_t(z)\, dt \tag{5.27}$$

and (5.24). □

We are heading towards

Theorem 5.5. *Let $V \in L^1_{\text{loc,unif}}(\mathbb{R})$ and let $H = -\frac{d^2}{dx^2} + V$. Then $C_s(H) \in \mathcal{R}_s$ and the associated kernel K_s of (5.14) obeys*

$$K_s(x, x+s) = -\tfrac{1}{4} \int_x^{x+s} V(u)\, du \tag{5.28}$$

and for each $t \in (0,\infty)$,

$$\sup_{x,y,|s|\le t} |K_s(x,y)| < \infty \tag{5.29}$$

In addition,

$$(S_s(H)f)(x) = \int_{x-s}^{x+s} L_s(x,y) f(y)\, dy \tag{5.30}$$

where

$$L_s(x, x+s) = \tfrac{1}{2} \tag{5.31}$$

Lemma 5.6. *It suffices to prove Theorem 5.5 for s small.*

Proof. Since $\cos(2u) = 2\cos^2(u) - 1$, one sees

$$C_{2s}(A) = 2C_s(A)^2 - \mathbf{1} \tag{5.32}$$

Thus, if $C_s \in \mathcal{R}_s$, one sees $C_{2s} \in \mathcal{R}_{2s}$ and

$$K_{2s}(x,y) = K_s(x, y+s) + K_s(x, y-s) + K_s(x+s, y) + K_s(x-s, y) \tag{5.33}$$

where $K(x,y) = 0$ if $|x-y| > s$. Thus

$$K_{2s}(x, x+2s) = K_s(x, y+s) + K_s(x+s, y+2s) \tag{5.34}$$

This shows that if the formula is known for $|s| \le T$, one gets it successively for $2T, 4T, 8T, \dots$.

Using (5.27), one sees that the result for $C_s(H)$ implies (5.30) and (5.31). □

Proof of Theorem 5.5. If A is a bounded self-adjoint operator on $\mathcal{H}$ which is bounded from below, and B is the operator on $\mathcal{H} \oplus \mathcal{H}$ given by

$$B = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -A & \mathbf{0} \end{pmatrix} \tag{5.35}$$

then

$$e^{sB} = \begin{pmatrix} C_t(A) & S_t(A) \\ -AS_t(A) & C_t(A) \end{pmatrix} \tag{5.36}$$

This formula can be checked by showing that the right side of (5.36) is a bounded semigroup whose derivative at $t = 0$ is B. DuHamel's formula for $A, \tilde{A}$ bounded says that

$$e^{t\tilde{B}} = e^{tB} + \int_0^t e^{sB}(\tilde{B} - B)e^{(t-s)\tilde{B}}\, ds \tag{5.37}$$

$$= e^{tB} + \int_0^t e^{s\tilde{B}}(\tilde{B} - B)e^{(t-s)B}\, ds \tag{5.38}$$

Using (5.36), we obtain

$$C_t(\tilde{A}) = C_t(A) - \int_0^t S_s(\tilde{A})(\tilde{A} - A)C_{t-s}(A)\, ds \tag{5.39}$$

$$= C_t(A) - \int_0^t S_s(A)(\tilde{A} - A)C_{t-s}(\tilde{A})\, ds \tag{5.40}$$

By taking limits, it is easy to obtain these formulae for $A = -\frac{d^2}{dx^2}$, $\tilde{A} = -\frac{d^2}{dx^2} + V$ with V bounded. By obtaining a priori bounds below depending only on certain L^1 norms of V, we get estimates for V in L^1 and so, using the lemma, prove the theorem.

By iterating (5.40), one gets an expansion (which converges if V is bounded and whose estimates then extend),

$$C_t\big(-\tfrac{d^2}{dx^2} + V(x)\big) = C_t^{(0)} + \sum_{n=1}^{\infty} C_t^{(n)} \tag{5.41}$$

$$C_t^{(n)} = (-1)^n \int_{0 \le s_1 + \cdots + s_n \le t} S_{s_1}^{(0)} V S_{s_2}^{(0)} \dots V S_{s_n}^{(0)} V C_{t-s_1-\cdots-s_n}^{(0)}\, ds_1, \dots ds_n \tag{5.42}$$

Apply the integrand in $C_t^{(n)}$ to a function f and evaluate at x for fixed $s_1, \dots, s_n$. Each $S_{s_j}^{(0)} V$ evaluates V at points and integrals using (5.25). The integrands in V are in the interval $(x-t, x+t)$, so if we take absolute values, we see this integrand is bounded by

$$\bigg(\frac{1}{2}\int_{x-t}^{x+t} |V(y)|\, dy\bigg)^n \big[\tfrac{1}{2}\, f(x + t - s_1 - \cdots - s_n) + \tfrac{1}{2}\, f(x - t + s_1 + \cdots + s_n)\big]$$

Now we can do the integral over $s_1, \dots, s_n$. For $t - s_1 - \cdots - s_n$ fixed, the new integrand is independent of $s_1, \dots, s_{n-1}$ and is bounded by t^{n-1}. We find

$$|(C_t^{(n)} f)(x)| \le t^{n-1} \bigg(\frac{1}{2}\int_{x-t}^{x+t} |V(y)|\, dy\bigg)^n \int_{x-t}^{x+t} |f(y)|\, dy \tag{5.43}$$

Moreover, $C_t^{(n)}$ has a continuous integral kernel $K_t^{(n)}(x,y)$ supported in $|x - y| \le t$. Since V is uniformly locally L^1, by taking t small, we can be sure $\sup_x \frac{1}{2}\int_{x-t}^{x+t} |V(y)|\, dy < 1$, which yields uniform convergence of $K_t^{(n)}$ to a uniformly bounded kernel.

By (5.27), we get (5.30) from $C_s(H) \in \mathcal{R}_s$, and (5.31) comes from noting that

$$|L_s(x,y) - \tfrac{1}{2}| \le (s - |x-y|) \sup_{x,y,u \le s} |K_u(x,y)| \tag{5.44}$$

Finally, using (5.39), we see that

$$K_t(x,x+t) = -\tfrac{1}{2}\int_0^t L_s(x,x+s)V(x+s)\,ds$$

proving (5.28). $\square$

To complete the proofs of Theorems 5.2 and 5.3 (and so Theorem 5.1), we need the transformation formulae of Delsarte, Levitan, Gel'fand, and Marchenko [73]:

Theorem 5.7. *If $V\in L^1([0,R])$ for $R<\infty$, then there exist functions K_N,K_D C^1 in $\{(y,x):0\le y\le x\le R\}$ so that for $0\le x\le R$,*

$$u_N(x,E) = C_x(E) + \int_0^x K_N(y,x)C_y(E)\,dy \tag{5.45}$$

$$u_D(x,E) = S_x(E) + \int_0^x K_D(y,x)S_y(E)\,dy \tag{5.46}$$

Moreover,

$$K_D(x,x) = K_N(x,x) = \tfrac{1}{2}\int_0^x V(t)\,dt \tag{5.47}$$

Remarks. 1. These formulae are in Marchenko [73, p. 9 and (1.2.28)]. He supposes V is continuous, but his proof works if V is L^1; indeed, see Remark 2.

2. Defining $\tilde u_X(x,k) = u_X(x,k^2)$ for $X=D,N$ and $Q_X(x,y)$ as the Fourier transform of $\tilde u_X$ in k, we see (5.3) becomes

$$\frac{\partial^2 Q}{\partial x^2} - \frac{\partial^2 Q}{\partial y^2} = VQ(x,y) \tag{5.48}$$

with initial conditions

$$\begin{aligned} &Q_N(x=0,y)=\delta(y) && Q_n'(x=0,y)=0 \\ &Q_D(x=0,y)=0 && Q_D'(x=0,y)=\delta(y) \end{aligned}$$

Thus, Theorem 5.7 is essentially Theorem 5.5 with a time-dependent V used.

3. By (5.6), (5.45), and (5.46), we obtain a critical representation for Δ:

$$\begin{aligned}\Delta(E) = 2C_x(E) &+ \int_0^L L_1(t)C_t(E)\,dt \\ &+ \int_0^L L_2(t)S_t(E)\,dt + K_D(L,L)S_L(E)\end{aligned} \tag{5.49}$$

where L_1,L_2 are continuous in $[0,L]$. Indeed,

$$L_1(t) = K_N(t,L) \qquad L_2(t) = \frac{\partial}{\partial x}K_D(t,x)\bigg|_{x=L}$$

Proof of Theorem 5.3. The analyticity is immediate from (5.49) as is (5.16) given

$$|C_x(E)| + |S_x(E)| \le C\exp\bigl(x\sqrt{|E|}\bigr)$$

Moreover, since for $t<L$,

$$\lim_{E\to-\infty}\frac{C_t(E)}{C_L(E)} = 0 \qquad \text{and} \qquad \lim_{E\to-\infty}\frac{S_L(E)}{C_L(E)} = 0$$

we have (5.17). $\square$

Proof of Theorem 5.2. By (5.49) and Theorem 5.5, $\frac{1}{2}\Delta(-\frac{d^2}{dx^2}+V)$ is in $\mathcal{R}_L$. Moreover, the only terms contributing to $K(x,x+L)$ come from $C_L(-\frac{d^2}{dx^2}+V)$ and $K_D(L,L)S_L(-\frac{d^2}{dx^2}+V)$. By (5.28), (5.31), and (5.47),

$$K(x,x+L) = -\tfrac{1}{4}\int_x^{x+L} V(y)\,dy + \tfrac{1}{2}\biggl(\tfrac{1}{2}\int_0^L V_0(y)\,dy\biggr)$$

which, given the periodicity of V_0, is (5.15). □

There is a second proof of (5.13). It depends on this analog of Naĭman's lemma:

Lemma 5.8. *If V is $L^1_{\text{loc,unif}}$ and $-\frac{d^2}{dx^2}+V$ commutes with S_L+S_{-L}, then*

$$V(x+L) = V(x) \tag{5.50}$$

Proof. Suppose first that V is bounded. Then S_L+S_{-L} leaves $D(-\frac{d^2}{dx^2})$ invariant and commutes with it, so S_L+S_{-L} commutes with V. If f is supported in a small neighborhood of x_0, $(x_0-\delta, x_0+\delta)$ with $|\delta|<L/2$, then $(S_L+S_{-L})(Vf)$ is two separate pieces $V(x-L)f(x-L)$ supported near x_0+L and $V(x+L)f(x+L)$ supported near x_0-L, while $V(S_L+S_{-L})f$ is two pieces $V(x)f(x-L)$ and $V(x)f(x+L)$. Since the pieces are disjoint,

$$V(x)f(x-L) = V(x+L)f(x-L)$$

which implies (5.50).

For general V, take $g\in\mathcal{C}_0^\infty(\mathbb{R})$ with $\int g(x)\,dx=1$ and note that $\int g(x)S_x(-\frac{d^2}{dx^2}+V)S_{-x}\,dx$ is $-\frac{d^2}{dx^2}+g*V$ and it commutes with S_L+S_{-L} also. But $g*V$ is bounded, so it is periodic. (5.50) follows by using an approximate δ-function. □

6. Block Jacobi Matrices and Matrix Orthogonal Polynomials

What the magic formula suggests is that the Jacobi matrix J has parameters that approach an isospectral torus if and only if $\Delta(J)$ approaches S^p+S^{-p}. $\Delta(J)$ is a matrix of width $2p+1$ (i.e., $\Delta(J)_{k\ell}=0$ if $k-\ell\notin\{0,\pm1,\dots,\pm p\}$) and S^p+S^{-p} is a matrix with 1's at the extremes.

A matrix of width $2\ell+1$ has the structure of a tridiagonal matrix if rewritten in terms of $\ell\times\ell$ blocks and $S^\ell+S^{-\ell}$ corresponds to $B_k=\mathbf{0}$, $A_k=\mathbf{1}$, the identity matrix, so $\Delta(J)\sim S^\ell+S^{-\ell}$, at the matrix level, approaches the 'free case.' This will allow us to reduce our main theorems to matrix analogs of the theorems on perturbations of the free case.

Of course, the association of the block matrix to orthogonal polynomials is critical—the orthogonality will be with respect to a matrix-valued measure. There is a huge literature on MOPRL (see, e.g., [16, 23, 31, 32, 33, 34, 35, 36, 37, 38, 55, 71, 72, 112, 113]) and MOPUC (see, e.g., [8, 24, 25, 26, 27, 28, 29, 42, 68, 70, 90, 107, 114]). In this section, our main purpose is to set notation and discuss the important notion of equivalent families of block Jacobi matrices, a notion discussed more explicitly in [20].

Given a semi-infinite complex matrix $M=\{m_{ij}\}_{1\le i,j\le\infty}$ and $\ell=1,2,\dots$, we define the $\ell\times\ell$ block decomposition as the family of $\ell\times\ell$ matrices $\{M_{pq}\}_{1\le p,q\le\infty}$ by

$$(M_{pq})_{ij} = m_{\ell(p-1)+i,\,\ell(q-1)+j} \qquad i,j=1,\dots\ell \tag{6.1}$$

Definition. A block Jacobi matrix is an M where

$$M_{pq} = \begin{cases} B_p & \text{if } q = p \geq 1 \\ A_p & \text{if } q = p+1,\, p \geq 1 \\ A_{p-1}^* & \text{if } q = p-1,\, p \geq 2 \\ 0 & |p-q| \geq 2 \end{cases} \tag{6.2}$$

with each A_p invertible and each B_p Hermitian.

We will start using $\mathcal{J}$ for such matrices.

In analogy with the scalar case, one may be tempted to require

$$A_p > 0 \tag{6.3}$$

but to include $\Delta(J)$, we do not want to do that exclusively. If (6.3) holds, we say that $\mathcal{J}$ is of type 1.

If instead

$$A_1 \dots A_n > 0 \tag{6.4}$$

for all n, we say $\mathcal{J}$ is of type 2.

An $\ell \times \ell$ matrix, K, is said to be in $\mathcal{L}$ if it is lower triangular with strictly positive diagonal elements, that is,

$$K_{ij} = \begin{cases} 0 & \text{if } i < j \\ > 0 & \text{if } i = j \end{cases} \tag{6.5}$$

If each $A_p \in \mathcal{L}$, we say that $\mathcal{J}$ is of type 3.

The calculations in Section 3 and 4 show:

Proposition 6.1. (i) *If Δ is the discriminant of a periodic Jacobi matrix, J_0, of period ℓ, then for any Jacobi matrix, J, $\Delta(J) = \mathcal{J}$ is a block Jacobi matrix of type 3.*

(ii) *If Δ is the discriminant of a periodic CMV matrix, $\mathcal{C}_0$, of even period ℓ, then for any CMV matrix, $\mathcal{C}$, of $\Delta(\mathcal{C}) = \mathcal{J}$ is a block Jacobi matrix of type 3.*

We will see that distinct $\mathcal{J}$'s may correspond to the same measure. Indeed, in the scalar case, the b_n's and $|a_n|$'s are fixed by the measure, but the $\arg(a_n)$'s are arbitrary. Thus, we define

Definition. Two block Jacobi matrices, $\mathcal{J}$ and $\widetilde{\mathcal{J}}$, are called equivalent if and only if there is an $\ell \times \ell$ block diagonal unitary $\mathcal{U} = \mathbf{1} \oplus U_2 \oplus U_3 \oplus \cdots$ (we will use U_1 for $\mathbf{1}$) so that

$$\tilde{\mathcal{J}} = \mathcal{U}\mathcal{J}\mathcal{U}^{-1} \tag{6.6}$$

This is equivalent to

$$\widetilde{B}_n = U_n B_n U_n^{-1} \tag{6.7a}$$

$$\tilde{A}_n = U_n A_n U_{n+1}^{-1} \tag{6.7b}$$

We will be interested in equivalence classes of $\mathcal{J}$'s.

Proposition 6.2 ([20])**.** *Each equivalence class of $\mathcal{J}$'s has exactly one element each of type 1, type 2, and type 3.*

Definition. The Nevai class is the set of $\mathcal{J}$'s for which

$$B_n \to 0 \qquad A_n^* A_n \to \mathbf{1} \tag{6.8}$$

The following is immediate from (6.7):

Proposition 6.3. *If some $\mathcal{J}$ is in the Nevai class, so are all equivalent $\mathcal{J}$'s.*

Damanik, Pushnitski, and Simon [20] prove that

Proposition 6.4 ([20])**.** *If $\mathcal{J}$ is in the Nevai class and is type 1, type 2, or type 3, then*

$$A_n \to \mathbf{1} \tag{6.9}$$

We will sometimes need the MOPRL, the matrix orthogonal polynomials. What we describe here are the left OPs. There are also right OPs (see [20]), which we do not need here. An ℓ-dimensional matrix-valued measure is a positive scalar measure $d\rho_t(x)$ and a nonnegative $\ell \times \ell$ matrix-valued function $M(x)$. The matrix-valued measure

$$d\rho(x) = M(x)\, d\rho_t(x) \tag{6.10}$$

can always be normalized by

$$\operatorname{Tr}(M(x)) = \ell \tag{6.11}$$

We will always assume $d\rho$ is normalized, that is,

$$\int d\rho(x) = \mathbf{1} \tag{6.12}$$

The proper notion of nontriviality is a little subtle; it is discussed in detail in [20]. For our purpose here, it is sufficient that $\langle\langle \cdot, \cdot \rangle\rangle_{\mathrm{L}}$ defined below is nondegenerate on polynomials.

If f, g are two ℓ-dimensional matrix-valued functions, we define

$$\langle\langle f, g \rangle\rangle_{\mathrm{L}} = \int g(x) M(x) f(x)^\dagger \, d\rho_t(x)$$

where, following [98, Section 2.13], we use † for Hermitian adjoint (so for MOPUC, one does not confuse Szegő dual Φ_n^* with adjoints). The name 'left' comes from the fact that if C is an $\ell \times \ell$ matrix, then

$$\langle\langle f, Cg \rangle\rangle_{\mathrm{L}} = C \langle\langle f, g \rangle\rangle_{\mathrm{L}} \tag{6.13}$$

$$\langle\langle Cf, g \rangle\rangle_{\mathrm{L}} = \langle\langle f, g \rangle\rangle_L C^\dagger \tag{6.14}$$

We will normally just write $\langle\langle \cdot, \cdot \rangle\rangle$ from now on.

Left orthonormal polynomials are of the form

$$p_n(x) = \kappa_n x^n + \text{lower order}$$

with matrix coefficients, defined by

$$\langle\langle p_n, p_m \rangle\rangle = \delta_{nm} \mathbf{1} \tag{6.15}$$

So long as $d\rho$ is nontrivial, the p_n exist. They are not unique since if $\{U_n\}_{n=1}^\infty$ are unitary $\ell \times \ell$ matrices,

$$\tilde{p}_n(x) = U_{n+1} p_n(x) \tag{6.16}$$

are also MOPRL. We demand $\kappa_0 = \mathbf{1}$, that is, $p_0(x) = \mathbf{1}$, and so $U_1 = \mathbf{1}$.

$\{p_j\}_{j=0}^n$ are a left module basis for matrix polynomials of degree n, that is, if f is any polynomial of degree n, then there are unique $\ell \times \ell$ matrices $f_0, \dots, f_n$ so that

$$f(x) = \sum_{j=0}^n f_j p_j(x)$$

Indeed,

$$f_j = \langle\!\langle p_j, f\rangle\!\rangle \tag{6.17}$$

For $n = 1, 2, \dots$, define

$$B_n = \langle\!\langle p_{n-1}, xp_{n-1}\rangle\!\rangle \qquad A_n = \langle\!\langle p_n, xp_{n-1}\rangle\!\rangle \tag{6.18}$$

Then, since $xp_j = \sum_{\ell=0}^{j+1} C_\ell p_\ell$ implies $\langle\!\langle p_j, xp_n\rangle\!\rangle = \langle\!\langle xp_j, p_n\rangle\!\rangle = 0$ if $j \le n-2$, we have

$$xp_n(x) = A_{n+1}p_{n+1}(x) + B_{n+1}p_n(x) + A_n^* p_{n-1}(x) \tag{6.19}$$

This implies $A_{n+1}\kappa_{n+1} = \kappa_n$ so

$$\kappa_n = (A_1 \dots A_n)^{-1} \tag{6.20}$$

and the type 2 condition is equivalent to $\kappa_n > 0$.

Looking at (6.19), we see that (6.16) holds for $\tilde p_n, p_n$ if and only if $\tilde A_n, \widetilde B_n$ are related to A_n, B_n by (6.7). Jacobi matrix equivalence is just a 'change of phase' in the MOPRL.

Given a block Jacobi matrix, we can view it as acting on the Hilbert space $\ell^2(\{1, 2, \dots\}, \mathbb{C}^\ell)$ with inner product

$$\langle f, g\rangle = \sum_{n=1}^\infty \langle f_n, g_n\rangle_{\mathbb{C}^\ell} \tag{6.21}$$

If $\{e_j\}_{j=1}^\ell$ is the standard basis of $\mathbb{C}^\ell$, then $\{\delta_{k;j}\}_{k=1}^\infty{}_{j=1}^\ell$, defined by

$$(\delta_{k;j})_n = \delta_{kn} e_j \tag{6.22}$$

is a basis. $\mathcal{J}$ acts on $\ell^2(\{1, 2, \dots\}, \mathbb{C}^\ell)$ via

$$(\mathcal{J} f)_n = A_{n-1}^* f_{n-1} + B_n f_n + A_n f_{n+1} \tag{6.23}$$

(with $A_0 = 0$).

The spectral measure for $\mathcal{J}$ is the $\ell \times \ell$ matrix-valued measure with

$$\langle \delta_{0;j}, f(\mathcal{J})\delta_{0;k}\rangle = \int f(x)\, d\rho_{jk}(x) \tag{6.24}$$

for any scalar-valued function f. It is easy to see (e.g., [20]) that this map from $\mathcal{J}$ to ρ inverts the one given by forming the MOPRL and defining $\mathcal{J}$ by (6.18). Moreover, $\mathcal{J}$ and $\widetilde{\mathcal{J}}$ are equivalent if and only if $d\tilde\rho \equiv d\rho$.

The m-function is defined by

$$m(E) = \int \frac{1}{x-E}\, d\rho(x) \tag{6.25}$$

$$= \langle \delta_{0;\,\cdot\,}, (\mathcal{J} - E)^{-1}\delta_{0;\,\cdot\,}\rangle \tag{6.26}$$

It is an $\ell \times \ell$ matrix-valued Herglotz function:

$$\operatorname{Im} E > 0 \Rightarrow \operatorname{Im} m(E) > 0 \tag{6.27}$$

that is, $\frac{1}{2i}(m - m^\dagger)$ is positive definite in the upper half-plane. For information on matrix Herglotz functions, see [3, 4, 11, 12, 40, 44, 47, 48, 51, 56, 62, 63, 92, 110]. Obviously, by (6.25), m is constant over equivalence classes.

As in the scalar case (see [98, Section 1.2]), one has that for a.e. $x \in \mathbb{R}$, $\lim_{\varepsilon\downarrow 0} m(x + i\varepsilon) \equiv m(x + i0)$ exists and

$$d\rho_{\mathrm{ac}}(x) = \pi^{-1} \operatorname{Im} m(x + i0)\, dx \tag{6.28}$$

Here

$$d\rho_{\mathrm{ac}}(x) = M(x)\, d\rho_{t;\mathrm{ac}}(x) \tag{6.29}$$

where $d\rho_{t;\mathrm{ac}}$ is the a.c. part of $d\rho_t$. Alternatively, $d\rho_{\mathrm{ac}}$ is the unique matrix-valued measure which is a.c. (i.e., $\rho_{\mathrm{ac}}(I) = 0$ for any set with $|I| = 0$) and where there is a set K with $|K| = 0$ so $(\rho - \rho_{\mathrm{ac}})(\mathbb{R} \setminus K) = 0$.

Given a block Jacobi matrix, $\mathcal{J}$, by $\mathcal{J}^{(n)}$ we mean the matrix with the top n (block matrix) rows and leftmost n columns removed, that is,

$$B_k^{(n)} = B_{k+n} \qquad A_k^{(n)} = A_{k+n} \tag{6.30}$$

We write $m^{(n)}(z)$ for the m-function associated to $\mathcal{J}^{(n)}$. Equivalent $\mathcal{J}$'s do not have the same $m^{(n)}$ for $n \geq 1$ (although they are unitarily related). We see $m^{(0)} \equiv m$.

We will need the following result of Aptekarev–Nikishin [6] (see also [20]), a matrix analog of the well-known Jacobi–Stieltjes recursion for OPRL:

Theorem 6.5 ([6, 20])**.** *We have that*

$$m^{(n)}(E)^{-1} = E - B_{n+1} - A_{n+1} m^{(n+1)}(E) A_{n+1}^* \tag{6.31}$$

for $n = 0, 1, 2, \dots$.

Next, we need to note the following analog of a well-known scalar result (see, e.g., [100]) proven in [20]:

Theorem 6.6. *Let $\mathcal{J}$ be a block Jacobi matrix with $\sigma_{\mathrm{ess}}(\mathcal{J}) \subset [a,b]$. Then, for any ε, there is a K so that for $k \geq K$,*

$$\sigma(\mathcal{J}^{(k)}) \subset [a - \varepsilon, b + \varepsilon] \tag{6.32}$$

Finally, we need to look at poles and zeros of $\det(m(z))$. In the scalar ($\ell = 1$) case, poles occur precisely at eigenvalues of $\mathcal{J}$ and zeros at eigenvalues of $\mathcal{J}^{(1)}$, the once stripped Jacobi matrix. In that scalar case, these eigenvalues are distinct.

In the matrix case, $\mathcal{J}$ and $\mathcal{J}^{(1)}$ can have eigenvalues in common (as can be easily arranged by taking a direct sum of suitable scalar J's) so there can be cancellations. We say a scalar meromorphic function, $f(z)$, has a zero/pole of order $k \in \mathbb{Z}$ at $z_0 \in \mathbb{C}$ if $(z - z_0)^{-k} f(z)$ has a finite nonzero limit as $z \to z_0$. We will need the following result from [20]:

Theorem 6.7 ([20])**.** *Let $x_0 \in \mathbb{R}$. Let q_0 be the multiplicity of x_0 as an eigenvalue of $\mathcal{J}$, and q_1 its multiplicity as an eigenvalue of $\mathcal{J}_1$. Then*

(a) $q_0 + q_1 \leq \ell$
(b) $\det(m(z))$ *has a zero/pole of order $q_1 - q_0$.*

We will also need the following result from Aptekarev–Nikishin [6]:

Theorem 6.8. *Let $\mathcal{J}$ be a block Jacobi matrix with $\sigma_{\mathrm{ess}}(\mathcal{J}) = [-2, 2]$, $\sigma(\mathcal{J}) \setminus \sigma_{\mathrm{ess}}(\mathcal{J})$ a finite set and with $g(x) = d\rho_{\mathrm{ac}}(x)/dx$ we have*

$$\int (4 - x^2)^{-1/2} \log(\det(g(x))\, dx > -\infty \tag{6.33}$$

Suppose $\mathcal{J}$ is type 2. Then

$$\lim_{n\to\infty} A_1 \dots A_n$$

exists and is a strictly positive matrix.

The authors of [20] hope to extend this result to allow point spectrum obeying $\sum_j (|E_j| - 2)^{1/2} < \infty$ before [20] appears.

7. A Denisov–Rakhmanov Theorem for MOPRL

As preparation for proving Theorem 1.2 in Section 8, in this section we will prove

Theorem 7.1. *Let $d\rho$ be a nondegenerate $\ell \times \ell$ matrix-valued measure on $\mathbb{R}$ with associated block Jacobi matrix $\mathcal{J}$ of type 3 so that*

$$\text{(i)} \qquad \sigma_{\text{ess}}(\mathcal{J}) = [-2,2] \tag{7.1}$$

$$\text{(ii)} \qquad d\rho = f(x)\,dx + d\rho_{\text{s}} \tag{7.2}$$

with $d\rho_{\text{s}}$ singular and

$$\det(f(x)) > 0 \tag{7.3}$$

a.e. on $[-2,2]$. Then

$$B_n \to \mathbf{0} \qquad A_n \to \mathbf{1} \tag{7.4}$$

Remark. (7.3) says the a.c. spectrum has multiplicity ℓ.

If (7.1) is replaced by the stronger $\sigma(J) = [-2,2]$ and type 3 by type 2, this is a theorem of Yakhlef–Marcellán [112]. We will prove Theorem 7.1 by modifying their proof.

The shift from type 2 to 3 is easy on account of Proposition 6.4. By applying the argument of [112], we get $\tilde{A}_n \to \mathbf{1}$ for the equivalent $\tilde{\mathcal{J}}$ of type 2, conclude the whole equivalence class is in the Nevai class, and see $A_n \to \mathbf{1}$. So we will only worry about the changes needed to go from $\sigma(J) = [-2,2]$ to $\sigma_{\text{ess}}(J) = [-2,2]$, where we follow Denisov's approach for the scalar case [30].

[112] relies on a matrix version of Rakhmanov's theorem proven by van Assche [107]. We need to extend it slightly to allow a.c. spectrum on a large subset of $\partial\mathbb{D}$ rather than all of $\partial\mathbb{D}$:

Theorem 7.2. *Let $d\mu$ be an $\ell \times \ell$ matrix-valued measure on $\partial\mathbb{D}$ and let $\{\alpha_n\}_{n=0}^\infty$ denote its matrix Verblunsky coefficients . Suppose*

$$d\mu = w(\theta)\,\frac{d\theta}{2w} + d\mu_{\text{s}} \tag{7.5}$$

where $d\mu_{\text{s}}$ is singular, and let

$$\Omega = \{\theta : \det(w(\theta)) > 0\} \tag{7.6}$$

Then

$$\limsup_{n\to\infty} \|\alpha_n\| \le 2\sqrt{2\ell}\left(1 - \left(\frac{|\Omega|}{2\pi}\right)^3\right)^{1/2} \tag{7.7}$$

Remarks. 1. For notation on MOPUC, see [20].

2. Where we use $\{\alpha_n\}_{n=0}^\infty$, van Assche [107] uses $\{H_n\}_{n=1}^\infty$ related to α_n by

$$H_n = -\alpha_{n-1}^\dagger \tag{7.8}$$

3. We follow notation from [107] and the variant of the scalar proof as in [99, Section 9.1] where a_n, b_n, c_n, d_n below all appear.

We define

$$a_n = \|\alpha_n\|$$

$$b_{n,q} = \frac{1}{2\pi}\int_0^{2\pi} \|[\varphi_n^{\text{L}}(e^{i\theta})\varphi_{n+q}^{\text{L}}(e^{i\theta})^{-1}]\,[\varphi_n^{\text{L}}(e^{i\theta})\varphi_{n+q}^{\text{L}}(e^{i\theta})^{-1}]^* - I\|\,d\theta$$

$$c_{n,q} = \frac{1}{2\pi\ell}\int_0^{2\pi} \operatorname{Tr}(\varphi_n^{\text{L}}(e^{i\theta})\varphi_{n+q}^{\text{L}}(e^{i\theta})^{-1}[\varphi_{n+q}^{\text{L}}(e^{i\theta})^*]^{-1}\varphi_n^{\text{L}}(e^{i\theta})^*)^{1/2}\,d\theta$$

$$d_n = \frac{1}{2\pi\ell} \int_0^{2\pi} \operatorname{Tr}([\varphi_n^{\mathrm{L}}(e^{i\theta}) w(\theta) \varphi_n^{\mathrm{L}}(e^{i\theta})^*]^{1/2})\, d\theta$$

Proposition 7.3. *For every $n \geq 0$, we have that*

$$a_n \leq b_{n,q} \qquad \text{for every } q \geq 1 \tag{7.9}$$

$$b_{n,q}^2 \leq 8\ell^2(1 - c_{n,q}) \qquad \text{for every } q \geq 1 \tag{7.10}$$

$$d_n^2 \leq \inf_{q \geq 1} c_{n,q} \tag{7.11}$$

Moreover, we have that

$$\left(\frac{|\Omega|}{2\pi}\right)^{3/2} \leq \liminf_{n\to\infty} d_n \tag{7.12}$$

Consequently,

$$\limsup_{n\to\infty} a_n \leq 2\sqrt{2}\,\ell \biggl(1 - \left(\frac{|\Omega|}{2\pi}\right)^3\biggr)^{1/2} \tag{7.13}$$

Proof. The second to last displayed formula on [107, p. 7] is (7.9). The estimates on the bottom half of [107, p. 12] show that

$$\begin{aligned}
b_{n,q}^2 &= \frac{1}{4\pi^2} \biggl(\int_0^{2\pi} \|[\varphi_n^{\mathrm{L}}(e^{i\theta})\varphi_{n+q}^{\mathrm{L}}(e^{i\theta})^{-1}]\, [\varphi_n^{\mathrm{L}}(e^{i\theta})\varphi_{n+q}^{\mathrm{L}}(e^{i\theta})^{-1}]^* - I\|\, d\theta\biggr)^2 \\
&\leq \frac{2\ell}{\pi} \int_0^{2\pi} \|([\varphi_n^{\mathrm{L}}(e^{i\theta})\varphi_{n+q}^{\mathrm{L}}(e^{i\theta})^{-1}]\, [\varphi_n^{\mathrm{L}}(e^{i\theta})\varphi_{n+q}^{\mathrm{L}}(e^{i\theta})^{-1}]^*)^{1/2} - I\|^2\, d\theta \\
&\leq \frac{2\ell}{\pi}\, (4\pi\ell - 4\pi\ell\, c_{n,q}) \\
&= 8\ell^2(1 - c_{n,q})
\end{aligned}$$

which is (7.10). The third displayed formula on [107, p. 14] is (7.11).

Now, mimicking the estimates on the bottom half of [107, p. 14],

$$\begin{aligned}
&\int_\Omega \operatorname{Tr}([f(\theta)w(\theta)f(\theta)^*]^{1/4})\, d\theta \\
&\qquad \leq \biggl(2\pi\ell \int_\Omega \operatorname{Tr}(f(\theta)\varphi_n^{\mathrm{L}}(e^{i\theta})^{-1}(\varphi_n^{\mathrm{L}}(e^{i\theta})^*)^{-1} f(\theta)^*)\, \frac{d\theta}{2\pi}\biggr)^{1/4} (2\pi\ell\, d_n)^{1/2}
\end{aligned}$$

Taking $n \to \infty$, we see that

$$\begin{aligned}
&\int_\Omega \operatorname{Tr}([f(\theta)w(\theta)f(\theta)^*]^{1/4})\, d\theta \\
&\qquad \leq \biggl(2\pi\ell \operatorname{Tr} \int_\Omega f(\theta) d\mu(\theta) f(\theta)^*\bigr)^{1/4} \bigl(2\pi\ell \liminf_{n\to\infty} d_n\bigr)^{1/2}
\end{aligned}$$

Removing the singular part as in [107], we obtain

$$\begin{aligned}
&\int_\Omega \operatorname{Tr}([f(\theta)w(\theta)f(\theta)^*]^{1/4})\, d\theta \\
&\qquad \leq \biggl(2\pi\ell \operatorname{Tr} \int_\Omega f(\theta)w(\theta)f(\theta)^* d\theta\biggr)^{1/4} \bigl(2\pi\ell \liminf_{n\to\infty} d_n\bigr)^{1/2}
\end{aligned}$$

Proceeding as in [107, pp. 15–16], it then follows that

$$|\Omega|\ell \leq (2\pi\ell\, |\Omega|\ell)^{1/4} \bigl(2\pi\ell \liminf_{n\to\infty} d_n\bigr)^{1/2}$$

which implies (7.12).

Putting these estimates together,

$$a_n \leq b_{n,1} \leq 2\sqrt{2}\,\ell(1-c_{n,1})^{1/2} \leq 2\sqrt{2}\,\ell(1-d_n^2)^{1/2}$$

and hence

$$\limsup_{n\to\infty} a_n \leq 2\sqrt{2}\,\ell\biggl(1-\biggl(\frac{|\Omega|}{2\pi}\biggr)^3\biggr)^{1/2}$$

which is (7.13). □

In particular, for $2\pi - |\Omega|$ small,

$$\limsup_{n\to\infty} a_n = O\biggl(\biggl(1-\biggl(\frac{|\Omega|}{2\pi}\biggr)^3\biggr)^{1/2}\biggr) = O((2\pi-|\Omega|)^{1/2})$$

as in the scalar case.

We have thus proven Theorem 7.2. To get Theorem 7.1, we follow [112] using the analog of Denisov's arguments for the case $\ell = 1$.

Proof of Theorem 7.1. By Proposition 6.4, we need only prove for the type 2 choice, for any $\varepsilon > 0$, we have

$$\limsup\,(\|\tilde{A}_n - 1\| + \|\tilde{B}_n\|) \leq \varepsilon \tag{7.14}$$

By the Szegő mapping and Geronimus connection formulae in [112], this holds by Theorem 7.2 so long as for any $\varepsilon' > 0$, we can find k so $\sigma(\mathcal{J}^{(k)}) \subset [-2-\varepsilon', 2+\varepsilon']$, and this is true by Theorem 6.6. □

8. A Denisov–Rakhmanov Theorem for Periodic OPRL

Our main goal in this section is to prove Theorem 1.2. We will also prove the 'hard' half of Theorem 1.1. The simplicity of the proof shows the magic in the magic formula!

Proof of Theorem 1.2. By a right limit of J, we mean a two-sided Jacobi matrix, J_r, (but with some a's allowed to vanish) so that for some subsequence $n_j \to \infty$ and any $k \in \mathbb{Z}$,

$$a_{n_j+k} \to (a_r)_k \qquad b_{n_j+k} \to (b_r)_k \tag{8.1}$$

By our standing convention, Jacobi parameters are uniformly bounded, so by compactness, if $d_m((a,b), \mathcal{T}_{J_0}) \nrightarrow 0$, there exists a right limit $J_r \notin \mathcal{T}_{J_0}$. Thus, it suffices to show that any right limit J_r has $J_r \in \mathcal{T}_{J_0}$.

By the hypotheses of Theorem 1.2, the spectral mapping theorem, and the fact that Δ maps $\sigma_{\text{ess}}(J_0)$ to $[-2,2]$ with a p-fold cover on $(-2,2)$, we see that

$$\Delta(J)_{\text{ess}} = [-2,2]$$

and $\Delta(J)$ has a.c. spectrum of multiplicity p. So thinking of $\mathcal{J} \equiv \Delta(J)$ as a block Jacobi matrix, $\mathcal{J}$ is of type 3 and the hypotheses of Theorem 7.1 apply. It follows that $A_n \to \mathbf{1}$, $B_n \to \mathbf{0}$. This means that

$$\Delta(J_r) = S^p + S^{-p}$$

so by the magic formula (Theorem 3.1), $J_r \in \mathcal{T}_{J_0}$. □

Rakhmanov's theorem is often related to issues of w-lim$|p_n|^2\,d\mu$ and to the density of zeros. We note that there are also results of that genre here:

Theorem 8.1. *If J_0 is a periodic Jacobi matrix of period p and J is a Jacobi matrix with bounded Jacobi parameters whose right limits all lie in $\mathcal{T}_{J_0}$ (in particular, if the hypotheses of Theorem 1.2 hold), then (with $d\mu$ the measure for J)*
(a)

$$\operatorname*{w-lim}_{n\to\infty} \frac{1}{p} \sum_{j=1}^{p} |p_{j+n}(x)|^2 \, d\mu(x) = d\nu \tag{8.2}$$

the density of zeros for J_0.
(b) *The density of zeros of $p_n(x)$ converges to $d\nu$.*

Proof. If $J_1 \in \mathcal{T}_{J_0}$, then the spectral measure $d\mu_k^{(J_1)}$ associated to δ_k has period p in k since J_1 is periodic. Thus $\lim_{N\to\infty} \frac{1}{2N+1} \sum_{|j|\leq N} d\mu_j^{(J_1)} = \frac{1}{p}\sum_{j=1}^{p} d\mu_j^{(J_1)}$, but the limit is $d\nu$ by the discussion in Subsection 2.8. Since

$$\int x^\ell |p_j^{(J)}(x)|^2 \, d\mu(x) = \langle \delta_j, J^\ell \delta_j \rangle \tag{8.3}$$

and $J_{j,j}, J_{j,j\pm 1}$ is very close to some $(J_1)_{j,j}$, $(J_1)_{j,j+1}$ for $|j-j_0| \leq M$ for fixed M and $j_0 \to \infty$, we see that moments of LHS of (8.2) are close to moments of $d\nu$. This proves (a).

If $J^{(n)}$ denotes the top left $n\times n$ submatrix of J, then

$$\int x^\ell \, d\nu_n = \frac{1}{n} \operatorname{Tr}((J^{(n)})^\ell)$$

so

$$\lim_{n\to\infty} \int x^\ell \, d\nu_n = \lim_{n\to\infty} \int x^\ell \biggl[\frac{1}{n} \sum_{j=0}^{n-1} p_j(x)^2 \, d\mu(x) \biggr]$$

and thus (a) implies (b). □

We also have

Theorem 8.2. *If $d_m((a,b), \mathcal{T}_{J_0}) \to 0$, then*

$$\sigma_{\text{ess}}(J) \subset \sigma_{\text{ess}}(J_0)$$

Proof. By the magic formula, compactness, and the fact that every right limit of J is in $\mathcal{T}_{J_0}$, we see that every right limit of $\Delta(J)$ is $S^p + S^{-p}$, that is, $A_n \to \mathbf{1}$, $B_n \to \mathbf{0}$. Thus, by Weyl's theorem, $\sigma_{\text{ess}}(\Delta(J)) = [-2,2]$. Since

$$\sigma_{\text{ess}}(\Delta(J)) = \Delta(\sigma_{\text{ess}}(J))$$

we see $\sigma_{\text{ess}}(J) \subset \Delta^{-1}([-2,2]) = \sigma_{\text{ess}}(J_0)$. □

Remarks. 1. Since Δ is p to 1, we cannot conclude that $\sigma_{\text{ess}}(J_0) = \sigma_{\text{ess}}(J)$ from $\Delta(\sigma_{\text{ess}}(J_0)) = \Delta(\sigma_{\text{ess}}(J))$.

2. That $\sigma_{\text{ess}}(J_0) \subset \sigma_{\text{ess}}(J)$ is a simple trial function argument given that J must have some right limits; see [66, 67].

9. Denisov–Rakhmanov Sets

In this section, we want to show how one can take suitable limits of Theorem 1.2 to get a 'cheap' proof of similar theorems in other nonperiodic cases. We will also present an insight into the proper general form of Denisov–Rakhmanov-type theorems.

The right limits we have discussed so far involve the weak product topology on the Jacobi parameters, so we will emphasize this fact by using the phrase 'weak right limits' in this section. We are also interested in limits in the ℓ^∞-topology for two-sided sequences, that is, $\{c_n^{(k)}\}_{n=-\infty}^\infty \to \{c_n^{(\infty)}\}_{n=-\infty}^\infty$ in this topology if and only if, as $k \to \infty$,

$$\sup_n |c_n^{(k)} - c_n^{(\infty)}| \to 0$$

In terms of weak limits, we note the following:

Proposition 9.1. *Let E be a closed set. Let J be a half-line Jacobi matrix with*

$$\Sigma_{\mathrm{ac}}(J) = \sigma_{\mathrm{ess}}(J) = E \tag{9.1}$$

Let J_r be a weak right limit of J. Then

$$\Sigma_{\mathrm{ac}}(J_r) = \sigma(J_r) = E \tag{9.2}$$

Remark. Note that (9.2) has $\sigma(J_r)$, not merely $\sigma_{\mathrm{ess}}(J_r)$.

Proof. By results in [66],

$$\sigma(J_r) \subset E \subset \Sigma_{\mathrm{ac}}(J_r)$$

Since $\Sigma_{\mathrm{ac}}(J_r) \subset \sigma(J_r)$ trivially, (9.2) holds. $\square$

Recall that a sequence $\{c_n\}_{n=-\infty}^\infty$ is called uniformly almost periodic (in the general theory of almost periodic functions, this defines 'almost periodic'—we add 'uniformly' because the term is sometimes used in a weaker sense in the spectral theory literature) if and only if $\{c^{(\ell)}\}_{\ell=-\infty}^\infty$ given by $(c^{(\ell)})_n = c_{n+\ell}$ has compact closure in the ℓ^∞-topology.

Definition. A set E is called *essentially perfect* if and only if E is closed, and for all $x \in E$ and $\delta > 0$, $|(E-\delta, E+\delta)| > 0$.

Remark. Essentially perfect sets are precisely the sets, E, for which there is a purely a.c. measure $d\rho$ with $\mathrm{supp}(d\rho) = E$.

Definition. A set E is said to be a *Denisov–Rakhmanov set* if and only if
(i) E is essentially perfect and bounded.
(ii) There is a set $\mathcal{T}_E$ compact in the uniform topology so that for any bounded Jacobi matrix, J, for which (9.1) holds, the set of right limits of J lies in $\mathcal{T}_E$.

The definition says nothing explicit about $\mathcal{T}_E$ being a torus, but by Proposition 9.1, if $J_r \in \mathcal{T}_E$, then (9.2) holds, and since $\mathcal{T}_E$ is closed under translations, each J_r in $\mathcal{T}_E$ is almost periodic. By Kotani theory (see [60, 61, 95, 99]), $\langle \delta_n, (J_r - x - i\varepsilon)^{-1}\delta_n\rangle$ has real boundary values for a.e x. In many cases and, in particular, if E is a finite union of closed intervals, Sodin–Yuditskii [104] (see also [5, 14]) proved there is a natural torus so that any almost periodic J_r with real boundary values lies in this torus. Thus for such cases, that E is a Denisov–Rakhmanov set can be connected to approach to an isospectral torus. In particular, our Theorem 1.2 implies the statement that $\sigma_{\mathrm{ess}}(J_0)$ is a Denisov–Rakhmanov set.

Given an essentially perfect set, E, we define $\mathcal{D}(E)$ to be the set of Jacobi matrices obeying (9.1).

The following two simple results will be the basis of our approximation theorems:

Proposition 9.2. *Let E be an essentially perfect set. Suppose there are uniformly compact sets $\{\mathcal{T}^{(n)}\}_{n=1}^\infty$ and $\mathcal{T}^{(\infty)}$ of two-sided Jacobi matrices so that*
(1) *If $J_n \in \mathcal{T}^{(n)}$ and $J_n \to J_\infty$ weakly, then $J_\infty \in \mathcal{T}^{(\infty)}$.*
(2) *For any weak right limit point J_r of some $J \in \mathcal{D}(E)$, there is $\tilde{J} \in \mathcal{T}^{(n)}$ so*

$$\sup_{|j|\le n} |a_j^{(r)} - \tilde{a}_j| + |b_j^{(r)} - \tilde{b}_j| \le \frac{1}{n} \tag{9.3}$$

Then E is a Denisov–Rakhmanov set.

Proof. Let J_n be the $\tilde{J}$ guaranteed by (9.3). Then clearly, J_n converges weakly to J_r so, by (1), $J_r \in \mathcal{T}^{(\infty)}$. Since $\mathcal{T}^{(\infty)}$ is uniformly compact, E is a Denisov–Rakhmanov set. □

Proposition 9.3. *Let J_0 be a fixed periodic Jacobi matrix with essential spectrum E_0. Then for any n, there is a $\delta > 0$ so that for any set E with*

$$\text{(a)} \qquad E \subset \{x : \operatorname{dist}(x, E_0) < \delta\} \tag{9.4}$$

$$\text{(b)} \qquad |E| > (1-\delta)|E_0| \tag{9.5}$$

and any $J \in \mathcal{D}(E)$, we have that any right limit, J_r, obeys (9.3) for some $\tilde{J} \in \mathcal{T}_{J_0}$.

Moreover, if p is fixed and C is a compact subset of $[(0,\infty)\times\mathbb{R}]^p$, then δ can be picked to work for all $J_0 = \{(a_n, b_n)\}_{n=1}^p \in C$.

Proof. The uniformity claimed in the last statement comes from noting that choices can be made uniformly in the proof below.

Let p be the period of J_0. We first claim that given δ_1, we can find δ so if E obeys (9.4)–(9.5) then

$$\operatorname{dist}(\Delta(E), [-2,2]) < \delta_1 \tag{9.6}$$

$$|\{x \in (-2,2) : \text{all } p \text{ solutions of } \Delta(y) = x \text{ lie in } E\}| > 4 - \delta_1 \tag{9.7}$$

This is immediate from the continuity of Δ and its derivatives.

Next, we note that given ε_1, we can find δ_1 so that if $\mathcal{J}$ is a $p\times p$ block Jacobi matrix so that

$$\sigma_{\text{ess}}(\mathcal{J}) \subset [-2-\delta_1, 2+\delta_1]$$

$$|\{x \in [-2,2] : \mathcal{J} \text{ has a.c. spectrum at } x \text{ of multiplicity } p\}| > 4 - \delta_1$$

then

$$\limsup_{k,m\to\infty} |\mathcal{J}_{km} - (S^p + S^{-p})_{km}| < \varepsilon_1$$

The proof of this is identical to the proof of the matrix Denisov–Rakhmanov theorem.

Combining these steps, we are reduced to showing for any n and ε, there is ε_1 so for all two-sided J_r with $\operatorname{dist}(\sigma(J_r), E) < \varepsilon$, we have that

$$\sup_{k,m} |\Delta(J_r) - (S^p + S^{-p})_{km}| < \varepsilon_1 \tag{9.8}$$

implies there is a $\tilde{J} \in \mathcal{T}_{J_0}$ so that (9.3) holds. To do this, we first follow the proof of Theorem 3.1 to note that for n, ε_2, and ε_3 fixed, we can find ε_1 so (9.8) implies there is a period p $J^\sharp$

$$\|\Delta(J^\sharp) - \Delta(J)\| < \varepsilon_2 \tag{9.9}$$

and

$$\sup_{|j|\leq n} |a_j^{(r)} - a_j^\sharp| + |b_j^{(r)} - \tilde{b}_j| \leq \varepsilon_3$$

Finally, a compactness argument shows that for any n, we can find ε_4 so for any periodic $J^\sharp$ with

$$\|\Delta(J^\sharp) - (S^p + S^{-p})\| < \varepsilon_4$$

there is a $\tilde{J} \in \mathcal{T}_{J_0}$ so that

$$\|J^\sharp - \tilde{J}\| \leq \frac{1}{2n}$$

Putting these together implies (9.3). □

Theorem 9.4. *Let $\ell_1, \ell_2, \dots$ be an arbitrary sequence in $(2, 3, 4, \dots)$. For any ℓ_1-periodic Jacobi matrix $J^{(0)}$, there exist $k_2, k_3, \dots$ so that for any limit periodic J with Jacobi coefficients*

$$a_n = a_n^{(0)} + \sum_{m=2}^\infty \operatorname{Re}[A_m e^{2\pi i n/\ell_1\ell_2\dots\ell_m}] \tag{9.10}$$

$$b_n = b_n^{(0)} + \sum_{m=2}^\infty \operatorname{Re}[B_m e^{2\pi i n/\ell_1\dots\ell_m}] \tag{9.11}$$

obeying

$$|A_m| + |B_m| \leq k_m \tag{9.12}$$

we have that $\sigma(J)$ is a Denisov–Rakhmanov set.

Remark. The study of limit periodic discrete Schrödinger operators with small tails was initiated by Avron–Simon [7] and Chulaevsky [19]. They prove purely a.c. spectrum.

Proof. As in [7, 19], one can pick the k_m's so the spectrum is purely a.c. and so that the union of all isospectral tori for the periodic approximates lie in a fixed ℓ^∞ compact sets. This implies the limit periodic potentials also have compact isospectral sets, and within this compact set, weak convergence implies norm convergence so hypothesis (1) of Proposition 9.2 holds. By decreasing the k_m's if necessary, Proposition 9.3, continuity of the spectrum in ℓ^∞ norm, and absolute continuity of periodic spectrum imply we can be sure that (9.3) holds. Thus Proposition 9.2 implies this theorem. □

Our final theorem in this section is the following:

Theorem 9.5. *Fix ℓ. Let $\mathcal{G} = \{(\alpha_1, \beta_1, \alpha_2, \dots, \alpha_{\ell+1}, \beta_{\ell+1}) \in \mathbb{R}^{2\ell+2} : \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots < \beta_{\ell+1}\}$. For $(\vec{\alpha}, \vec{\beta}) \in \mathcal{G}$, define*

$$E(\vec{\alpha}, \vec{\beta}) = \bigcup_{j=1}^{\ell+1} [\alpha_j, \beta_j] \tag{9.13}$$

Then $\{(\vec{\alpha}, \vec{\beta}) : E(\vec{\alpha}, \vec{\beta})$ is a Denisov–Rahkmanov set$\}$ contains a dense G_δ.

Remarks. 1. As we have seen, the $E(\vec{\alpha},\vec{\beta})$ which arise from periodic problems is precisely those where the harmonic measure of each $e_j = [\alpha_j, \beta_j]$ is rational. In particular, if we fix $\vec{\alpha}$ and $\beta_{\ell+1}$, the set of β's that are periodic is countable, and so certainly not a G_δ. We show that the family that leads to Denisov–Rahkmanov sets is uncountable.

2. It is a reasonable conjecture that every E is a Denisov–Rakhmanov set, so this result is weak. We include it because it is such a 'cheap' way to go beyond the periodic case using only that case.

Proof. For each $(\vec{\alpha},\vec{\beta})$, it is known [104] that there is an isospectral torus $\mathcal{T}$ of almost periodic J's where (whole line) spectrum is precisely $E(\vec{\alpha},\vec{\beta})$. It follows from the construction in [104] that if $(\vec{\alpha}^{(n)},\vec{\beta}^{(n)}) \in \mathcal{G}$ converge to $(\vec{\alpha}^{(\infty)},\vec{\beta}^{(\infty)}) \in \mathcal{G}$, then condition (1) of Proposition 9.2 holds.

Let $\mathcal{G}_p$ be the subset of $\mathcal{G}$ coming from periodic problems—this is dense in $\mathcal{G}$. For $(\vec{\alpha}^{(0)},\vec{\beta}^{(0)}) \in \mathcal{G}_p$, pick $J(\vec{\alpha}^{(0)},\vec{\beta}^{(0)})$ periodic with $E(\vec{\alpha}^{(0)},\vec{\beta}^{(0)})$ as spectrum and pick $\delta_n(\vec{\alpha}^{(0)},\vec{\beta}^{(0)})$ via Proposition 9.3 requiring $\delta_n < \frac{1}{2}\min(|\beta_j^{(0)} - \alpha_j^{(0)}|, |\alpha_{j+1}^{(0)} - \beta_j^{(0)}|)$. Let $U^{(n)}(\vec{\alpha}_0,\vec{\beta}_0) = \{(\vec{\alpha},\vec{\beta}) : E(\vec{\alpha},\vec{\beta})$ obeys (9.4)/(9.5) for $E = E(\vec{\alpha},\vec{\beta})$, $E_0 = E(\vec{\alpha}^{(0)},\vec{\beta}^{(0)})$ and $\delta = \delta_n\}$, and let

$$U^{(n)} = \bigcup_{\mathcal{G}_p} U^{(n)}(\alpha^{(0)},\beta^{(0)})$$

This is dense and open. Then $\cap_n U^{(n)}$ is a dense G_δ whose points, by construction and Proposition 9.2, correspond to Denisov–Rakhmanov sets. $\square$

10. Sum Rules for MOPRL

In this section, our main goal is to prove the following two theorems about block Jacobi matrices:

Theorem 10.1 (P_2 Sum Rule for MOPRL)**.** *Let $\mathcal{J}$ be a block Jacobi matrix with $\ell\times\ell$ Jacobi parameters $\{A_n\}_{n=1}^\infty, \{B_n\}_{n=1}^\infty$ and matrix measure $d\rho$. Let $m(E)$ be given by (6.25) and suppose $\sigma_{\text{ess}}(\mathcal{J}) = [-2,2]$. Define for $z \in \mathbb{D}\setminus\{z = E + E^{-1} : E \in \sigma(\mathcal{J})\setminus[-2,2]\}$*

$$M(z) = -m(z+z^{-1}) \tag{10.1}$$

Let F, G be the functions

$$F(\beta+\beta^{-1}) = \tfrac{1}{4}\,[\beta^2 - \beta^{-2} - \log\beta^2] \tag{10.2}$$

for $\beta \in \mathbb{R}\setminus[-1,1]$, that is, $E = \beta+\beta^{-1} \in \mathbb{R}\setminus[-2,2]$ and

$$G(a) = a^2 - 1 - \log(a^2) \qquad a \in (0,\infty) \tag{10.3}$$

Then $\lim_{r\uparrow 1} M(re^{i\theta})$ exists for a.e. θ and

$$\begin{aligned} &\frac{1}{2\pi}\int \log\biggl(\frac{\sin^\ell\theta}{\det(\operatorname{Im} M(e^{i\theta}))}\biggr)\sin^2\theta\,d\theta \\ &\qquad + \sum_{E\in\sigma(\mathcal{J})\setminus[-2,2]} F(E) = \sum_{n=1}^\infty \operatorname{Tr}(\tfrac{1}{4}\,B_n^2 + \tfrac{1}{2}\,G(|A_n|)) \end{aligned} \tag{10.4}$$

Remarks. 1. All terms are positive (since F and G are positive, this is evident for two terms; positivity of the integral will be seen below), so this sum rule always makes sense, although some terms may be $+\infty$.

2. Recall that $|A_n| = \sqrt{A_n^* A_n}$; although since the formula for $G(a)$ only involves a^2, one does not need to take a square-root.

3. Because of the trace and absolute value, $\mathrm{Tr}(\frac{1}{4}B_n^2 + \frac{1}{2}G(|A_n|))$ is constant over equivalence classes of Jacobi matrix parameters.

4. In the type 1 case, the RHS of (10.4) is finite if and only if $\mathcal{J} - S^p - S^{-p}$ is Hilbert–Schmidt. This is also true when $\mathcal{J}$ is of type 3; see Proposition 11.11.

Theorem 10.2 (Sharp Case C_0 Sum Rule for MOPRL)**.** *Consider the three quantities:*

$$Z(\mathcal{J}) = \frac{1}{4\pi}\int_0^{2\pi} \log\biggl(\frac{\sin^\ell\theta}{\det(\operatorname{Im} M(e^{i\theta}))}\biggr)\,d\theta \tag{10.5}$$

$$\mathcal{E}_0(\mathcal{J}) = \sum_{E\notin\sigma(\mathcal{J})\setminus[-2,2]} \log(|\beta|) \tag{10.6}$$

where β is related to E by

$$\beta\in\mathbb{R}\setminus[-1,1] \qquad E = \beta+\beta^{-1} \tag{10.7}$$

and

$$A_0(\mathcal{J}) = \lim_{N\to\infty} -\sum_{n=1}^N \log(\det(|A_n|)) \tag{10.8}$$

which we suppose exists but it may be $+\infty$ or $-\infty$. Then

(i) *If any two of $Z, \mathcal{E}_0, A_0$ are finite, then so is the third.*

(ii) *If all are finite, then*

$$Z(J) = A_0(J) + \mathcal{E}_0(J) \tag{10.9}$$

(iii) *If all are finite, then*

$$\lim_{N\to\infty}\sum_{n=1}^N \mathrm{Tr}(B_n) \tag{10.10}$$

exists.

Remark. We will prove (and actually use it to prove Theorem 1.3) that if $\mathcal{E}_0(J) < \infty$, then $Z(J) < \infty$ so long as

$$\underline{A}_0(\mathcal{J}) = \liminf_N \biggl(-\sum_{n=1}^N \log(\det(|A_n|))\biggr) < \infty \tag{10.11}$$

Theorem 10.1 is a matrix-valued analog of the OPRL P_2 sum rule of Killip–Simon [59], and Theorem 10.2 of the OPRL Case C_0 sum rule by Simon–Zlatoš [103]. Both were refinements of sum rules of Case [17, 18] who in turn was motivated by earlier KdV and Toda sum rules. Case only considered short-range $|a_n - 1| + |b_n|$, while [59, 103] considered the necessary techniques to go up to the borderline of validity. [103] had some simplifications of [59], and [96] further simplified, although each of the later two proofs depends heavily on the earlier ones. Here, following Simon [96], we will prove a nonlocal step-by-step sum rule. As there, the key is a suitable representation theorem for meromorphic Herglotz functions—in this case, extended to matrix-valued functions.

For $a \in (-1,1)$, we define Blaschke factors as usual by

$$b(z,a) = \begin{cases} \frac{a-z}{1-az} & 0 < a < 1 \\ \frac{z-a}{1-az} & -1 < a \leq 0 \end{cases} \tag{10.12}$$

Proposition 10.3. *Let $f(z)$ be an $\ell \times \ell$ matrix-valued meromorphic function on $\mathbb{D}$ so that* ($\operatorname{Im} A \equiv (A - A^*)/2i$)

$$\text{(i)} \qquad \pm \operatorname{Im} f(z) > 0 \text{ when } \pm \operatorname{Im} z > 0 \tag{10.13}$$

$$\text{(ii)} \qquad \lim_{z \to 0} f(z) z^{-1} = \mathbf{1} \tag{10.14}$$

Then

(a) *For a.e. θ, $\lim_{r \uparrow 1} f(re^{i\theta}) \equiv f(e^{i\theta})$ exists.*

(b) $\log|\det(f(e^{i\theta}))| \in \cap_{1 \leq p < \infty} L^p(\partial\mathbb{D}, d\theta/2\pi)$

(c) *All the zeros and poles of* $\det(f(z))$ *lie on* $(-1,1)$ *and are of finite order. Let $\{z_j\}_{j=1}^\infty$ and $\{p_j\}_{j=1}^\infty$ be those zeros and poles of* $\det(f(z))$ *repeated up to multiplicity* (*it can also happen that both sets are finite*). *$z = 0$ is not included in $\{z_j\}$. Then*

$$B_\infty(z) = \lim_{r \uparrow 1} \frac{\prod_{|z_j| < r} b(z, z_j)}{\prod_{|p_j| < r} b(z, p_j)} \tag{10.15}$$

exists and obeys:

(i) *B_∞ is analytic and nonvanishing on $\mathbb{C} \setminus \{z_j\} \cup \{p_j\} \cup \{z_j^{-1}\} \cup \{p_j^{-1}\} \cup \{\pm 1\}$*

(ii) *$|B_\infty(e^{i\theta})| = 1$ on $\partial\mathbb{D} \setminus \{\pm 1\}$*

(iii)

$$|\arg(B_\infty(z))| \leq 2\pi\ell \tag{10.16}$$

for $|z| < 1$ with $\arg$ *normalized by* $\arg B_\infty(0) = 0$.

(d) *We have the representation*

$$\det(f(z)) = z^\ell B_\infty(z) \exp\biggl(\int \frac{e^{i\theta} + z}{e^{i\theta} - z} \log|\det(f(e^{i\theta}))| \frac{d\theta}{\pi}\biggr) \tag{10.17}$$

Remarks. 1. It should be possible to prove that $0 < \arg(B_\infty(z)) < \pi\ell$ for $\operatorname{Im} z > 0$; we settle for the weaker result.

2. (10.14) is not central for a result of this type, but it is true in applications and simplifies the notation.

3. This result for $\ell = 1$ is in [96]. $\ell > 1$ has some subtleties, but the basic strategy we use is that of [96].

We will prove this result in a sequence of lemmas:

Lemma 10.4. $\det(f(z))$ *is analytic and nonvanishing in $\Omega \equiv \{z : z \in \mathbb{D}, \operatorname{Im} z > 0\}$, and* $\arg(\det(f(z)))$ *can be chosen in that region to be continuous so that*

$$0 < \arg(\det(f(z))) < \pi\ell \tag{10.18}$$

Proof. In Ω, all matrix elements $\langle \varphi, f(z)\varphi \rangle$ are analytic and have a.e. boundary values (since they are scalar Herglotz functions), so by polarization, $f(z)$ is analytic on Ω and has a.e. boundary values. Thus $\det(f(z))$ as a polynomial in matrix elements is analytic on Ω.

Consider

$$P(\lambda, z) = \det(\lambda \mathbf{1} - f(z)) \tag{10.19}$$

which is a polynomial in λ with analytic coefficients away from the poles of f. It follows, for z near any z_0 about which f is analytic, that the roots $P(\lambda, z) = 0$ written as a function of z are analytic functions in $(z - z_0)^{1/k}$ for some k depending on z_0. It then follows that near any fixed z_0, all roots are analytic, that is, singularities are isolated.

Pick $x_0 \in (0, \varepsilon)$ so that $x_0 f(x_0) > 0$, so all eigenvalues $\lambda_1(x_0), \dots, \lambda_\ell(x_0)$ are in $(0, \infty)$. Let $z \in \Omega$ be a point about which all eigenvalues are analytic, and let $\gamma(z)$ be a simple closed path from x_0 to z which avoids the discrete set where eigenvalues are not analytic and lie in Ω except for x_0 with, say, $\gamma(0) = x_0$, $\gamma(1) = z$. By analytically continuing eigenvalues, we get function $\{\lambda_j(z)\}_{j=1}^{\ell}$, so $\lambda_j(z)$ are all the eigenvalues of $f(\gamma(t))$ and $\lambda_j(0) \in (0, \infty)$. By $\operatorname{Im} f > 0$, $\operatorname{Im} \lambda_j(z) > 0$, so if we define $\arg(\lambda_j(z))$ with $\arg(\lambda_j(0)) = 0$, we have

$$0 < \arg(\lambda_j(z)) < \pi$$

Thus

$$\arg(\det(f(z))) = \sum_{j=1}^{\ell} \arg(\lambda_j(z))$$

normalized by $\arg(\det(f(x_0))) = 0$ obeys (10.18).

By analyticity of $\det(f(z))$ and the fact that it is nonvanishing, $\arg(\det(f(z)))$ is uniquely defined as a continuous function on Ω with $\lim_{\varepsilon \downarrow 0} \arg(\det(f(x_0 + i\varepsilon))) = 0$. By the above, (10.18) holds at all points z in Ω where all eigenvalues are analytic and so, by continuity and the open mapping theorem for analytic functions, all points. $\square$

Lemma 10.5. *Let $a < b$ lie in $(-1, 1)$ so that both a and b are neither a zero nor a pole of $\det(f(z))$. Let $Z(a, b), P(a, b)$ be the number of zeros, poles of $\det(f(z))$ in (a, b) counting multiplicity. Then*

$$|Z(a, b) - P(a, b)| \leq \ell \tag{10.20}$$

Proof. By the argument principle, $2\pi(Z - P)$ is the change of $\arg(\det(f(z))$ along the circle through a and b centered at $\frac{1}{2}(a + b)$. By Lemma 10.4, this is at most $2(\ell\pi)$. $\square$

Lemma 10.6. *The sets of zeros and poles (with multiplicity) of $\det(f(z))$, including the ℓ-fold zero at $z = 0$, can be written as ℓ subsets $z_j^{(k)}, p_j^{(k)}$ with $k = 1, \dots, \ell$ and $-\tilde{N}_k < j < N_k$ (with N_k and $\tilde{N}_k$ among $1, 2, \dots, \infty$) so that $z_0^{(k)} = 0$ and*

$$z_j^{(k)} < p_j^{(k)} < z_{j+1}^{(k)} \tag{10.21}$$

for all allowed values of j.

Remarks. 1. If there are infinitely many z in $(-1, 0)$ and in $(0, 1)$, then $\tilde{N}_k = N_k = \infty$ for all k. The awkwardness requiring $N_k, \tilde{N}_k$ is only needed if there are finitely many zeros.

2. To avoid notational complexity, we slightly lied if N_k or $\tilde{N}_k$ is finite. If N_k is finite, $z_j^{(k)}$ runs to $j = N_k$. $p_j^{(k)}$ can then run to either N_k or $N_k - 1$.

Proof. Construct $S_1, S_2, \dots, S_\ell$ as follows: Set $z_0^{(1)} = 0$. Let $p_0^{(1)}$ be the first pole larger than $z_0^{(1)}$, $z_1^{(1)}$ the first zero larger than $p_0^{(1)}$, $p_1^{(1)}$ the next pole, etc. This either continues indefinitely, in which case we set $N_1 = \infty$, or stops because there is no next zero or pole. Then do the same to the left of 0, that is, $p_{-1}^{(1)}$ is the first pole smaller than $z_1^{(1)}$, etc. Clearly, the points in S_1 obey (10.21). Now remove the points of S_1 (or decrease their multiplicity by 1) and repeat the construction (starting with $z_0^{(2)} = 0$) to make $S_2, S_3, \dots, S_\ell$.

We claim that after we construct ℓ S_j's, we have exhausted all the poles and zeros. Let us show this is true for $(0,1)$; the argument for $(-1,0)$ is similar (and since 0 has multiplicity ℓ, it is removed after ℓ steps).

Suppose $\tilde z$ is a zero that is left and it is closer to zero than any leftover zero or pole. If $\tilde z$ lies in some $(p_j^{(k)}, z_{j+1}^{(k)})$, $j = 0, 1, \dots$, we could have used it as $z_{j+1}^{(k)}$ so it cannot lie in any such interval. Put differently, there are only matched zeros and poles in $(0,\tilde z) \cap \cup_{j=1}^{\ell} S_j$. By the choice of $\tilde z$, there are no other poles in $(0,\tilde z)$. Thus, for small δ, the interval $(-\delta, \tilde z + \delta)$ has $\ell + 1$ extra zeros over poles, violating Lemma 10.5. So the closest leftover point is not a zero.

Suppose $\tilde p$ is a pole that is left and it is closer to zero than any other leftover zero or pole. As above, $\tilde p$ cannot lie in any $(z_j^{(k)}, p_j^{(k)})$, $j = 0, 1, \dots$, so there are only matched zeros and poles in $[0, \tilde p) \cap \cup_{j=1}^{\ell} S_j$. But then, for small δ, $(\delta, \tilde p + \delta)$ has $\ell + 1$ extra poles, violating Lemma 10.5. Thus $\cup_{j=1}^{\ell} S_j$ includes all zeros and poles. $\square$

Lemma 10.7. *The limit $B_\infty(z)$ of* (10.15) *exists and obeys conditions* (i)–(iii) *of Proposition 10.3*(c).

Proof. Renumber the $p_j^{(k)}$ into a single sequence $p_1, p_2, \dots$, so $|p_1| \le |p_2| \le \cdots$ and let z_m be the correspondingly paired $z_{j+1}^{(k)}$ (paired to the $p_j^{(k)}$ that is p). Since $\{(p_j^{(k)}, z_{j+1}^{(k)})\}_{j=1}^{N_j}$ are disjoint subsets of $(0,1)$ for each fixed k,

$$\sum_{j=1}^{\infty} |z_{j+1}^{(k)} - p_j^{(k)}| = \sum_{j=1}^{\infty} z_{j+1}^{(k)} - p_j^{(k)} < 1$$

so we see that

$$\sum_{j=1}^{\infty} |z_j - p_j| \le 2\ell$$

The existence of B_∞ then follows by Proposition 13.8.2 of [99], as do (i) and (ii).

To get (iii), we note that just taking the zeros and poles in a single S_j yields a set obeying (13.8.5) and (13.8.6) of [99]. So, by (13.8.10), that product has arg bounded by 2π. The ℓ-fold product thus obeys (10.16). $\square$

Proof of Proposition 10.3. Given Lemma 10.7, the proof is essentially that of Theorem 13.8.3 of [99].

$$g(z) \equiv \frac{\det(f(z))}{z^\ell B_\infty(z)} \tag{10.22}$$

is analytic and nonvanishing on $\mathbb{D}$ with $g(0) > 0$ (since $B_\infty(0) > 0$). Moreover, by (10.18) and (10.16),

$$|\arg g(z)| \le 4\pi\ell \tag{10.23}$$

so, by M. Riesz' theorem, $\log(g(z)) \in \cap_{p<\infty} H^p(\partial\mathbb{D})$ from which (a), (b) of the theorem are immediate and (d) follows from the Poisson representation for $\log(g(z))$ since $\log(|g(e^{i\theta})|) = \log(|\det(f(e^{i\theta}))|$. $\square$

Now we turn to block Jacobi matrices where we obtain:

Theorem 10.8 (Nonlocal Step-by-Step Sum Rule for Block Jacobi Matrices)**.** *Let $\mathcal{J}$ be a block Jacobi matrix with $\sigma_{\text{ess}}(\mathcal{J}) \subset [-2,2]$ and Jacobi parameters $\{A_n, B_n\}_{n=1}^\infty$. Let $\mathcal{J}^{(1)}$ denote this Jacobi matrix with the top row of blocks and left-most column of blocks removed. Let $m(E), m^{(1)}(E)$ be the m-functions given by* (6.25). *Let $M, M^{(1)}$ be defined on $\mathbb{D}$ by*

$$M(z) = -m(z+z^{-1}) \tag{10.24}$$

with poles at $\{p_i\}_{i=1}^N$ where $p_i + p_i^{-1}$ are eigenvalues of $\mathcal{J}$. We repeat each p_i a number of times equal to the multiplicity of the eigenvalues (*equivalently, the rank of the residue*)*. Let $\{z_i\}_{i=1}^{N-1}$ be the corresponding points for $\mathcal{J}^{(1)}$. Then*

(a) *The Blaschke product, $B_\infty(z)$, defined by the $\{z_i\} \cup \{p_i\}$ via* (10.15) *exists and obeys* (i)–(iii) *of Proposition 10.3*(c).

(b) *$M(z)$ and $M^{(1)}(z)$ have limits $M(e^{i\theta})$ and $M^{(1)}(e^{i\theta})$ as $r \uparrow 1$ for $z = re^{i\theta}$ for a.e. θ in $\partial\mathbb{D}$ and*

$$\log\left[\frac{\det(\operatorname{Im} M)(e^{i\theta})}{\det(\operatorname{Im} M^{(1)}(e^{i\theta}))}\right] \in \bigcap_{1\le p<\infty} L^p\left(\partial\mathbb{D}, \frac{d\theta}{2\pi}\right) \tag{10.25}$$

(c)

$$\det\left(\frac{|A_1| M(z)}{z}\right) = B_\infty(z) \exp\left(\int \frac{e^{i\theta}+z}{e^{i\theta}-z} \log\left[\frac{\det(\operatorname{Im} M(e^{i\theta}))}{\det(\operatorname{Im} M^{(1)}(e^{i\theta}))}\right]\frac{d\theta}{4\pi}\right) \tag{10.26}$$

Remark. As in the case $\ell = 1$, it can happen (although not in examples where sum rules are finite) that $\det(\operatorname{Im} M(e^{i\theta})) = \det(\operatorname{Im} M^{(1)}(e^{i\theta})) = 0$ for θ in a set of positive measure. (b) and (c) are shorthand for the more precise

(i) For a.e. θ, $\det(\operatorname{Im} M(e^{i\theta})) = 0$ if and only if $\det(\operatorname{Im} M^{(1)}(e^{i\theta})) = 0$.

(ii) There is an a.e. positive function $g(\theta)$ on $\partial\mathbb{D}$, equal to $\det(\operatorname{Im} M(e^{i\theta}))/\det(\operatorname{Im} M^{(1)}(e^{i\theta}))$ when the ratio is not 0/0 so that (10.25) and (10.26) hold if the formal ratio is replaced by $g(\theta)$.

Proof. Given Proposition 10.3, this is essentially identical to the proof of Theorem 13.8.4 of [99] with care given to matrix issues. We begin by noting that (6.31) for $n \to n+1$ first implies near $z = 0$

$$M^{(n+1)}(z)^{-1} = z^{-1} + O(1) \tag{10.27}$$

and then by (6.25) that

$$\left(\frac{M^{(n)}(z)}{z}\right)^{-1} = 1 - B_{n+1}z - (A_{n+1}^* A_{n+1} - 1)z^2 + O(z^3) \tag{10.28}$$

Since $M^{(n)}(z)/z$ is near 1 for z small, we can compute its determinant using

$$\det(C) = \exp(\operatorname{Tr}(\log(C))) \tag{10.29}$$

which holds if $\|C - 1\| < 1$. Thus

$$\begin{aligned} &\log\det\left(\frac{M^{(n)}(z)}{z}\right) \\ &\quad = \operatorname{Tr}(B_{n+1})z + [\operatorname{Tr}\{([A_{n+1}^* A_{n+1} - 1] + \tfrac12 B_{n+1}^2)\}]z^2 + O(z^3) \end{aligned} \tag{10.30}$$

In addition, (6.31) implies

$$\operatorname{Im}[M(z)^{-1}] = \operatorname{Im}(z+z^{-1}) - A_1 \operatorname{Im} M_1(z) A_1^* \tag{10.31}$$

so at points where $M(z)$ has radial limits (a.e. θ, see below),

$$-[M(e^{i\theta})^*]^{-1}\operatorname{Im} M(e^{i\theta})[M(e^{i\theta})]^{-1} = -A_1 \operatorname{Im} M(e^{i\theta}) A_1^* \tag{10.32}$$

which, using (on account of $\det(|C|)^2 = \det(C^*)\det(C)$)

$$|\det(A_1)| = \det(|A_1|)$$

yields

$$|\det(|A_1| M(e^{i\theta}))|^2 = \frac{\det(\operatorname{Im} M(e^{i\theta}))}{\det(\operatorname{Im} M_1(e^{i\theta}))} \tag{10.33}$$

We now apply Proposition 10.3 to $M(z)$ which obeys (10.13) (since $\operatorname{Im}(z+z^{-1}) < 0$ on $\mathbb{D}$ and (10.24) has a minus sign) and (10.14) by (10.27).

By Theorem 6.7, our $B_\infty(z)$ here (after perhaps canceling some zeros and poles) is the $B_\infty(z)$ of Proposition 10.3. (a) and (b) immediately follow from Proposition 10.3. We get (10.26) from (10.17) by using (10.33) (noting (10.26) has a $1/4\pi$ while (10.17) a $1/2\pi$ on account of the square on the left side of (10.33)). We also use that if c is a positive constant,

$$\exp\biggl(\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\log(c^2)\frac{d\theta}{4\pi}\biggr) = c \tag{10.34}$$

$\square$

As in [96], we can get step-by-step P_2 (originally in [59]), C_0, C_1 (originally in [103]) sum rules immediately from Taylor expansion of the log of (10.26). We let $\beta_j(\mathcal{J})$ be the numbers in $(-1,1)\setminus\{0\}$ for which $E_j \equiv \beta_j + \beta_j^{-1}$ are eigenvalues of $\mathcal{J}$ counting multiplicities.

Theorem 10.9 (C_0, C_1, P_2 Step-by-Step Sum Rules)**.**
(i)

$$\begin{aligned}\frac{1}{4\pi}\int_0^{2\pi} &\log\biggl(\frac{\det(\operatorname{Im} M^{(1)}(e^{i\theta}))}{\det(\operatorname{Im} M(e^{i\theta}))}\biggr)d\theta \\ &- \sum_j \log(|\beta_j(\mathcal{J})|) - \log(|\beta_j(\mathcal{J}^{(1)})|) = -\log(\det|A_1|)\end{aligned} \tag{10.35}$$

(ii)

$$\begin{aligned}-\frac{1}{2\pi}\int_0^{2\pi} &\log\biggl(\frac{\det(\operatorname{Im} M^{(1)}(e^{i\theta}))}{\det(\operatorname{Im} M(e^{i\theta}))}\biggr)\cos\theta \\ &+ \sum_j [\beta_j(\mathcal{J}) - (\beta_j^{-1}(\mathcal{J})^{-1})] - [\beta_j(\mathcal{J}^{(1)}) - (\beta_j(\mathcal{J}^{(1)}))^{-1}] \equiv \operatorname{Tr}(B_1)\end{aligned} \tag{10.36}$$

(iii)

$$\begin{aligned}\frac{1}{4\pi}\int &\log\biggl(\frac{\det(\operatorname{Im} M^{(1)}(e^{i\theta}))}{\det(\operatorname{Im} M(e^{i\theta}))}\biggr)\sin^2(\theta)\phi(\theta) \\ &+ \sum F(E_j(\mathcal{J})) - F(E_j(\mathcal{J}^{(1)})) = \operatorname{Tr}\tfrac14\bigl(B_1^2 + \tfrac12 G(|A_1|)\bigr)\end{aligned} \tag{10.37}$$

where F is given by (10.2) *and G by* (10.3).

Remark. The $E_j(\mathcal{J})$ in $(-\infty,-2)$ and $(2,\infty)$ and $E_j(\mathcal{J}^{(1)})$ interlace in the $\ell=1$ case. In the general ℓ case, we have at most ℓ fewer eigenvalues of $\mathcal{J}^{(1)}$ on any $(-\infty,-E_0)$ or (E_0,∞) so, as in Lemma 10.6, one can decompose into ℓ interlacing subsets. This and the monotonicity of functions like F show the eigenvalue sums in (10.35)–(10.37) are conditionally convergent. Similarly, the integrals are always convergent.

Proof. Apply log to both sides of (10.26) and take Taylor coefficients. The constant term is (10.35) and the first derivative is (10.36). If $L(z)$ is the log of the left side and $R(z)$ of the right, then

$$L(0)+\tfrac{1}{2}\,L''(0)=R(0)+\tfrac{1}{2}\,R''(0)$$

is (10.37). $\square$

The proofs of Theorems 10.1–10.2 are now identical to those of the scalar case; see, for example, the discussion of Theorems 13.8.6 and 13.8.8 of [99]. In particular, $Z(\mathcal{J})$ and $Q(\mathcal{J})$ (the integral on the right of (10.4)) are negatives of relative entropies, and so, lower semi-continuous.

11. Szegő and Killip–Simon Theorems When All Gaps Are Open

Our goal here is to prove Theorems 1.3 and 1.4. Our strategy, of course, will be to translate Theorems 10.1 and 10.2 for $\Delta(J)$ to statements about J. Firstly, we need to relate the a.c. part of the matrix measure for $\Delta(J)$ to the a.c. part of the (scalar) measure for J. And secondly, to relate ℓ^2 norms of coefficients of $\Delta(J)$ to the distance of J's Jacobi parameters to the isospectral torus. We begin with the first question. Thus we take

$$d\rho_J(x)=\omega(x)\,dx+d\mu_{J,\mathrm{s}}(x) \tag{11.1}$$

with $d\mu_{J,\mathrm{s}}$ singular and ω supported precisely on $\sigma_{\mathrm{ess}}(J_0)$. By this assumption and the spectral mapping theorem, $\Delta(J)$ has a.c. spectrum precisely on $[-2,2]$ so the matrix measure for $\Delta(J)$ has the form

$$d\rho_{\Delta(J)}(E)=W(E)\,dE+d\rho_{\Delta(J),\mathrm{s}}(E) \tag{11.2}$$

Proposition 11.1. *Let J_0 be a periodic Jacobi matrix with period p and J a Jacobi matrix with Jacobi parameters $\{a_n,b_n\}_{n=1}^\infty$ and measure $d\rho_J$ of the form* (11.1) *with ω supported on $\sigma_{\mathrm{ess}}(J_0)$. Let Δ be the discriminant for J_0 and $W(E)$ the a.c. part of the $p\times p$ matrix-valued measure $d\rho_{\Delta(J)}$ associated to $\Delta(J)$ (so W is a $p\times p$ matrix). Then for $E\in(-2,2)$ and $\Delta^{-1}(E)=\{x_1,\dots,x_p\}$,*

$$\det(W(E))=\biggl(\prod_{j=1}^p a_j^{2p-2j}\bigl[a_j^{(0)}\bigr]^p\biggr)\biggl(\prod_{j=1}^p\omega(x_j)\biggr) \tag{11.3}$$

Proof. In the block Jacobi form, $d\rho_{\Delta(J)}$ has jk matrix element equal to the spectral measure of the operator $\Delta(J)$ associated to δ_j,δ_k, that is, $\int F(x)(d\rho_\Delta(x))_{jk}=\langle\delta_j,F(\Delta(E))\delta_k\rangle$. But $\delta_j=p_{j-1}(J)\delta_1$. It follows that

$$W_{kj}(E)=\sum_{\ell=1}^p\omega(x_\ell)\bigl(|\Delta'(x_\ell)|\bigr)^{-1}p_{k-1}(x_\ell)p_{j-1}(x_\ell) \tag{11.4}$$

Note that the factors of $1/\Delta'$ arise from the Jacobian $\frac{dE}{dx} = \Delta'(x)$. We can re-write (11.4) as $W_{kj}(E) = (MAM^t)_{kj}$ where A is the diagonal matrix

$$A_{\ell m} = \delta_{\ell m}\omega(x_\ell)\big(|\Delta'(x_\ell)|\big)^{-1} \tag{11.5}$$

and M is the matrix

$$M_{k\ell} = p_{k-1}(x_\ell) \qquad k = 1,\dots,p;\ \ell = 1,\dots,p \tag{11.6}$$

Next we compute $\det(M)$; $\det(A)$ is easy. Note that

$$p_{k-1}(x_\ell) = \bigg(\prod_{j=1}^{k-1} a_j\bigg)^{-1} x_\ell^{k-1} + \text{lower order} \tag{11.7}$$

Moreover, inductively one sees that the lower order terms can be neglected in the determinant—they can be removed by subtracting a multiple of rows above (i.e., smaller values of k). Thus,

$$\det(M) = \bigg[\prod_{k=1}^{p}\bigg(\prod_{j=1}^{k-1} a_j\bigg)^{-1}\bigg]\det(x_\ell^{k-1}) = \bigg(\prod_{j=1}^{p} a_j^{p-j}\bigg)\bigg(\prod_{j>k}(x_j - x_k)\bigg) \tag{11.8}$$

by the well-known formula for Vandermonde determinants. This can be simplified further. The points x_j are precisely the zeros of the polynomial $\Delta(x) - E$; hence, invoking (2.14),

$$\Delta(x) - E = \bigg(\prod_{j=1}^{p} a_j^{(0)}\bigg)^{-1}\bigg(\prod_{k=1}^{p}(x - x_k)\bigg)$$

In this way we discover that

$$\det(M)^2 = \bigg(\prod_{j=1}^{p} a_j^{2p-2j}\,\big[a_j^{(0)}\big]^p\bigg)\bigg(\prod_{k=1}^{p}|\Delta'(x_k)|\bigg) \tag{11.9}$$

Multiplying this by $\det(A)$ gives (11.3). □

Corollary 11.2. *If J_0 has all gaps open and $\alpha > -1$, then*

$$\int_{-2}^{2}(4 - E^2)^\alpha|\log\det(W(E))|\,dE < \infty \tag{11.10}$$

if and only if

$$\int_{\sigma_{\text{ess}}(J_0)} \operatorname{dist}(x, \mathbb{R}\setminus\sigma_{\text{ess}}(J_0))^\alpha|\log\omega(x)|\,dx < \infty \tag{11.11}$$

When $\alpha = -\frac{1}{2}$, the same conclusion holds even if some gaps are closed.

Remark. Since $\alpha > -1$, $(4 - E^2)^\alpha$ (resp., $\operatorname{dist}(\dots)^\alpha$) are in L^p for some $p > 1$, so the $\log_+(\quad)$ is always integrable and these conditions are equivalent to the integral without $|\cdot|$ being larger than $-\infty$.

Proof. Changing variables via $E = \Delta(x)$ and applying Proposition 11.1 shows that (11.10) holds if and only if

$$\int_{\sigma_{\text{ess}}(J_0)} |\log\omega(x)|\,(4 - \Delta(x)^2)^\alpha\,|\Delta'(x)|\,dx < \infty \tag{11.12}$$

If all gaps are open then $|\Delta'(x)|$ is strictly positive on $\sigma_{\text{ess}}(J_0)$, while $4 - \Delta(x)^2$ is a polynomial with a simple zero at each band edge (and no others). This proves the first claim.

At a closed gap, $4-\Delta(x)^2$ has a double zero and $\Delta'(x)$ a simple zero. When $\alpha=-\frac{1}{2}$, these cancel exactly. $\square$

Next we turn to the ℓ^2 issue. Given any two-sided periodic matrix $\tilde{J}$ with Jacobi parameters $\{a_n,b_n\}_{n=1}^p$ and fixed periodic J_0, let $B_{J_0}(\tilde{J}), A_{J_0}(\tilde{J})$ be the constant $p\times p$ blocks in $\Delta_{J_0}(\tilde{J})$. We are heading towards showing that $\|B_{J_0}(\tilde{J})\|_2^2+\|A_{J_0}(\tilde{J})-\mathbf{1}\|_2^2$ is comparable to $\operatorname{dist}((a_n,b_n)_{n=1}^p,\mathcal{T}_{J_0})$. This will be the key to showing ℓ^2 tails in the matrix pieces of $\Delta(J)-S^p-S^{-p}$ for general J is equivalent to (1.28).

The crucial fact will be that the polynomial coefficients of $\Delta_{J_0}-\Delta_{\tilde{J}}$ are comparable to $\operatorname{dist}((a_n,b_n)_{n=1}^p,\mathcal{T}_{J_0})$. For this we need the following, which is a simple application of the implicit function theorem and compactness:

Lemma 11.3. *Let F be a C^∞ map of an open set $U\subset\mathbb{R}^n$ to $\mathbb{R}^\ell$ with $\ell<n$. Suppose $\mathcal{T}=F^{-1}(y_0)$ is a smooth manifold of dimension $n-\ell$ and compact for some $y_0\in\mathbb{R}^\ell$, and*

$$\operatorname{rank}((\nabla F)(x_0))=\ell \tag{11.13}$$

for all $x_0\in\mathcal{T}$. Then for any compact neighborhood, K, of $\mathcal{T}$, there are $c_k,d_k\in(0,\infty)$ so for all $x\in K$,

$$c_k|F(x)-y_0|\le\operatorname{dist}(x,\mathcal{T})\le d_k|F(x)-y_0| \tag{11.14}$$

One can restate (11.13) in a more illuminated way in terms of the components $F_1,\dots,F_\ell$ of $\mathcal{T}$. Of course, $\nabla F_j(x_0)$ is orthogonal to $\mathcal{T}$ at x_0. The condition (11.13) is equivalent to saying that $\{\nabla_j F(x_0)\}_{j=1}^\ell$ span the normal bundle to $\mathcal{T}$. This is equivalent to saying they are linearly independent. Notice that if J_0 has all gaps open, $\mathcal{T}_{J_0}$ is of dimension $p-1=2p-(p+1)$ and Δ_{J_0} is a polynomial of degree p, hence with $p+1$ coefficients. Thus the following shows we can use Lemma 11.3:

Theorem 11.4. *Suppose all gaps are open for some periodic J_0. Then at any point in $\mathcal{T}_{J_0}$, the gradients of the derivatives of the coefficients of Δ_J span the normal bundle of $\mathcal{T}_{J_0}$ in $\mathbb{R}^{2p}$.*

Proof. Δ_{J_0} has the form

$$\Delta_{J_0}(x)=(a_1\dots a_p)^{-1}\prod_{j=1}^p(x-\lambda_j)=\sum_{j=0}^p c_jx^j$$

where λ_j are the roots. The coefficients thus obey

$$c_p^{-1}=a_1\dots a_p \tag{11.15}$$

$$c_\ell c_p^{-1}=\sum_{1\le k_1\le\dots\le k_{p-\ell}\le p}\lambda_{k_1}\dots\lambda_{k_{1-\ell}}\equiv s_{k-\ell}\qquad \ell<p \tag{11.16}$$

It is well known that if

$$t_\ell=\sum_{j=1}^p\lambda_j^\ell \tag{11.17}$$

then t_ℓ is ℓs_ℓ plus a polynomial in $\{s_j\}_{j=1}^{\ell-1}$, so $\{\nabla t_j\}_{j=1}^\ell$ and $\{\nabla s_j\}_{j=1}^\ell$ span the same space. It follows that we need only show the gradients of c_p^{-1} and t_ℓ span the normal bundle of $\mathcal{T}_{J_0}$.

Let

$$\mathcal{M}_0=\{(a_n,b_n)_{\ell=1}^p : a_1a_2\dots a_p=a_1^{(0)}a_2^{(0)}\dots a_p^{(0)};\ b_1+\dots+b_p=b_1^{(0)}+\dots+b_p^{(0)}\} \tag{11.18}$$

We know $\mathcal{M}_0 \supset \mathcal{T}_{J_0}$. Clearly, ∇c_p^{-1} and ∇t_1 span the normal bundle to $\mathcal{M}_0$ since $t_1 = \sum_{j=1}^p b_j$ (see (2.14)). Thus we need only show the projections of $\{\nabla t_\ell\}_{\ell=2}^p$ into the tangent space of $\mathcal{M}_0$ span the normal bundle of $\mathcal{T}_{J_0}$ in $\mathcal{M}_0$.

Studies of the Toda flows show that $\mathcal{M}_0$ is a symplectic manifold with $\{t_\ell\}_{\ell=2}^p$ Poisson commuting. Since the symplectic form on $\mathcal{M}_0$ is nondegenerate, to say $\{\nabla t_j\}_{j=2}^p$ span the normal bundle is the same as saying that the Hamiltonian flows generated by $\{t_j\}_{j=2}^p$ span the tangent bundle of $\mathcal{T}_{J_0}$, or equivalently, given $\dim(\mathcal{T}_{J_0}) = p-1$, that these Hamiltonian flows are independent.

This independence is a theorem of van Moerbeke [108, Theorem 5.2] or [102]. $\square$

Lemma 11.5. *For any compact subset, K, of period p Jacobi matrices, there exist constants c_K and d_K in $(0,\infty)$ so for all $J \in K$,*

$$c_K \biggl\| \sum_{\ell=0}^p \alpha_\ell J^\ell \chi_p \biggr\|_2 \leq \sum_{\ell=0}^p |\alpha_\ell|^2 \leq d_K \biggl\| \sum_{\ell=0}^p \alpha_\ell J^\ell \chi_p \biggr\|_2 \tag{11.19}$$

where χ_p is the projection onto the p-dimensional space spanned by $\{\delta_j\}_{j=1}^p$.

Proof. $\{J^\ell \chi_p\}_{\ell=0}^p$ are independent since J^ℓ has strictly positive elements in the ℓ-th diagonal and $\{J^k\}_{k<\ell}$ only has zero elements there. Hence, the matrix

$$\operatorname{Tr}(\chi_j J^\ell J^k \chi_p)\big|_{\ell,k=1,\dots,p} \tag{11.20}$$

is strictly positive so (11.19) holds for each fixed J. The optimal constants are clearly continuous so uniformly bounded above and below on K. $\square$

Proposition 11.6. *Let J_0 be a periodic Jacobi matrix with all gaps open. For any compact neighborhood K of $\mathcal{T}_{J_0}$ in $(0,\infty)^p \times \mathbb{R}^p$, there are constants c_K and d_K in $(0,\infty)$ so that for all $J \in K$,*

$$\begin{aligned} c_K(\|A_{J_0}(J) - \mathbf{1}\|^2 + \|B_{J_0}(J)\|^2)^{1/2} &\leq \operatorname{dist}(J, \mathcal{T}_{J_0}) \\ &\leq d_K(\|A_{J_0}(J) - \mathbf{1}\|^2 + \|B_{J_0}(J)\|^2)^{1/2} \end{aligned}$$

Proof. We have that

$$2\|A_{J_0}(J) - 1\|^2 + \|B_{J_0}(1)\|^2 = \|[\Delta_{J_0}(J) - (S^p + S^{-p})]\chi_p\|^2 \tag{11.21}$$

But by the magic formula,

$$\Delta_J(J) = S^p + S^{-p} \tag{11.22}$$

so

$$[\Delta_{J_0}(J) - (S^p + S^{-p})]\chi_p = \sum_{\ell=0}^p c_\ell J^\ell \chi_p \tag{11.23}$$

where c_ℓ is the difference of coefficients for J and J_0. By Lemma 11.5,

$$\text{RHS of (11.21)} \sim \sum_{\ell=0}^p |c_\ell|^2 \tag{11.24}$$

where $\sim$ means the ratio is bounded above and away from zero on compact subsets.

By Lemma 11.3 and Theorem 11.4,

$$\sum_{\ell=0}^p |c_\ell|^2 \sim \operatorname{dist}(J, \mathcal{T}_{J_0})^2 \tag{11.25}$$

Combining this with (11.24) proves the proposition. $\square$

Now we take a general J not periodic and form $\Delta_{J_0}(J)$ which is a one-sided block Jacobi matrix with block elements $A_{n,J_0}(J), B_{n,J_0}(J)$.

Lemma 11.7. *$\Delta_{J_0}(J)_{k\ell}$ for $k \leq \ell$ depends only on $\{b_j\}_{j=k-\alpha}^{\ell+\alpha}$ and $\{a_j\}_{j=k-\alpha}^{\ell+\alpha-1}$ where $\alpha = \lfloor \frac{1}{2}(p-(\ell-k)) \rfloor$ is the greatest integer less than or equal to $\frac{1}{2}[p-(\ell-k)]$.*

Proof. Each factor of J changes index by at most one. In order to get from k to ℓ, $\ell - k$ steps are needed. The remainder cannot go below $\ell - \alpha$ or above $k + \alpha$ and get back to k in p steps. □

Lemma 11.8. *Let J have Jacobi parameters $\{a_n, b_n\}_{n=1}^{\infty}$. Let $\tilde{J}$ be periodic with period p and suppose $b_n = \tilde{b}_n$ for $kp - p \leq n \leq kp + 2p$ and $a_n = \tilde{a}_n$ for $kp - p \leq n \leq kp + 2p - 1$. Then*

$$A_{k,J_0}(J) = A_{J_0}(\tilde{J}) \qquad B_{k,J_0}(J) = B_{J_0}(\tilde{J})$$

Proof. Immediate from Lemma 11.7. □

Lemma 11.9. *Let $k \leq \ell$ and $\alpha = [\frac{1}{2}(p - (\ell - k))]$. For any two J and $\tilde{J}$ and any K, there is C_K so that*

$$|\Delta_{J_0}(J)_{k\ell} - \Delta_{J_0}(\tilde{J})_{k\ell}| \leq C_K \sup_{k-\alpha \leq j \leq \ell+\alpha} [|b_j - \tilde{b}_j| + |a_j - \tilde{a}_j|] \tag{11.26}$$

so long as

$$\sup[|b_j| + |\tilde{b}_j| + |a_j| + |\tilde{a}_j|] \leq K \tag{11.27}$$

Proof. Immediate from Lemma 11.7 and the fact that Δ_{J_0} has matrix elements that are fixed (given J_0) polynomials in a's and b's. □

Lemma 11.10. *If*

$$\sum_{k=1}^{\infty} |A_{k,J_0}(J) - \mathbf{1}|^2 + |B_{k,J_0}(J)|^2 < \infty \tag{11.28}$$

then

$$\sum_{k=1}^{\infty} |A_{k+1,J_0}(J) - A_{k,J_0}(J)|^2 + |B_{k+1,J_0}(J) - B_{k,J_0}(J)|^2 < \infty \tag{11.29}$$

Proof. The difference of two ℓ^2 vectors is in ℓ^2. □

Our next preliminary is to relate $A \in \mathcal{L}$ to

$$|A| = \sqrt{A^* A} \tag{11.30}$$

Proposition 11.11. *The map $A \mapsto |A|$ from $\mathcal{L}$ to positive definite matrices is a diffeomorphism. In particular, for A's in $\mathcal{L}$ with $\|A - 1\| < \frac{1}{2}$, there exist constants C_1 and C_2 so that*

$$C_1 \|A - 1\|_2 \leq \| \, |A| - 1\|_2 \leq C_2 \|A - 1\|_2 \tag{11.31}$$

Proof. By (11.30), $A \mapsto |A|$ is a smooth map.

The inverse map (strictly $|A|^2 \mapsto A$) is known as the Cholesky factorization; see [50, 109]. Given $B > 0$, apply the Gram–Schmidt procedure to the (linearly independent) columns of B working from right to left. This gives a factorization $B = QA$ with Q unitary and $A \in \mathcal{L}$. Note that $|A| = B$ and that because the columns of B are linearly dependent, $B \mapsto A$ is also a smooth map. □

Theorem 11.12. *Let J_0 be a two-sided p-periodic Jacobi matrix with all gaps open and let Δ_{J_0} denote its discriminant. For a Jacobi matrix with parameters (a_n, b_n), the following are equivalent:*

(i) $\Delta_{J_0}(J) - S^p - S^{-p}$ *is a Hilbert–Schmidt operator on* $\ell^2(\{0,1,2,\dots\})$.

(ii) $\sum_n \operatorname{Tr}\{B_n^2 + |A_n - 1|^2\} < \infty$.

(iii) $\sum_n \operatorname{Tr}\{B_n^2 + (|A_n| - 1)^2\} < \infty$.

(iv) $\sum_n \operatorname{Tr}\{B_n^2 + G(|A_n|)\} < \infty$.

(v) $\sum_m d_m((a,b), \mathcal{T}_{J_0})^2 < \infty$.

(vi) $\sum_m \tilde{d}_m((a,b), \mathcal{T}_{J_0})^2 < \infty$.

Here we have adopted the abbreviations $A_n := A_{n,J_0}(J)$ and $B_n := B_{n,J_0}(J)$

Proof. (i)⇔(ii) amounts to the definition of the Hilbert–Schmidt norm.

(ii)⇔(iii) follows from Proposition 11.11.

(iii)⇔(iv) Notice that G, defined in (10.3), obeys

$$c_\varepsilon (x-1)^2 \le G(x) \le c'_\varepsilon (x-1)^2 \qquad \forall\, x \in (\varepsilon, \varepsilon^{-1})$$

Applying this to the eigenvalues of $|A_n|$ yields this equivalence.

(v)⇔(vi) is the $q = 2$ case of Proposition 3.5.

(vi)⇒(i) By Lemma 11.7, each matrix element of $\Delta_{J_0}(J) - S^p - S^{-p}$ is a smooth function of p consecutive pairs (a_n, b_n); moreover, by the magic formula, all of these smooth functions vanish if the corresponding p-tuple belongs to $\mathcal{T}_{J_0}$. The implication now follows from the fact that smooth functions are Lipschitz.

(ii)⇒(v) Define $J^{(k)}$ to be the p-periodic Jacobi matrix that equals J on block k, that is,

$$b_\ell^{(k)} = b_{kp+\ell} \qquad a_\ell^{(k)} = a_{kp+\ell} \tag{11.32}$$

for $\ell = 1, 2, \dots, p$. Obviously, $J^{(k)} = J$ on block k and, by (11.29), the difference on blocks $k-1$ and $k+1$ are in ℓ^2, that is,

$$\sum_k \Big[\sup_{(k-1)p \le j \le (k+2)p-1} |b_j - b_j^{(k)}| + |a_j - a_j^{(k)}| \Big]^2 < \infty \tag{11.33}$$

This implies (v) via Lemma 11.9. □

Proof of Theorem 1.4. We will refer to the three statements (i)–(iii) of Theorem 1.4 simply by their numbers.

Suppose first that

$$d_m\big((a,b), \mathcal{T}_{J_0}\big) \in \ell^2 \tag{11.34}$$

Then (i) holds by Theorem 1.1. Moreover, by Theorem 11.12 and the hypothesis that all gaps are open, the RHS of (10.4) is finite. Therefore, the LHS is finite. Next we use this fact to prove (ii) and (iii).

As Δ' is nonvanishing at all band edges,

$$\sum_j F(\Delta(E_j)) < \infty \iff \sum_{j=1}^N \operatorname{dist}\big(E_j, \sigma_{\mathrm{ess}}(J)\big)^{3/2} < \infty \tag{11.35}$$

which verifies (iii). By Corollary 11.2,

$$\text{Leftmost term in (10.4)} < \infty \iff \text{(ii) holds} \tag{11.36}$$

This completes the proof of (i)–(iii).

Conversely, if (i)–(iii) hold, then by (11.35), (11.36), and (10.4), we see that the RHS of (10.4) is finite. By Theorem 11.12, this implies (11.34). □

Proof of Theorem 1.3. Let β_j be the β's associated to $\Delta(J)$, that is, $|\beta_j| > 1$, $\beta_j + \beta_j^{-1} = E_j$ with E_j the eigenvalues of $\Delta(J)$ in $\mathbb{R}\setminus[-2,2]$. Then $\log|\beta_j| \sim |\beta_j| - 1$ as $\beta \to \pm 1$ small and $|\beta_j| - 1 \sim (|E_j| - 2)^{1/2}$. Therefore,

$$(10.6) < \infty \iff \sum_j (|E_j| - 2)^{1/2} \iff (1.24) \tag{11.37}$$

By Corollary 11.2,

$$(10.5) < \infty \iff (1.25) \tag{11.38}$$

Finally, if $\{A_n, B_n\}_{n=1}^\infty$ are the $p \times p$ blocks in $\Delta(J)$, we have $A_n = U|A_n|$ for some U with $|\det(U)| = 1$, so

$$\det|A_n| = |\det(A_n)| = \prod_{j=(n-1)p+1}^{np} \prod_{k=j}^{j+p=1} \left[\frac{a_k}{a_k^{(0)}}\right] \tag{11.39}$$

by (3.3) and (2.14). Thus

$$\sum_{n=1}^{N} \log(\det(|A_n|)) - p \sum_{k=1}^{Np} \log\left[\frac{a_k}{a_k^{(0)}}\right]$$

is bounded. Thus (1.27) is equivalent to (10.11).

By Theorem 10.2, we see that when (1.24) holds, then (1.25) ⇔ (1.27), and if they hold, (1.26) has a limit.

Moreover, if they hold, the hypotheses of Theorem 1.4 hold, so (1.28) is true. That (1.29) holds is a theorem of Peherstorfer–Yuditskii [84], but see the remark below. □

Remark. Using Theorem 6.8, we can provide a proof that $\{a_n\}_{n=1}^\infty$ has a periodic limit (of necessity, part of at least one element of $\mathcal{T}_{J_0}$) without using the machinery of [84], but we have not succeeded in showing $\{b_n\}_{n=1}^\infty$ has a limit in this manner! Moreover, because of the current limitation of Theorem 6.8, we are restricted to requiring that $\{E_j : E_j \in \sigma(J_0)\}$ is finite. The proof is the following. Since $\mathcal{L}$ is a group, $A_1 \dots A_n$ is a lower triangular, so given via an adjoint of a QR algorithm. Let $\tilde{A}_j$ be the equivalent type 2. Our A_j are type 3. By the continuity of the QR algorithm, $A_1 \dots A_n$ has a limit since $\tilde{A}_1 \dots \tilde{A}_n$ does. Looking at the first two diagonal elements, this implies that $\prod_{n=1}^{mp} \frac{a_j}{a_j^{(0)}}$ and $\prod_{n=2}^{mp+1} \frac{a_j}{a_j^{(0)}}$ have nonzero limits. But their ratio is $\frac{a_{mp+1}}{a_1}$ so a_{mp+1} has a limit. Similarly, $a_{mp+\ell}$ has a limit.

12. Szegő and Killip–Simon Theorems When Some Gaps Are Closed

Here we want to examine what might replace Theorems 1.3 and 1.4 if J_0 is periodic but with some closed gaps. The Szegő-type theorem is almost the same as Theorem 1.3:

Theorem 12.1. *Let J_0 be any two-sided periodic Jacobi matrix with Jacobi parameters $\{a_n^{(0)}, b_n^{(0)}\}_{n=-\infty}^\infty$, and J a one-sided Jacobi matrix with Jacobi parameters*

$\{a_n, b_n\}_{n=1}^{\infty}$ and spectral measure $d\rho$. Suppose that (1.22) holds, and that

$$\sum_{m=1}^{N} \operatorname{dist}(E_m, \sigma_{\mathrm{ess}}(J))^{1/2} < \infty \tag{12.1}$$

if $\{E_m\}_{m=1}^{N}$ is a labelling of the eigenvalues of J outside $\sigma_{\mathrm{ess}}(J)$. Then

$$-\int_{\sigma_{\mathrm{ess}}(J_0)} \log\biggl(\frac{d\rho_{\mathrm{ac}}}{dx}\biggr) \operatorname{dist}(x, \mathbb{R} \setminus \sigma_{\mathrm{ess}}(J_0))^{-1/2}\, dx < \infty \tag{12.2}$$

implies

$$\lim \biggl(\sum_{j=1}^{pN} \log\biggl(\frac{a_j}{a_j^{(0)}}\biggr)\biggr) \tag{12.3}$$

exists and lies in $(-\infty, \infty)$. Conversely, (12.2) holds so long as

$$\limsup \biggl(\sum_{j=1}^{N} \log\biggl(\frac{a_j}{a_j^{(0)}}\biggr)\biggr) > -\infty \tag{12.4}$$

and then the limit in (12.3) exists and lies in $(-\infty, \infty)$.

Moreover, if (12.2) or (12.4) holds, then there is $J_1 \in \mathcal{T}_{J_0}$, so

$$d_m(J, J_1) \to 0 \tag{12.5}$$

Remark. All that is missing is (1.28) which we do not claim. However, since (12.1)/(12.2) imply (i)–(iii) of Theorem 12.3 below, we have (12.7).

Proof. As noted, even with closed gaps, (1.25) is equivalent to (11.10) (see Corollary 11.2). Once one notes this, the proof of Theorem 1.3 provides all the results stated as Theorem 12.1. □

Theorem 1.4 used open gaps in two ways. First, in the translation of a matrix pseudo-Szegő condition, (11.10) with $\alpha = \frac{1}{2}$ to the original spectral measure of J, and second, translating a Hilbert–Schmidt bound on $\Delta(J) - S^p - S^{-p}$ to ℓ^2 approach to the isospectral torus. The second issue can be finessed if we leave things as a Hilbert–Schmidt condition, which reduces to a sum of translates of an explicit positive polynomial in the a_n's and b_n's being finite. As for translating (11.10) with $\alpha = \frac{1}{2}$, the argument that proved Corollary 11.2 translates immediately to

Lemma 12.2. *Suppose $\sigma(J_0)$ has closed gaps at $\{y_j\}_{j=1}^{\ell} \subset \sigma(J_0)$. Then (11.10) holds with $\alpha = \frac{1}{2}$ if and only if*

$$\int_{\sigma_{\mathrm{ess}}(J_0)} \operatorname{dist}(x, \mathbb{R} \setminus \sigma(J_0))^{1/2} \prod_{j=1}^{\ell} |x - y_j|^2 |\log(\omega(x))|\, dx < \infty \tag{12.6}$$

Plugging this into our proof of Theorem 1.4 immediately yields

Theorem 12.3. *Let J_0 be a two-sided periodic Jacobi matrix with closed gaps at $\{y_j\}_{j=1}^{\ell} \subset \sigma(J_0)$ and let J be a Jacobi matrix. Then*

$$\operatorname{Tr}((\Delta_{J_0}(J) - S^p - S^{-p})^2) < \infty \tag{12.7}$$

if and only if

(i) (1.22) *holds.*

(ii) (1.24) *holds with $\frac{1}{2}$ replaced by $\frac{3}{2}$.*

(iii) (12.6) *holds.*

Example 12.4. Take J_0 to be the two-sided free Jacobi matrix but think of it as period 2. Then

$$\Delta_J(x) = x^2 - 2$$

and a direct calculation of $J^2 - S^2 - S^{-2}$ shows that (12.7) is equivalent to the three conditions

$$\sum_n (a_n^2 + b_n^2 + a_{n+1}^2 - 2)^2 < \infty \tag{12.8}$$

$$\sum_n (a_{n+1}(b_n + b_{n+1}))^2 < \infty \tag{12.9}$$

$$\sum_n (a_n a_{n+1} - 1)^2 < \infty \tag{12.10}$$

If $b_n = 0$ and

$$a_n = 1 + (-1)^n (n+1)^{-\beta} \tag{12.11}$$

then (12.8)–(12.10) hold if and only if $\beta > \frac{1}{4}$ while, of course, (1.13) requires $\beta > \frac{1}{2}$. This is one of many known extensions of the (J_0-free) Killip–Simon theorem (see, e.g., Laptev et al. [65], Kupin [64], and Nazarov et al. [78]). Some of these results have MOPRL analogs which, via the magic formula, lead to variants of Theorems 1.4 and 12.3.

13. Eigenvalue Bounds for MOPRL

There are Birman–Schwinger kernels for MOPRL and it should be possible to extend the proofs of most bounds on the number of eigenvalues outside $[-2, 2]$ or on moments of $|E_j| - 2$ from the scalar to matrix case with optimal constants. But if one is willing to settle for less than optimal constants (but still not awful constants), there is a simple method to go from the scalar to matrix case. It depends on the following:

Theorem 13.1. *Let $\mathcal{J}$ be an $\ell \times \ell$ block Jacobi matrix in the Nevai class with Jacobi parameters $\{A_n, B_n\}_{n=1}^\infty$. Let $E_j^\pm(\mathcal{J})$ denote its eigenvalues counting multiplicity outside $[-2, 2]$, that is, $E_1^+ \geq E_2^+ \geq \cdots > 2 > -2 > \cdots \geq E_2^- \geq E_1^-$. Let $J_\pm$ be the scalar Jacobi matrix with $a_n \equiv 1$ and*

$$b_n^\pm = \pm\|B_n\| \pm \|A_{n-1} - 1\| \pm \|A_n - 1\| \tag{13.1}$$

and let $J_\pm^{(\ell)}$ be an ℓ-fold direct sum of $J_\pm$. Then

$$|E_j^\pm(\mathcal{J})| \leq |E_j^\pm(J_\pm^{(\ell)})| \tag{13.2}$$

Proof. The matrix analog of the observation of Hundertmark–Simon [53] extended to $2\ell \times 2\ell$ matrices (with $\ell \times \ell$ blocks) says that

$$\begin{pmatrix} -|A_n - \mathbf{1}| & \mathbf{1} \\ \mathbf{1} & -|A_n - \mathbf{1}| \end{pmatrix} \leq \begin{pmatrix} 0 & A_n \\ A_n^* & 0 \end{pmatrix} \leq \begin{pmatrix} |A_n - \mathbf{1}| & \mathbf{1} \\ \mathbf{1} & |A_n - \mathbf{1}| \end{pmatrix} \tag{13.3}$$

since

$$\left\| \begin{pmatrix} 0 & C \\ C^* & 0 \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} C^*C & 0 \\ 0 & CC^* \end{pmatrix} \right\| = \|C\|^2 \tag{13.4}$$

Thus writing $J_\pm^{(\ell)}$ as $\ell \times \ell$ blocks with each block a multiple of $\mathbf{1}$,

$$J_-^{(\ell)} \le \mathcal{J} \le J_+^{(\ell)} \tag{13.5}$$

from which (13.2) is immediate. □

Corollary 13.2. *For any block Jacobi matrix, $\mathcal{J}$, in Nevai class,*

$$\sum_{j,\pm} (E_j^\pm(\mathcal{J})^2 - 4)^{1/2} \le 2\ell \sum_n \|B_n\| + 4\ell \sum_n \|A_n - 1\| \tag{13.6}$$

Remark. In particular, this implies if the RHS of (13.6) is finite, so is the LHS.

Proof. Hundertmark–Simon [53] proved

$$\sum_j (E_j^\pm(J_\pm)^2 - 4)^{1/2} \le \sum_n b_n^\pm \tag{13.7}$$

from which (13.6) follows by (13.2). □

14. The Analog of Nevai's Conjecture

Proof of Theorem 1.5. By (1.32), $J - J_0$ is trace class. Thus $J^\ell - J_0^\ell = \sum_{k=0}^{\ell-1} J^k(J - J_0)J^{\ell-1-k}$ is trace class, so $\Delta_{J_0}(J) - \Delta_{J_0}(J_0) = \Delta_{J_0}(J) - (S^p + S^{-p})$ is trace class.

It follows that if (1.32) holds and $\Delta(J)$ has matrix Jacobi parameters $\{A_n, B_n\}_{n=1}^\infty$ that

$$\sum_{n=1}^\infty \|B_n\| + \sum_{n=1}^\infty \|A_n - 1\| < \infty \tag{14.1}$$

By Corollary 13.2, the eigenvalues $\Delta(J)$ obey

$$\sum_{j=1}^\infty (|E_j^\pm| - 2)^{1/2} < \infty \tag{14.2}$$

(14.1) also implies

$$\sum_{n=1}^\infty |\log(\det|A_n|)| < \infty \tag{14.3}$$

We can apply Theorem 10.2 and conclude that $Z(J)$ is finite, that is,

$$\int (4 - E^2)^{-1/2} \log(\det(W(E)))\, dE > -\infty \tag{14.4}$$

By Corollary 11.2, we obtain (1.25). □

15. Perturbations of Periodic OPUC

In this final section, we want to present the translations of our results to OPUC. Since the magic formula maps periodic OPUC to MOPRL, the changes needed in the proofs will be minor, although for the analog of Theorem 1.4, there is one significant change. It is interesting to note the sequence of mappings for the OPUC periodic Rakhmanov's theorem. We map OPUC to MOPRL using the magic formula and them map that to MOPUC using the Szegő map.

The OPUC version of Theorem 1.1 is already in Last–Simon [66]. As for Theorem 1.2:

Theorem 15.1. *Let $\mathcal{C}_0$ be a two-sided periodic CMV matrix. Let $\mathcal{C}$ be an ordinary CMV matrix with Verblunsky coefficients $\{\alpha_n\}_{n=0}^\infty$. Suppose*

$$\Sigma_{\rm ac}(\mathcal{C}) = \sigma_{\rm ess}(\mathcal{C}_0) \tag{15.1}$$

Then, as $m\to\infty$,

$$d_m(\alpha, \mathcal{T}_{\mathcal{C}_0}) \to 0 \tag{15.2}$$

Proof. Let $\mathcal{C}_r$ be a right limit of $\mathcal{C}$. By Theorem 7.1, $\Delta(\mathcal{C}_r) = S^p + S^{-p}$. Thus, by Theorem 4.1, $\mathcal{C}_r \in \mathcal{T}_{\mathcal{C}_0}$. □

As for the analog of Theorem 1.3, if we drop discussion of ℓ^2 convergence, it holds, similar to Theorem 12.1.

Theorem 15.2. *Let $\mathcal{C}_0$ be a two-sided periodic CMV matrix with Verblunsky coefficients $\{\alpha_j^{(0)}\}_{j=-\infty}^\infty$, and $\mathcal{C}$ a one-sided CMV matrix with Verblunsky coefficients $\{\alpha_j\}_{j=0}^\infty$ and spectral measure $d\mu$. Suppose that $\sigma_{\rm ess}(\mathcal{C}) = \sigma(\mathcal{C}_0)$ and*

$$\sum_{m=1}^N \operatorname{dist}(E_m, \sigma_{\rm ess}(\mathcal{C}))^{1/2} < \infty \tag{15.3}$$

where $\{E_m\}_{m=1}^N$ is a labelling of the eigenvalues of $\mathcal{C}$ outside $\sigma_{\rm ess}(\mathcal{C}_0)$. Then

$$-\int_{\sigma_{\rm ess}(\mathcal{C}_0)} \log\biggl(\frac{d\mu_{\rm ac}}{d\theta}\biggr) \operatorname{dist}(\theta, \mathbb{R}\setminus\sigma_{\rm ess}(\mathcal{C}_0))^{-1/2}\, \frac{d\theta}{2\pi} < \infty \tag{15.4}$$

implies

$$\lim_{N\to\infty} \biggl(\sum_{j=1}^{pN} \log\biggl(\frac{\rho_j}{\rho_j^{(0)}}\biggr)\biggr) \tag{15.5}$$

exists and lies in $(-\infty,\infty)$. Conversely, (1.25) *holds so long as*

$$\limsup_{N\to\infty} \biggl(\sum_{j=1}^{N} \log\biggl(\frac{\rho_j}{\rho_j^{(0)}}\biggr)\biggr) > -\infty \tag{15.6}$$

and then the limit in (15.5) *exists and lies in $(-\infty,\infty)$.*

Remarks. 1. We have not stated that the α_n have a limit in $\mathcal{T}_{\mathcal{C}_0}$. We suspect that the methods of [84] extend to the OPUC but have not checked this and they do not explicitly mention it.

2. Of course, if this theorem is applicable and $\mathcal{C}_0$ obeys the conditions of Theorem 15.3 below, then we have a result on ℓ^2 convergence to $\mathcal{T}_{\mathcal{C}_0}$.

3. One can replace $\rho_j^{(0)}$ by the logarithmic capacity of $\sigma(\mathcal{C}_0)$.

Proof. At open gaps, $\Delta'(e^{i\theta}) \neq 0$, so (15.3) is equivalent to

$$\sum_{\substack{E\notin[-2,2]\\ E\in\sigma(\Delta(\mathcal{C}))}} (|E|-2)^{1/2} < \infty \tag{15.7}$$

Moreover, by (4.4)–(4.7), we have that

$$\log(\det|A_n|) = \log(\det(A_n)) = \log\biggl[\prod_{j=(n-1)p+1}^{np} \prod_{k=j}^{j+p^{-1}} \frac{\rho_k}{\rho_k^{(0)}}\biggr] \tag{15.8}$$

Now just follow the proof of Theorem 12.1. □

In carrying over Theorem 1.4 to OPUC, one runs into a serious roadblock: van Moerbeke's theorem [108] that the Hamiltonian flows generated by the coefficients of the t_j (given by (11.17)) are independent is not known for OPUC. Instead, we use a weaker result of Simon [99, Section 11.10] that proves the derivatives of coefficients of Δ span the normal bundle for a dense open set that is 'most' of the points with all gaps open. We will call the isospectral tori in this dense open set the generic independence tori. Then by mimicking the arguments in Section 11, we obtain

Theorem 15.3. *Let $\mathcal{C}_0$ be a two-sided CMV matrix in a generic independence torus with Verblunsky coefficients $\{\alpha_j^{(0)}\}_{j=-\infty}^{\infty}$. Let $\mathcal{C}$ be a CMV matrix with Verblunsky coefficients $\{\alpha_j\}_{j=0}^{\infty}$. Then*

$$\sum_{m=0}^{\infty} d_m(\alpha, \mathcal{T}_{\mathcal{C}_0})^2 < \infty \tag{15.9}$$

if and only if
(i) $\sigma_{\text{ess}}(\mathcal{C}) = \sigma(\mathcal{C}_0)$
(ii) *For the eigenvalues $\{E_j\}_{j=1}^{N}$ not in $\sigma(\mathcal{C}_0)$,*

$$\sum_{j=1}^{N} \text{dist}(E_j, \sigma(\mathcal{C}_0))^{3/2} < \infty \tag{15.10}$$

(iii) *If μ is the spectral measure for $\mathcal{C}$, then*

$$-\int_{\sigma(\mathcal{C}_0)} \log\biggl(\frac{d\mu_{\text{ac}}}{d\theta}\biggr) \text{dist}(\theta, \partial\mathbb{D} \setminus \sigma_{\text{ess}}(\mathcal{C}_0))^{1/2} \, \frac{d\theta}{2\pi} < \infty \tag{15.11}$$

Our last result is a periodic OPUC version of Nevai's conjecture.

Theorem 15.4. *Let $\mathcal{C}_0$ be a two-sided p-periodic CMV matrix and let $\mathcal{C}$ be a CMV matrix with Verblunsky coefficients $\{\alpha_j\}_{j=0}^{\infty}$. Then*

$$\sum_{m=0}^{\infty} d_m(\alpha, \mathcal{T}_{\mathcal{C}_0}) < \infty \tag{15.12}$$

implies (15.4)

All the above results assume the period p is even. However, by using sieving (see Example 1.6.14 of [98]) to map period p to period $2p$, one can extend Theorems 15.1 and 15.2 to p odd. In particular, we obtain the $p = 1$ results of [10, 2, 9] as very special cases of ours.

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