Four Amazing Positivities with Dimers/i-Matchings

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Abstract

We collect a number of striking recent results in a study of dimers on infinite regular bipartite lattices and also on regular bipartite graphs. We clearly separate rigorously proven results from conjectures. A primary goal is to show people: here is a field which is ripe for further interesting research. We separate four classes of endeavor, of which we here extract two items to whet one's appetite. Primo, for hyper-rectangular lattices of dimension d, $1 \leq d \leq 10$, the first 20 virial coefficients are positive. (One has no understanding of this yet!) Secondo, all regular bipartite graphs with less than 14 vertices satisfy graph positivity, defined below. (Here there is some understanding.)

Most of this work was joint with P. Butera and M. Pernici. Most of the computer work was by M. Pernici. As said above, we have four sections, in each we separate the proved portion from the conjectures. Notes at the end present some further material. Proofs may be computer based, analytic, or analytic with computer assistance. (All computations are in integer arithmetic.)

In each section we italicize the conjecture in it towards which the effort is centered. The general references are for I, [1], for II, [3], for III, [2] and for IV, [3]. Additional references are separately noted.

Section I

I. Proved

1. For an infinite r-regular lattice the dimer entropy, λ, as a function of the dimer density, p, for p small enough is of the form

$$\lambda(p) = \frac{1}{2}(p\ln(r) - p\ln(p) - 2(1-p)\ln(1-p) - p) + \sum_{k=2}^{\infty} a_k p^k. \tag{1}$$

   This result was first obtained by me [6]. It is an easy consequence of previously known results.

2. For infinite hyper-rectangular lattices of dimension d, $1 \leq d \leq 10$, the $a_k(d)$ are positive for $k \leq 20$.

3. Note 1 below lists other infinite lattices for which all computed a_k are positive.

I . Conjectured

4. *For an infinite regular bipartite lattice all a_k are positive.*

5. In Note 2 a "certainly true" conjectured form of the $a_k(d)$ above in I.2 is given that extends the result of I.2 to all dimensions.

Section II

II . Proved

1. For a dimer gas on an infinite hyper-rectangular lattice of dimension d, $1 \leq d \leq 10$, the first 20 virial coefficients are positive.

2. Note 1 lists other infinite lattices for which all computed virial coefficients are positive (for a dimer gas).

II . Conjectured

3. *For a dimer gas on an infinite regular bipartite lattice, all virial coefficients are positive.*

4. The conjecture in Note 2 extends the result of II.1 above to all dimensions.

Section III

Definitions

We deal with r-regular bipartite graphs with $v = 2n$ vertices. We define $d(i)$ a function of the number of i-matchings, m_i

$$d(i) \equiv \ln\left(\frac{m_i}{r^i}\right) - \ln\left(\frac{\overline{m}_i}{(v-1)^i}\right)$$

where $\overline{m}_i$ is the number of i-matchings on the complete (not bipartite complete) graph on the same vertices.

$$\overline{m}_i = \frac{v!}{(v-2i)!i!2^i}.$$

We let Δ be the finite difference operator

$$\Delta d(i) = d(i+1) - d(i).$$

We say a graph satisfies graph positivity if all the meaningful $\Delta^k d(i) \geq 0$. (Loosely speaking, the conditions $\Delta^k d(i) \geq 0$ are a generalization of the positivity of the a_k in eq.(1) from infinite lattices to general graphs.)

III . Proved

1. All graphs with $v < 14$ satisfy graph positivity.

2. When $r = 4$ the first violations occur when $v = 22$, in 2 graphs of 2806490 with $v = 22$.

3. For $r = 3$ the fraction of graphs not satisfying graph positivity monotonically decreases between $v = 14$ and $v = 30$. There is a single violation at $v = 14$.

4. If $r \leq 10$, $i+k \leq 100$, $k \leq 27$ then

$$\text{Prob}(\Delta^k d(i) \geq 0) \xrightarrow[n\to\infty]{} 1$$

This is proved in [4].

III . Conjectured

5. *As n goes to infinity the fraction of graphs that satisfy graph positivity approaches one.*

Section IV

Definitions

Again, dealing with r-regular bipartite graphs with $v = 2n$ vertices. m_i the number of i-matchings. We define

$$u_i = -\ln(i!m_i).$$

We say a graph satisfies virial positivity if $\Delta^k u(i) \geq 0$ for all meaningful $k \geq 2$. (Loosely speaking a generalization of the positivity of the virial coefficients from infinite lattices to all graphs.)

IV . Proved

1. All graphs with $v \leq 18$ satisfy virial positivity.

2. When $r = 4$, the first violation occurs when $v = 20$ in a single graph among 62611 graphs with $v = 20$.

3. When $r = 3$ the fraction of graphs not satisfying virial positivity decreases monotonically between 18 and 30 with a single violation at 18.

4. If $r \leq 10$, $i+k \leq 100$, $2 \leq k \leq 27$ then

$$\text{Prob}(\Delta^k u(i) \geq 0) \xrightarrow[n\to\infty]{} 1.$$

This is proved in [5].

IV . Conjectured

5. *As $n \to \infty$ the fraction of graphs that satisfy virial positivity approaches one.*

Note 1

Additional regular bipartite lattices for which computed coefficients are positive

a) Hexagonal lattice, through 7^{th} coefficient.

b) Tetrahedral lattice, through 19^{th} coefficient.

c) Hyper-body-centered-cubic lattices of dimension $d = 3, 4, 5, 6, 7$ through 24^{th} coefficient.

These results are given in [1] with other references there cited.

Note 2

For $a_s(d)$ of I.2 we have (by a very clever, but formal argument) derived the expression

$$a_s(d) = \frac{c_{s,r}}{d^r} + \frac{c_{s,r+1}}{d^{r+1}} + \cdots + \frac{c_{s,s-1}}{d^{s-1}}$$

with $r \geq s/2$, [7],[8]. We so wish someone would give a rigorous proof of this result!

Note 3

I am old school. If someone invites me to give a Zoom slide presentation of these results, say the IAMP, it would take me a while to familiarize myself with the technical details of the computer.

References

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