Shapes of growing droplets — a model of escape from a metastable phase

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ABSTRACT: Nucleation from a metastable state is studied for an Ising ferromagnet with nearest and next nearest neighbour interaction and at very low temperatures. The typical escape path is shown to follow a sequence of configurations with a growing droplet of stable phase whose shape is determined by dynamical considerations and differs significantly from equilibrium shape corresponding to the instanantenous volume. Primary 82A05; secondary 82A25 The research has been partially supported by CNR – GNFM.

. Introduction

Relaxation to equilibrium of a system close to a first order transition is a problem whose rigorous treatment is recently receiving a lot of attention S 1, 2, NS 1, 2, MOS. In particular, an interesting question is to grasp the escape pattern of a system relaxing from a metastable starting configuration toward a stable equilibrium state. The escape is through a formation of small droplets or crystals of stable phase that are stabilized once they reach certain critical size. Here, we are interested in the shape of such crystal during the process of growth. It seems that the mechanism of growth depends on the size of a crystal. While for large supercritical crystals one has to take into account the transport of matter and heat around the crystal, the growth of tiny subcritical crystals should be governed in a more direct way by the instanantenous microscopic dynamics.

Our aim in the present paper is to discuss the shape of a growing crystal modeled by the Ising model. Namely, we are considering the Ising model with ferromagnetic nearest neighbour and next nearest neighbour interactions in the presence of a small positive external field. Starting from the configuration $-\underline{1}$, minus spins at all lattice sites in a fixed finite volume under periodic boundary conditions, we study the relaxation pattern of the stochastic process yielded by a standard Glauber dynamics. In particular, we are interested in the typical configurations during the first excursion from the configuration $-\underline{1}$ to the configuration $+\underline{1}$ with all sites occupied by plus spins. We present a detailed description of the escape pattern in the asymptotic region of vanishing temperatures. It turns out that in this asymptotics (and in a finite volume) one can consider a single droplet of plus spins, playing the role of the crystal, that grows in a very particular manner to the critical droplet.

On the first sight one could suppose that growth is through crystals that minimize the surface tension under fixed instanantenous volume Z. The shape of such crystals is the equilibrium shape yielded by the Wulff construction W, RW, DKS. For our model at low temperatures the Wulff construction can be easily shown to lead to a shape that closely follows an octagon with coordinate and oblique sides proportional to the nearest neighbour and next nearest neighbour interactions, respectively. The main result of the present paper (Theorem 3) asserts that the typical growth of subcritical crystals is through a sequence of particular shapes that significantly differ from the equilibrium Wulff octagons. This is in agreement with a similar result [KO 1] concerning Ising model with anisotropic nearest neighbour interaction. Notice that the fact that we are considering the asymptotics of vanishing temperature with fixed (small) external field is technically crucial for our proofs. It would be interesting (and difficult) to extend these type of results also to the region of vanishing external field under a fixed (small) temperature or to the region where both external field and temperature vanish in such a way that their ratio is fixed. The escape time and asymptotics of the metastable state in the latter region has been recently discussed S 2 in the case of the nearest neighbour d-dimensional Ising model.

We believe that a difference between dynamical and equilibrium shapes would occur alrady for the simplest nearest neighbour Ising model. But while it would reveal itself only as a higher order effect at low temperatures, for the model with an additional next nearest neighbour interaction discussed here it appears already in the first order of the low temperature asymptotics. The main effect is thus captured in a situation that is sufficiently simple to be studied in a rigorous way.

The paper is organized in the following way. In Section 2 we introduce the model, the dynamics and the notation concerning octagonal shapes that yield local minima of considered interaction (Lemma 2.1). Then we summarize our results in Theorem 1 and 2 (asymptotics of the hitting time to the configuration $+\underline{1}$) and Theorem 3 (describing the sequence of droplet shapes on the escape path).

It turns out that the evolution of the crystal can be evaluated by discussing "the movement in the energy landscape". Section 3 is devoted to a detailed investigation of passages between neighbouring (octagonal) local minima. An important role is played by a characterization of different basins of attraction and saddle points on their boundaries. A closely linked fact is the existence of two different important time scales in the problem. The shorter one is a typical time needed for a passage from a particular octagonal local minimum to a close octagon with identical circumscribed rectangle. The height of the corresponding saddle point is proportional to the value of the next nearest neighbour coupling. Only when passing to an octagon with larger circumscribed rectangle one has to overcome a saddle point whose height is proportional to the nearest neighbour coupling and the typical passage time is correspondingly longer. (We are supposing that the nearest neighbour coupling is stronger than the next nearest neighbour one.) In terms of this longer time scale we observe a growing octagonal shape whose oblique sides are "breathing" around "equilibrium positions" when observed on a shorter time scale.

The global saddle point, a configuration minimizing the maximal energies on pathes from $-\underline{1}$ to $+\underline{1}$, is discussed in Section 4. For its investigation it is useful to introduce a "global basin of attraction of the configuration $-\underline{1}$" — resp. its subset $\mathcal{A}$ characterized by the fact that starting from a configuration in $\mathcal{A}$, a typical path first hits the configuration $-\underline{1}$ before reaching the global minimum $+\underline{1}$. The goal is to choose the set $\mathcal{A}$ small enough to satisfy this condition but, on the other side, large enough to make the minimum on its boundary to coincide with the global saddle point.

The results of Section 3 and 4 are than merged in Section 5 to the proofs of Theorems 1-3. The basic estimate in getting a typical escape time is the lower bound on the probability to reach a global saddle point. A crucial observation is that when passing through an octagonal local minimum configuration, one has a freedom to stay in its local basin of attraction for a random time in a time interval characterized by the minimal saddle point on its boundary. The resulting estimate can thus be multiplied by corresponding "resistance times". It leads to a correct lower bound only if the path enters the local basin of attraction through lowest saddle point. This criterion determines the optimal escape path and is responsible for a particular dynamically optimal sequence of shapes mentioned above.

The results of the present paper, as well as those from KO 1, were announced in KO 2.

. Setting and results

We will consider a *discrete time Metropolis dynamics* for a totally ferromagnetic two dimensional *Ising model* with *nearest neighbours* and *next nearest neighbours interaction.*

The choice of a discrete instead of a continuous time evolution is made only for the sake of simplicity of the exposition. It will appear clear to the reader that, with some minor changes, all our results can be extended to the continuous case. Our dynamics will be given by a Markov chain whose space of states is $\Gamma = \{-1,1\}^{\Lambda}$ where Λ is a two dimensional torus — namely, Λ is the square $\{1,\dots,M\}^2$ with periodic boundary conditionsFor instance $x=(M+1,b)$ is, for every $b \le M$, identified with $x'=(1,b)$ and so on..

A configuration σ is a function

$$\sigma\colon \Lambda \to \{-1,1\}, \tag{2.1}$$

i.e. $\sigma \in \Gamma = \{-1,1\}^{\Lambda}$. The value $\sigma(x)$ is the *spin* at the site x. The *energy* of a configuration σ is

$$H(\sigma) = -\frac{\tilde{J}}{2} \sum_{<x,y>\subset\Lambda} \sigma(x)\sigma(y) - \frac{K}{2} \sum_{\ll x,y\gg\subset\Lambda} \sigma(x)\sigma(y) - \frac{h}{2}\sum_{x\in\Lambda} \sigma(x), \tag{2.2}$$

where we suppose that $K,\ \tilde{J},\ h > 0$. Here $<x,y>$ denotes a pair of nearest neighbours in Λ: $|x-y|=1$ and $\ll x,y\gg$ denotes a pair of next nearest neighbours in Λ: $|x-y|=\sqrt{2}$. Our dynamics is defined with the help of the following *updating rule* :

Given the configuration σ at time t we first choose at random (with uniform probability) a site $x \in \Lambda$. Then we flip the spin at the site x with probability

$$\exp(-\beta(\Delta_x H(\sigma))^{+}), \tag{2.3}$$

where

$$\Delta_x H(\sigma) = H(\sigma^{(x)}) - H(\sigma) \tag{2.4}$$

with

$$\sigma^{(x)}(y) = \begin{cases} \sigma(y) & \text{whenever } y \neq x, \\ -\sigma(x) & \text{whenever } y = x. \end{cases} \tag{2.5}$$

Here, for every $c \in \mathrm{R}$ we denote $(c)^+ = \min(c, 0)$; β is the inverse temperature.

The *transition probabilities* are then given by

$$P(\sigma \to \eta) = \begin{cases} \frac{1}{|\Lambda|} \exp(-\beta(\Delta_x H(\sigma))^+) & \text{if } \eta = \sigma^{(x)} \text{ for some } x, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

The *space of the trajectories* of our process is

$$\Omega = \Gamma^{\mathrm{N}} \equiv (\{-1, 1\}^{\Lambda})^{\mathrm{N}}. \tag{2.7}$$

An element in Ω is denoted by ω, it is a function

$$\omega : \quad \mathrm{N} \to \Gamma.$$

If

$$\omega = \sigma_0, \sigma_1, \dots, \sigma_t, \dots,$$

we set

$$\omega_t \equiv \omega_t = \sigma_t.$$

The dynamics is *reversible* with regard to the Gibbs measure in Λ in the sense that

$$P(\sigma \to \eta) e^{-\beta H(\sigma)} \equiv P(\eta \to \sigma) e^{-\beta H(\eta)}. \tag{2.8}$$

We will discuss a behaviour at very low temperatures. Thus it is natural to describe configurations in terms of contours. Namely, for every $\sigma \in \Gamma$ consider the union $C(\sigma)$ of all closed unit squares centered at lattice sites x for which $\sigma(x) = +1$. The boundary ∂C of C is a polygon with vertices on the dual lattice $\mathrm{Z}^2 + (\frac{1}{2}, \frac{1}{2})$ such that in any vertex of the dual lattice an even number (0, 2, or 4) of unit segments belonging to this polygon meet. Any connected component γ of the boundary ∂C is called a *contour* of the configuration σ.

It is easy to see that if a configuration σ has the boundary $\partial C(\sigma)$ consisting of a single contour γ, the energy of σ is

$$H(\sigma) - H(-\underline{1}) = J|\gamma| - K|A(\gamma)| - h|I(\gamma)|. \tag{2.9}$$

Here

$$J = \tilde{J} + 2K, \tag{2.10}$$

$|\gamma|$ is the length of γ, $|I(\gamma)|$ is the cardinality (area) of the interior $I(\gamma) \equiv C(\sigma)$. Finally, $|A(\gamma)|$ is the number of corners (right angles) of γ. Notice that in the situation like that on Fig. 2.1 we count 4 corners.

Fig. 2.1

A contour γ is said to be *isolated* if it lies at a distance at least $\sqrt{2}$ from other contours.

A relevant role will be played by a particular class of contours that we call octagons. An *octagon* is a closed contour inscribed in a rectangle R with edges parallel to the lattice axes. Call P_1, P_2, P_3, P_4 the vertices of R. The octagon contains four straight edges with extremes $x_i, y_i,\ i = 1, \dots, 4 : (x_1, y_1) \subseteq P_1P_2$, $(x_2, y_2) \subseteq P_2P_3$, $(x_3, y_3) \subseteq P_3P_4$, and $(x_4, y_4) \subseteq P_4P_1$, called *coordinate edges*, and four *oblique edges* that have a local staircase structure with extremes y_1x_2, y_2x_3, y_3x_4, y_4x_1. (See Fig. 2.2)

When referring to a *stable octagon* we suppose that $|x_i - y_i| \geq 2$, $|y_i - x_{i+1}| \geq \sqrt{2}$, $i = 1, \dots, 4$. When discussing a particular octagon Q, it will be useful to introduce

$$L_i = |x_i - y_i|, \quad i = 1, \dots, 4, \tag{2.11}$$

Fig. 2.2

for the lengthes of its coordinate edges,

$$l_i = 1 + \frac{1}{\sqrt{2}}|y_i - x_{i+1}|, \quad i = 1, \ldots, 4, \ (x_5 \equiv x_1), \tag{2.12}$$

for the lengths of its oblique edges,

$$\begin{aligned} D_1 &= \overline{P_1P_2} = L_1 + l_4 - 1 + l_1 - 1 = \overline{P_4P_3} = L_3 + l_3 - 1 + l_2 - 1, \\ D_2 &= \overline{P_2P_3} = L_2 + l_1 - 1 + l_2 - 1 = \overline{P_4P_1} = L_4 + l_3 - 1 + l_4 - 1, \end{aligned} \tag{2.13}$$

for the lengthes of the sides of its *rectangular envelope* $R(Q) = P_1P_2P_3P_4$, and, finally,

$$\begin{aligned} d_1 &= L_1 + L_4 + 2(l_4 - 1) = L_2 + L_3 + 2(l_2 - 1), \\ d_2 &= L_1 + L_2 + 2(l_1 - 1) = L_4 + L_3 + 2(l_3 - 1) \end{aligned} \tag{2.14}$$

for the distances (in units of $1/\sqrt{2}$) between pairs of opposite parallel oblique edges. We will use

$$Q(D_1, D_2, l_1, l_2, l_3, l_4) \tag{2.15}$$

to denote the corresponding octagon. Another way of characterize it, is by specifying, say, d_1, d_2, L_1, L_2, L_3, L_4.

The previous definitions made no reference to the *location of the octagon*. Sometimes, in the sequel, we will consider a canonical location. We say that an octagon Q is *centered* if the upper left corner x_1 of Q is the point $(-1/2, 1/2)$ of the dual lattice (namely, the uppermost left + spin, the first in lexicographic order, is the origin). Sometimes, when not specifying a location of an octagon Q, we tacitly assume it to be centered.

We will often use the same symbol Q also to denote the set of all spin configurations σ giving rise to a unique closed contour (full of pluses) consisting, up to a translation on the torus, of the octagon Q; sometimes we write $\sigma \in Q$. We set

$$H(Q) \equiv H(\sigma) \ , \quad \text{for } \sigma \in Q \ .$$

Clearly, for $Q = Q(D_1, D_2, (l_i)_{i=1,\dots,4})$ we have

$$H(Q) = 2J(D_1 + D_2) - hD_1D_2 - \sum_{j=1}^{4}[K(2l_j - 1) - \frac{1}{2}hl_j(l_j - 1)]. \tag{2.16}$$

Let us define

$$\begin{aligned} l^* &= \left[\frac{2K}{h}\right] + 1 \ , \quad \text{i.e. } h(l^* - 1) < 2K < hl^* \ , \\ D^* &= \left[\frac{2J}{h}\right] + 1 \ , \quad \text{i.e. } h(D^* - 1) < 2J < hD^* \ , \\ L^* &= D^* - 2(l^* - 1) = \left[\frac{2J}{h}\right] + 1 - 2\left[\frac{2K}{h}\right] . \end{aligned} \tag{2.17}$$

Here $[x]$ denotes the integer part of a real number x. We suppose also that $\frac{2K}{h}$ and $\frac{2J}{h}$ are not integers. Notice, that if η is defined as

$$\eta = \frac{2K}{h} - (l^* - 1) = \frac{2K}{h} - \left[\frac{2K}{h}\right] , \tag{2.18}$$

then $\dfrac{2\tilde{J}}{h} + 2\eta$ is not an integer and

$$L^* = \left[\frac{2\tilde{J}}{h} + 2\eta\right] + 1. \tag{2.19}$$

We always suppose in the present paper that $K \le \frac{J}{10}$ and $0 < h < fracK7$. These are essentially technical assumptions that make proofs simpler. We did not attempt to optimalize them and it might turn out that weaker conditions would also work.

We define a *standard octagon* as an octagon with

$$\min_{i=1,\dots,4} L_i \ge l^* \text{ and } l_j = l^*, \quad j = 1, \dots, 4 \ .$$

In this case we simply write

$$Q(D_1, D_2) \equiv Q(D_1, D_2, l^*, l^*, l^*, l^*) \ . \tag{2.20}$$

We call an octagon *regular* when

$$L_1 = L_2 = L_3 = L_4 = l_1 = l_2 = l_3 = l_4 = l$$

and denote it by $Q(l)$. We will only be interested in regular octagons with $l \leq l^*$. For configurations $\sigma \in Q(D_1, D_2, l_1, l_2, l_3, l_4)$ we often write

$$D_j = D_j(\sigma) \ , \quad j = 1, 2 \ ,$$

$$l_j = l_j(\sigma) \ , \quad j = 1, \ldots, 4 \ .$$

The assumption about smallness of h made above assures, in particular, that for every spin flip the energy changes at least by h. (See Lemma 3.1 in Section 3.)

The importance of octagons stems from the fact that they yield the set of all local minima of energy.

Lemma 2.1 Let σ be a configuration whose energy increases under every spin flip. Then the set $\{x, \sigma(x) = +1\}$ is a union of isolated stable octagons.

The proof of this lemma will be given in Section 3.

Given any set of configurations $\Delta \subset \Gamma$ we define the *first hitting time* to Δ as

$$\tau_\Delta = \inf\{t \geq 0 : \sigma_t \in \Delta\}. \tag{2.21}$$

Sometimes we use the notation $P_\eta(\cdot)$ to denote the probability distribution over the process starting at $t = 0$ from a configuration η.

As we mentioned in the Introduction we shall discuss the asymptotic behavior, in the limit $\beta \to \infty$, of typical paths of first escape from $-\underline{1}$ to $+\underline{1}$. We refer to NS, S, KO1, KO2 for a general introduction to this problem. Also in the present case, local minima will play a crucial role in the description of the transition from the metastable situation to the stable one. It will happen, similarly to the case of nearest neighbour Ising model, that small octagons have a tendency to shrink whereas large ones tend to grow. Again, the dynamical mechanism responsible for this behaviour relies on the competition between the creation of a suitable stable protuberance and the erosion of an edge. The main difference here, with respect to the nearest neighbour Ising model, comes from the presence of the oblique edges in stable isolated clusters — the octagons. The new phenomenon, namely the growth or shrinking in the oblique direction, is still governed by a competition between the creation of certain protuberance and the erosion of an edge. However, three new features should be mentioned:

" i)" the corresponding time scales are different with respect to the ones appearing in the growth-shrinking mechanism in the coordinate directions;

" ii)" for sufficiently large octagons, a sort of "equilibrium" in the oblique direction is established, and

" iii)" for "relevant" octagons, the global tendency to shrink or grow — to disappear or to invade the whole volume — is governed by the size of the "circumscribed rectangle". All these mechanisms will be described in the next sections starting from the "elementary" events, taking place at microscopic level and involving a single spin flip, then analyzing the transitions between neighbouring minima, and finally considering the global transition between $-\underline{1}$ to $+\underline{1}$. A very important role will be played by standard octagons that can be considered as the most stable ones among the octagons with given, sufficiently large, circumscribed rectangle (see Section 3). The *basin of attraction* of a local minimum will be defined in Section 3 in a natural way and neigbouring minima, namely the ones with a nonempty intersection of the boundaries of their basins of attraction, will be considered.

To define a *saddle point* between two neighbouring local minima, say Q and Q', we consider the minimax

$$\min_{\omega\colon\, Q\to Q'}\ \max_{\sigma\in\omega} H(\sigma) \tag{2.22}$$

(where we use $\omega\colon \sigma \to \tau$ to denote a generic path by subsequent spin flips starting from σ and ending in τ). A saddle point is any configuration $\bar\sigma$ for which the above minimax is attained. Namely, considering any $\omega_0\colon Q \to Q'$ for which the minimum in (2.22) is reached, any configuration $\bar\sigma$ for which $\max_{\sigma\in\omega_0} H(\sigma)$ is attained is a saddle point.

It will become clear that, for sufficiently large sizes, it is interesting to consider not only the jumps between neighbouring generic octagons (with related basins of attraction and saddle configurations) but also the transitions between standard octagons $Q(D_1, D_2)$ with corresponding "domains of attraction" (see Section 3) that group together all the basins of attraction of octagons inscribed in the same rectangle R as $Q(D_1, D_2)$. These transitions at "intermediate level" (corresponding to the exit from domains of attraction instead of basins of attraction) involve an intermediate time scale between the one referring to the transition between neighbouring generic octagons and the global one referring to the transition between $-\underline{1}$ and $+\underline{1}$.

A *global saddle point* is any configuration $\bar\sigma$ for which the minimax

$$\min_{\omega\colon\, -\underline{1}\to+\underline{1}}\ \max_{\sigma\in\omega} H(\sigma) \tag{2.23}$$

is attained. Let $\mathcal{P}$ be the set of all configurations obtained from a standard octagon $Q(D^*, D^*-1)$ or $Q(D^*-1, D^*)$ (D^* has been defined in the equation (2.17)) by attaching, to one of its longer coordinate edges a unit square protuberance (see Fig. 2.3). We will see that $\mathcal{P}$ coincides with the set of all global saddle points.

For any $\bar\sigma \in \mathcal{P}$ one has

$$H(\bar\sigma) - H(-\underline{1}) \equiv E^* = H(Q(D^*, D^*-1)) + 2\widetilde{J} - h = H(Q(D^*, D^*)) + h(D^*-1) - 4K \tag{2.24}$$

for the "height" of the global saddle point.

Fig. 2.3

We shall prove that the first excursion from $-\underline{1}$ to $+\underline{1}$ typically passes through a configuration from $\mathcal{P}$ and the time needed for this to happen is of the order $\exp(\beta E^*)$.

To present this statement in a formal way, we use $\bar{\tau}_{-\underline{1}}$ to denote the last instant in which $\sigma_t = -\underline{1}$ before $\tau_{+\underline{1}}$,

$$\bar{\tau}_{-\underline{1}} = \max\{t < \tau_{+\underline{1}} \colon \sigma_t = -\underline{1}\}, \tag{2.25}$$

and introduce

$$\bar{\tau}_{\mathcal{P}} = \min\{t > \bar{\tau}_{-\underline{1}} \colon \sigma_t \in \mathcal{P}\}. \tag{2.26}$$

Theorem 1

$$\lim_{\beta\to\infty} P_{-\underline{1}}(\bar{\tau}_{\mathcal{P}} < \tau_{+\underline{1}}) = 1. \tag{2.27}$$

Theorem 2

$$\lim_{\beta\to\infty} P_{-\underline{1}}(\exp[\beta(E^* - \varepsilon)] < \tau_{+\underline{1}} < \exp[\beta(E^* + \varepsilon)]) = 1 \tag{2.28}$$

for every $\varepsilon > 0$.

In addition, we are getting a much more detailed information about a typical path followed by our process σ_t during its first excursion from $-\underline{1}$ to $+\underline{1}$, or, in other words, between the moments $\bar{\tau}_{-\underline{1}}$ and $\tau_{+\underline{1}}$. Theorem 3 below states, roughly speaking, that with high probability for large β, the path σ_t sticks to a certain *tube* of trajectories. A precise definition involves a lot of other preliminary definitions and notions and we will be able to present it only at the end of Section 5. There, we will introduce the concept of an *ϵ-typical path* that wil be determined in terms of its geometrical properties, but also with a specified times of passing through certain configurations. For the moment we only say that, roughly speaking, the typical trajectories during the first excursion begin by following a sequence of almost regular octagons up to an edge l^*; after that the oblique edge stay almost constant at the value l^* while the coordinate edges grow, keeping the rectangular envelope almost squared, from the value l^* up to the value L^* corresponding to the critical nucleus. This first part of the first excursion can be viewed as a nucleation phenomenon and it involves, in average, "ascending" transitions with growing energy. Finally, the oblique edges stay further almost constant at the value l^* whereas, the coordinate edges continue to grow, with larger fluctuations, still preserving, however, the average squared shape of the rectangular envelope until the whole volume is invaded by pluses. This "supercritical growth" is, in average, a descent in energy.

Theorem 3 For every $\epsilon > 0$ one has

$$\lim_{\beta \to \infty} P_{-\underline{1}}(\{\sigma_t\}_{t \in [\bar{\tau}_{-\underline{1}}, \tau_{+\underline{1}}]} \text{ is an } \epsilon\text{-typical path}) = 1. \tag{2.29}$$

3

. Passage between neighbouring local minima

To be able to discuss in detail the growth or shrinking of a droplet we first introduce some "elementary events" — namely, certain particular spin-flips.

Notice first that the energy increment, when flipping the spin of a configuration σ at a site x, can be expressed in the form

$$\Delta_x H = H(\sigma^{(x)}) - H(\sigma) = (\tilde{J} M_{\tilde{J}}^{(x)}(\sigma) + K M_K^{(x)}(\sigma) + h)\sigma(x), \tag{3.1}$$

where

$$M_{\tilde{J}}^{(x)}(\sigma) = \sum_{y:|x-y|=1} \sigma(y)\ , M_K^{(x)}(\sigma) = \sum_{y:|y-x|=\sqrt{2}} \sigma(y).$$

We are interested in the region of phase diagram where K is small with respect to $\tilde{J}$ (next nearest neighbour interaction is a perturbation of nearest neighbour interaction). Recall that we are assuming that

$$0 < 7h < K \leq \frac{J}{10}. \tag{3.2}$$

Then every spin flip leads to a change of energy by a minimal amount of at least h. Lemma 3.1 Suppose that $0 < h < K$ and $0 < K \leq \frac{1}{2}(\tilde{J} - h)$. Then, for a single spin flip,

$$|\Delta_x H| \geq h. \tag{3.3}$$

Proof Observe that $M_{\tilde{J}}^{(x)}, M_K^{(x)} \in \{-4, -2, 0, 2, 4\}$. By inspection one verifies that the minimal value is attained for $M_{\tilde{J}} = M_K = 0$. ∎

We will consider five particular classes of spin flips at a site x. Namely:

" a)" an h-erosion if $\sigma(x) = +1$ and $M_{\tilde{J}}^{(x)} = M_K^{(x)} = 0$,

" b)" an h-recovery if $\sigma(x) = -1$ and $M_{\tilde{J}}^{(x)} = M_K^{(x)} = 0$,

" c)" a K-protuberance if $\sigma(x) = -1$, $M_{\tilde{J}}^{(x)} = 0$, and $M_K^{(x)} = -2$,

" d)" a K-erosion if $\sigma(x) = +1$, $M_{\tilde{J}}^{(x)} = 0$, and $M_K^{(x)} = 2$, and finally,

" e)" a J-protuberance if $\sigma(x) = -1$, $M_{\tilde{J}}^{(x)} = -2$, and $M_K^{(x)} = 0$. Notice that according to (3.1), the energy increment in these cases are, respectively,

(3.4a) $$\Delta_x H(\sigma) = +h \ ,$$

(3.4b) $$\Delta_x H(\sigma) = -h \ ,$$

(3.4c) $$\Delta_x H(\sigma) = 2K - h \ ,$$

(3.4d) $$\Delta_x H(\sigma) = 2K + h \ ,$$

(3.4e) $$\Delta_x H(\sigma) = 2\tilde{J} - h \ .$$

For a site x adjacent to an isolated contour, an h-erosion is possible iff x is adjacent to a convex angle as in Figure. 3.1a (figures are always drawn modulo reflections and rotations),

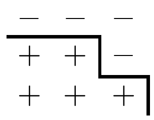

Fig. 3.1a

— the site x is the central one. Similarly an h-recovery is possible iff x is adjacent to a concave angle as in Fig. 3.1b,

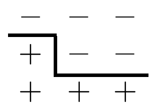

Fig. 3.1b

for a K-proturberance the situation is necessarily that on Fig. 3.1c,

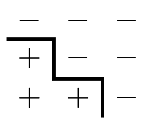

Fig. 3.1c

for a K-erosion that on Fig. 3.1d,

$$\begin{array}{ccc} + & - & - \\ + & + & - \\ + & + & + \end{array}$$

Fig. 3.1d

and finally, for a J-proturberance that on Fig. 3.1e.

$$\begin{array}{c|cc} + & - & - \\ + & - & - \\ + & - & - \end{array}$$

Fig. 3.1e

Before discussing the "full dynamics" at non-zero temperature, we first consider a deterministic dynamics where only spin flips with decreasing energy are allowed. What we get is actually certain generalization of *bootstrap percolation*. Consider finite sequences $S = \{x_1, \ldots, x_n\}$ of lattice sites. A sequence S is called a *sweep* if it contains every site from Λ. S is called *standard* if it contains at least

$$(8\tilde{J} + 8K + 2h)\frac{M^2}{h} \tag{3.5}$$

of consecutive sweeps. We use $\mathcal{S}$ to denote the set of all standard sequences.

Every sequence S yields a map on configurations (denoted again S): visit the sites in the order of S and flip the corresponding spin if it leads to a decrease of energy. Notice that the order in which the sites are visited is crucial. The particular number (3.5) of sweeps in a standard sequence S assures that we always finally reach a local minimum of energy — a configuration that does not allow any more a spin flip decreasing energy. Indeed, in every step when the energy changes, it decreases by at least h, the difference of absolute minimum and absolute maximum in energies is at most $(8\tilde{J} + 8K + 2h)M^2$, and finally, if no change during a sweep is made, the configuration is a local minimum.

As stated in Lemma 2.1, octagons are configurations forming local minima of the energy. Before we shall proceed with the investigation of the saddle points between octagons, we prove this lemma. However, first we introduce some concepts to be used in the proof and later.

We call *droplet* any connected set C consisting of a union of unit squares centered at lattice sites. Clearly, for every droplet C there exists a configuration σ such that the droplet C is given by the set $\{x, \sigma(x) = +1\}$.

To every droplet we assign three envelopes — $M(C)$, $Q(C)$, and $R(C)$. The sets $R(C)$ and $Q(C)$, *rectangular* and *octagonal envelope*, are smallest rectangle and octagon, respectively, containing C. The set $M(C)$, the *monotone envelope* of C, is the smallest monotone droplet containing C, where a droplet is called *monotone* if if its boundary has the same length as its rectangular envelope. (See Fig. 3.2.)

Fig. 3.2

Observe that, if the rectangle $R = R(C)$ is not winding around the torus, the monotone envelope $M(C)$ is the complement of the union of all those open right angles of the form $\{(x,y) \in R; x > x_0, y > y_0\}$, $\{(x,y) \in R; x > x_0, y < y_0\}$, $\{(x,y) \in R; x < x_0, y > y_0\}$, and $\{(x,y) \in R; x < x_0, y < y_0\}$ with (x_0, y_0) points of dual lattice, that do not intersect the set C. Notice that if $R(C)$ winds around the torus, then $M(C) = Q(C) = R(C)$.

Proof of Lemma 2.1 Consider an isolated connected component C of the set $\{x, \sigma(x) = +1\}$ and the octagon Q circumscribed to it. We will show that, supposing that the energy of σ increases under every spin flip, one necessarily has $C \equiv Q$. Consider first the *outer* *boundary* $\partial_{\rm out} C$. To avoid ambiguities, it can be defined by taking the outer boundary of an ϵ-neighbourhood of C in the limit $\epsilon \to 0$. Taking into account that for $\epsilon > 0$ one has a selfavoiding curve, we can consider (as will be useful later) a path winding around along all the boundary $\partial_{\rm out} C$. Our aim now is to prove first that $\partial_{\rm out} C \equiv \partial Q$ (see Fig. 3.3).

Fig. 3.3 The set C is shaded, thickened line denotes $\partial_{\rm out} C$.

To this end we inspect a catalogue of locally stable configurations (with respect to spin flip at x). We say that a configuration σ is *stable at* x if the spin flip $\sigma(x) \to -\sigma(x)$ increases the energy. Consider thus a configuration σ and a site x with $\sigma(x) = -1$. Whether the configuration σ is stable at x depends only on its value at the nearest and next nearest neighbour sites of x. Namely, it is stable at x whenever either $M^{(x)}_{\tilde J} < 0$ or $M^{(x)}_{\tilde J} = 0$ and $M^{(x)}_{K} < 0$. As a result we get the following catalogue (up to rotations and reflections) of stable situations around x (dots stand for an arbitrary spin):

•

−

•

Suppose now that $\partial_{\text{out}}C \neq \partial Q$. Then, considering a path along ∂Q oriented in the same sense as that along $\partial_{\text{out}}C$, there exist two points $A, B \in \partial_{\text{out}}C \cap \partial Q$ such that the paths along $\partial_{\text{out}}C$ and ∂Q between these two points do not have a common unit segment and such that either

" a)" the points A, B belong to the same side of Q (see Fig. 3.4A) or

" b)" they belong to two neighbouring sides of Q (see Fig. 3.4B).

Fig. 3.4a

Fig. 3.4b

The easiest case to tackle is A) with the side in question being, say, horizontal. Namely, consider the path γ between A and B and the lowermost horizontal line l touching γ. (See Fig. 3.5)

Fig. 3.5

At the point where γ for the first time (going from A to B) touches l, we have the configuration $l\ \begin{array}{cc} + & \lfloor - \\ \hline & + \end{array}$. For the spin $-$ to be stable it must be $l\ \begin{array}{ccc} + & - & - \\ \hline & + & \end{array}$, and, l being the lowermost line touching γ, we must have $l\ \begin{array}{ccc} + & - & - \\ \hline & + & + \end{array}$. In view of e) from our catalogue we necessarily have $l\ \begin{array}{ccc} - & - & - \\ + & - & - \\ \hline - & + & + \end{array}$ — otherwise the spin $-$ in the centre would not be stable. The spin $+$ above the line l is, according to a) from our catalogue, always unstable (it has three $-$ nearest neighbours) and we get a contradiction. In the remaining cases shown in Fig. 3.4 we get a contradiction by the same reasoning, once the lowermost horizontal line l touching γ does not pass through the point B.

The argument above can be also interpreted in the following way: whenever we encounter a concave corner $\begin{array}{cc} + & \lfloor - \\ & + \end{array}$, then, for the spin $-$ to be stable, we must have $\begin{array}{ccc} + & - & - \\ & + & \end{array}$ and hence $\begin{array}{ccc}+&-&-\\ &+&-\end{array}$, because supposing the configuration $\begin{array}{ccc}+&-&-\\ &+&+\end{array}$ leads to a contradiction. Discussing in the same fashion also the value of the spin above the upper $+$, we conclude that the configuration necessarily is $\begin{array}{ccc}-&-& \\ +&-&-\\ &+&-\end{array}$ and our concave corner is surrounded by two convex corners. We can use this observation in the following way. Consider the horizontal line l passing through B and the point B' where γ first hits l. Knowing already that γ does not pass below l, it proceeds from B' horizontaly so that a concave corner is formed: $\begin{array}{cc}+&-\\ B'&+\end{array}$. According to the above observation there must be two convex corner attached: $l \quad B'$. But since the line γ does not, before reaching B, pass below of l, the point B' must coincide with B.

Consider now the point $\overline{B}$ on γ, two units backward from B and repeat the above argument with the curve $\overline{\gamma}$ joining A with $\overline{B}$ and the horizontal line $\bar{l}$ passing through $\overline{B}$ (see Fig. 3.6).

Fig. 3.6

If $\bar{l}$ also passes through A, the curve $\overline{\gamma}$ has to be just a horizontal segment joining A and $\overline{B}$. If not we can repeat the argument and get a point $\overline{\overline{B}}$. Iterating this process we eventually get a line passing through A. As a result the curve γ actually follows the boundary ∂Q and we can conclude that $\partial_{\text{out}} \equiv \partial Q$.

To show, finally, that $\partial C = \partial_{\text{out}} C$ (there are no holes in C) consider the set $\partial C \setminus \partial_{\text{out}} C$ and the lowermost horizontal line touching it. Taking the most left touching point, we necessarily have a concave corner $l\ \begin{array}{cc}+&-\\ &+\end{array}$ and thus also $l\ \begin{array}{ccc}+&-&-\\ &+&-\end{array}$. The newly attached bond lies below l and in the same time does not belong to $\partial_{\text{out}} C$; this would be possible only if the upper right spin were + which is not the case. Hence we are getting a contradiction with the fact that the set $\partial C \setminus \partial_{\text{out}} C$ does not reach below l.

The fact that the octagon C has to be stable ($L_i, l_i \geq 2$) is obvious. Finally, if the configuration σ contained two octagonal components of mutual distance one, there would always exist, as it is easy to convince oneself by inspection of possible cases, a minus spin between them, whose flipping would lead to a decrease of energy. ∎

Among octagons with the same circumscribed rectangle, $Q = Q(D_1, D_2, l_1, \dots, l_4)$, the standard ones, or at least those with oblique sides as close as possible to l^*, minimize the energy. This is stated, in a slightly more general form, in the following lemma.

Lemma 3.2 Let R be a rectangle with sides $D_1 \geq D_2$ and consider the set $\mathcal{M}(D_1, D_2)$ of all monotone droplets with connected interiorNotice that even though a monotone envelope is connected, its interior might split into disjoint components in a situation like that illustrated on Fig. 2.1. Cf. Lemma 4.3.w hose circumscribed rectangle is R. Let σ_0 be the configuration corresponding to

" a)" the octagon $Q(D_1, D_2, l^*)$ if $D_2 \geq 2l^* - 1$,

" b)" the octagon $Q(D_1, D_2, l_1 = l_2 = l_3 = l_4 = \frac{1}{2}(D_2 + 1))$ if $D_2 < 2l^* - 1$ and it is odd, and

" c)" the octagon $Q(D_1, D_2, l_1 = l_2 = \frac{D_2}{2}, l_3 = l_4 = \frac{D_2}{2} + 1)$ if $D_2 < 2l^* - 1$ and it is even. Then

$$\min_{\sigma \in \mathcal{M}(D_1, D_2)} H(\sigma) = H(\sigma_0).$$

Proof Let $\sigma \in \mathcal{M}(D_1, D_2)$ and consider its circumscribed octagon $Q(D_1, D_2, l_1, \ldots, l_4)$. The configuration $\overline{\sigma}$ represented by Q has clearly lower energy than σ because it occupies larger area then $C(\sigma)$ and, taking into account that in every site of the dual lattice at most two its edges meet, its boundary necessarily has at least that number of corners as $C(\sigma)$ (this is clearly true for every its oblique side separately). The energy of $\overline{\sigma}$ is

$$H(\overline{\sigma}) = H(-\underline{1}) + 2J(D_1 + D_2) - hD_1D_2 + \sum_{a=1}^{4} F(l_a).$$

Notice that the function

$$F(l) \equiv -K(2l - 1) + \frac{1}{2}hl(l - 1) \tag{3.6}$$

is minimized for $l = l^*$. Indeed, for $l \in \mathrm{R}$, the parabola F has minimum at $l = \frac{2K}{h} + \frac{1}{2}$ and this point is closer to l^* than to either $l^* - 1$ or $l^* + 1$. Hence, whenever $D_2 \geq 2l^* - 1$, the energy of $\overline{\sigma}$ is larger or equal to that of σ_0 minimizing every term $F(l_a)$ separately.

If $D_2 < 2l^* - 1$, we first fix $\sum l_a$ and under this condition minimize $\sum F(l_a)$. The minimum is achieved for a maximally symetric quadruple $l'_1, \ldots, l'_4$, for which $\max_a l'_a - \min_a l'_a \leq$

1, such that $\sum l'_a = \sum l_a$. If $\sum l_a$ is not divisible by four, there is some freedom in the choice of $l'_1, \ldots, l'_4$ and, in particular, the conditions

$$|l'_1 + l'_4 - (l'_2 + l'_3)| \leq 1 \text{ and } |l'_1 + l'_2 - (l'_3 + l'_4)| \leq 1$$

can be satisfied. Given that $\sum l_a \leq 2(D_2 + 1)$, we have also

$$\max(l'_1 + l'_4, l'_2 + l'_3) \leq D_2 + 1 \text{ and } \max(l'_1 + l'_2, l'_3 + l'_4) \leq D_2 + 1 \leq D_1 + 1$$

and thus the octagon $Q(D_1, D_2, l'_1, l'_2, l'_3, l'_4)$ exists. We can further decrease its energy by increasing one by one $l'_1, \ldots, l'_4$ maintaining maximal possible symmetry. The resulting octagon depends on the parity of D_2. ∎

With every octagon Q we associate two *basins of attraction* — a narrow one

$$\mathcal{B}(Q) = \{\sigma : S\sigma = Q \text{ for every standard } S\}, \tag{3.7}$$

and a wide one

$$\widehat{\mathcal{B}}(Q) = \{\sigma : \text{ there exists a standard } S \text{ such that } S\sigma = Q\}. \tag{3.8}$$

Clearly,

$$\widehat{\mathcal{B}}(Q) \supset \mathcal{B}(Q)$$

and the sets $\mathcal{B}(Q)$ are disjoint for different Q's.

When discussing a growth of a droplet, we are studying a passage between close standard octagons with different rectangular envelops. It is possible to pass between different octagons with the same rectangular envelope at the cost of overcomming energetical barriers between them. A crucial circumstance here is that the barriers are much lower than those one has to pass when changing the rectangular envelope — the corresponding time scales are much shorter and on the scale relevant for the passage to a "neighbouring" standard octagon, these barriers can be overcome with a nonvanishing probability. If we choose to disregard the lower barriers, we can introduce "domains of attraction" associated with every standard octagon $Q = Q(D_1, D_2)$ specified by its rectangular envelope $R(D_1, D_2)$. Namely, we combine different octagons with the same rectangular envelope and introduce a broad *domain of attraction* $\widehat{\mathcal{D}}(D_1, D_2)$ and a naturally restricted one $\mathcal{D}(D_1, D_2)$. We thus define

$$\widehat{\mathcal{D}}(D_1, D_2) = \bigcup_{Q:(D_1(Q),D_2(Q))=(D_1,D_2)} \widehat{\mathcal{B}}(Q) \tag{3.9}$$

for every $(D_1, D_2) \in \mathbb{Z}_+^2$. For (D_1, D_2) such that $\min(D_1, D_2) \geq 3l^* - 2$ we define

$$\begin{aligned}(3.10)\quad \mathcal{D}(D_1, D_2) = \{\sigma :\ &\text{for every standard } S, S\sigma = Q \text{ with } Q \text{ such that} \\ &(D_1(Q), D_2(Q)) = (D_1, D_2) \text{ and } \min(L_1(Q), L_2(Q)) \geq l^*\}.\end{aligned}$$

Notice that the condition $\min(D_1, D_2) \geq 3l^* - 2$ assures that the standard octagon $Q(D_1, D_2) \in \mathcal{D}(D_1, D_2)$. Let us introduce also the *boundaries*

$$\partial\mathcal{D} = \{\sigma \notin \mathcal{D} \text{ such that there exists } x \text{ such that } \sigma^{(x)} \in \mathcal{D}\} \tag{3.11}$$

and

$$\partial\widehat{\mathcal{D}} = \{\sigma \in \widehat{\mathcal{D}} \text{ such that there exists } x \text{ such that } \sigma^{(x)} \notin \widehat{\mathcal{D}}\}. \tag{3.12}$$

Notice that, even though we take $\partial\mathcal{D}$ outside $\mathcal{D}$, clearly $\partial\mathcal{D} \subset \widehat{\mathcal{D}}$ and thus all configurations in $\partial\mathcal{D}$ and $\partial\widehat{\mathcal{D}}$ can "fall down" to a minimum "inside" $R(D_1, D_2)$. To see that $\partial\mathcal{D} \subset \widehat{\mathcal{D}}$ we observe that if $\xi \in \mathcal{D}$ and $\sigma = \xi^{(x)} \in \partial\mathcal{D}$, then $H(\xi) < H(\sigma)$. Indeeed, if σ were stable at x, the flip $\xi \to \sigma$ would decrease energy and the sequence $\{x, S\}$ with S such that $S\sigma$ leads to Q with $(D_1(Q), D_2(Q)) \neq (D_1, D_2)$ or with $\min(L_1(Q), L_2(Q)) < l^*$ — such sequence S exists since $\sigma \notin \mathcal{D}(D_1, D_2)$ — would map the configuration ξ to this Q in contradiction with the assumption $\xi \in \mathcal{D}$.

Whenever $\min(D_1, D_2) \geq 2l^* - 1$, we introduce the energy $\widehat{E}(D_1, D_2)$ by

$$\widehat{E}(D_1, D_2) = \begin{cases} h(\min(D_1, D_2) - 1) - 4K & \text{if } \min(D_1, D_2) < D^* \\ 2J - 4K - h & \text{if } \min(D_1, D_2) \geq D^* \end{cases} \tag{3.13}$$

and, for $\min(D_1, D_2) \geq 3l^* - 2$, also

$$E(D_1, D_2) = \widehat{E}(D_1, D_2) + \max(0, 3l^* - \min(D_1, D_2))\eta h, \tag{3.14}$$

where η is the constant defined by (2.18). For sufficiently large $\min(D_1, D_2)$ we have $E(D_1, D_2) = \widehat{E}(D_1, D_2)$, and according to the following Lemma, this energy actually yields the height, above the value $H(Q(D_1, D_2))$ of energy of the standard octagon, of the lowest point on the boundary of the domains of attraction $\mathcal{D}(D_1, D_2)$ and $\widehat{\mathcal{D}}(D_1, D_2)$ as well as heights of the saddle points for escaping paths. If $\min(D_1, D_2) < 3l^*$, the situation is more complex. A path leaving $\mathcal{D}$ through a saddle point is actually heading to an octagon Q still belonging to $\widehat{\mathcal{D}}$ — the saddle point is not on the boundary of $\widehat{\mathcal{D}}$.

Lemma 3.3 For every $(D_1, D_2) \in \mathrm{Z}_+^2$ such that $\min(D_1, D_2) \geq 2l^* - 1$ one has

$$\min_{\omega:\sigma_0 \to \mathcal{M}^c} \sup_{\sigma \in \omega} H(\sigma) > \min_{\sigma \in \partial\widehat{\mathcal{D}}(D_1, D_2)} H(\sigma) = H(Q(D_1, D_2)) + \widehat{E}(D_1, D_2), \tag{3.15}$$

where the minimum is over all paths starting at any octagon $Q = Q(D_1, D_2, l_1, l_2, l_3, l_4)$ and leaving the set $\mathcal{M}$ of all monotonous configurations.

If, moreover, $\min(D_1, D_2) \geq 3l^* - 2$, then also

$$\min_{\omega:\sigma_0 \to \mathcal{D}^c} \sup_{\sigma \in \omega} H(\sigma) = \min_{\sigma \in \partial\mathcal{D}(D_1, D_2)} H(\sigma) = H(Q(D_1, D_2)) + E(D_1, D_2), \tag{3.16}$$

where the minimum is over all paths starting at the standard octagon $Q(D_1, D_2)$ and leaving the set $\mathcal{D}(D_1, D_2)$.

Proof Let $\sigma \in \partial\widehat{\mathcal{D}}$, $\xi = \sigma^{(x)} \notin \widehat{\mathcal{D}}$. Since $\sigma \in \widehat{\mathcal{D}}$, there exists a standard S such that $S\sigma = Q$ and $(D_1(Q), D_2(Q)) = (D_1, D_2)$. Consider the sequence of configurations obtained by applying S on σ. Taking it in the opposite order, we get a path $\omega : Q \to \sigma$ such that the energy increases in every step.

With $Q = Q(D_1, D_2, l_1, l_2, l_3, l_4)$, $D_1 \geq D_2$, and using the function F defined in (3.6), we shall prove that

$$H(\sigma) \geq H(Q) + \min[h(D_2 - 1) - 4K, 2J - 4K - h] - \sum_{a=1}^{4}[F(l_a) - F(l^*)]. \tag{3.17}$$

On the other side, particular cases of $\sigma \in \partial\widehat{\mathcal{D}}$ yielding equality in (3.17) can be displayed. Namely, if $D_2 \leq D^*$, we take the standard octagon $Q(D_1, D_2)$ and cut, except one spin, all row of $(D_2 - 1)$ spins; the resulting droplet corresponds to $\sigma \in \partial\widehat{\mathcal{D}}$ since cutting the last spin we get $\xi \notin \widehat{\mathcal{D}}$. If $D_2 > D^*$, we get $\sigma \in \partial\widehat{\mathcal{D}}$ just by attaching one plus spin to the coordinate side.

Taking into account that

$$H(Q) = H(-\underline{1}) + 2J(D_1 + D_2) - hD_1 D_2 + \sum_{a=1}^{4} F(l_a), \tag{3.18}$$

we get from (3.17) the sought minimum for $\partial\widehat{\mathcal{D}}(D_1, D_2)$,

$$\min_{\sigma \in \partial\widehat{\mathcal{D}}(D_1, D_2)} H(\sigma) = H(Q(D_1, D_2)) + \widehat{E}(D_1, D_2).$$

To prove (3.17), notice first that, without a loss of generality, we can suppose that the path ω consists of at most $D_2 - 1 - \max(l_1 + l_2 - 2, 2l^* - 2)$ steps. (For concreteness we suppose that $\max(l_1 + l_2, l_2 + l_3, l_3 + l_4, l_4 + l_1) = l_1 + l_2$.) Indeed, in every step the energy increases by at least h and

$$h(D_2 - (2l^* - 1)) > h(D_2 - 1) - 4K$$

since $h(l^* - 1) < 2K$. Hence, the number of steps can be taken to be at most $D_2 - 1 - 2(l^* - 1)$. If $l_1 + l_2 > 2l^*$, we have even stronger restriction on the number of steps. Indeed, it suffices to prove a lower bound on the increase of energy,

$$h[D_2 - 1 - (l_1 - 1) - (l_2 - 1)] \geq h(D_2 - 1) - 4K - \sum_{a=1}^{4} [F(l_a) - F(l^*)]. \tag{3.19}$$

This is true, once we verify that

$$F(l) - F(l^*) \geq h(l - 1) - 2K \tag{3.20}$$

for any l. To see it, we observe that the line (as function of l) on the right hand side above touches the parabola on the left hand side in the point $l = l^* + 1$ and is below of it for $l = l^*$

and $l = l^* + 2$:

$$
\begin{aligned}
l = l^* : &\quad F(l) - F(l^*) = 0 > h(l^* - 1) - 2K, \\
l = l^* + 1 : &\quad F(l) - F(l^*) = -2K + \frac{1}{2} h l^* 2 = -2K + h l^*, \\
l = l^* + 2 : &\quad F(l) - F(l^*) = -4K + h(2l^* + 1) > -2K + h(l^* + 1).
\end{aligned}
$$

(Notice that equality in (3.19) is attained only for $l_1 = l_2 = l^* + 1$.)

Fig. 3.7

Next, we can suppose that all droplets $\zeta \in \omega$ (including the last one, σ) are monotone with connected interior. Indeed, starting from the octagon Q and supposing that after $n \leq D_2 + 1 - l_1 - l_2$ steps we still have such a monotone droplet, minimal energy flips leading to a non-monotone droplet are those shown in Fig. 3.7. (Given number of steps is clearly not sufficient to allow any configuration like

with subsequent splitting into disjoint components etc.) The increase of energy in the case a), b), and c) is $2J - 4K + h$, $2J - 2K + h$, and $2J - 4K - h$, respectively. In all three cases this value is at least as large as $\min[h(D_2 - 1) - 4K, 2J - 4K - h]$ and already this single step would suffice for our claim. This remark proves, in particular, the first inequality in (3.15).

Observe now that the spin flip $\sigma \to \sigma^{(x)} = \xi$ necessarily decreases energy (otherwise one would have $\xi \in \widehat{\mathcal{D}}$). Taking into account that σ is monotone with connected interior, the droplet ξ is also monotone with interior consisting of at most two components. Suppose first that ξ has connected interior. For such ξ, there clearly exists a standard sequence leading to its octagonal envelope $Q(\xi)$ and, since $\xi \notin \widehat{\mathcal{D}}$, one has $(D_1(\xi), D_2(\xi)) \neq (D_1, D_2)$. As a consequence

either $(D_1(\sigma), D_2(\sigma)) \neq (D_1, D_2)$ (actually $D_1(\sigma) > D_1$ or $D_2(\sigma) > D_2$)

or $(D_1(\sigma), D_2(\sigma)) = (D_1, D_2)$ and $\bar{d}_a = 1$ for some $a = 1, \ldots, 4$

(we use here $\bar{d}_a(\sigma)$ to denote the lengths of segments along which the set $C(\sigma)$ intersects the sides of the circumscribed rectangle $R(\sigma)$).

Consider the first configuration ζ in ω with this property and its predecessor $\overline{\zeta}$. If $D_1(\zeta) > D_1$ or $D_2(\zeta) > D_2$, then

$$H(\sigma) - H(Q) \geq H(\zeta) - H(\overline{\zeta}) \geq 2J - 4K - h \geq \min[h(D_2 - 1) - 4K, 2J - 4K - h]. \quad (3.21)$$

If $(D_1(\zeta), D_2(\zeta)) = (D_1, D_2)$ and in the same time $\bar{d}_a = 1$, one had to cut at least $L_a - \bar{d}_a = L_a - 1$ spins touching the side of $R(Q)$ to reach this configuration, and thus

$$H(\zeta) - H(Q) \geq h(L_a - 1) \geq h(D_2 - 1 - (l_1 - 1) - (l_2 - 1)). \quad (3.22)$$

Hence, using again (3.19) we get (3.17).

If the interior of ξ consists of two components, than the path ω reaching the configuration σ should consist of at least $L_a - 1$ steps. Indeed, consider the horizontal (or vertical) line passing through the point in which the closures of these two components intersect. In the configuration Q there is at least $2L_a$ plus spins at sites of distance $1/2$ from this line, while in the configuration ξ at lest L_a of them is missing ("two quadrants filled with minuses touch at the considered intersection point"). As a result, the inequality (3.22) holds for σ and we get (3.17).

To get the bound for $\partial\mathcal{D}(D_1, D_2)$, consider first $\sigma \in \partial\mathcal{D}(D_1, D_2) \setminus \partial\widehat{\mathcal{D}}(D_1, D_2)$. Notice that then there exists a standard sequence S mapping σ onto an octagon Q, $S\sigma = Q$, such that $\min(L_1(Q), L_2(Q)) < l^*$. Indeed, by the same argument as above we can show that all ζ on the (reversed) path from Q to σ are monotone, $(D_1(\sigma), D_2(\sigma)) = (D_1, D_2)$, and $\bar{d}_a(\sigma) \geq 2$, $a = 1, \ldots, 4$ (otherwise we would have $\sigma \in \partial\widehat{\mathcal{D}}(D_1, D_2)$). Hence, there does not exist any standard sequence S mapping σ onto an octagon Q such that $(D_1(Q), D_2(Q)) \neq (D_1, D_2)$

Moreover, for at least one ζ on our path it is $Q(\zeta) \supset Q$ and $Q(\zeta) \neq Q$. There exists a standard sequence mapping ζ into Q and thus $Q(\zeta) \supset Q$. If one had $Q(\zeta) = Q$ for every ζ including σ, then one could use the fact that, since $\sigma \in \partial\mathcal{D}$, there exists a standard sequence $\bar{S}$ mapping σ into $\bar{Q}$ such that $(D_1(\bar{Q}), D_2(\bar{Q})) = (D_1, D_2)$ and $\min(L_1(\bar{Q}), L_2(\bar{Q})) \geq l^*$ to get a contradiction. Indeed, for such $\bar{Q}$ one cannot have $\bar{Q} \subset Q(\sigma) = Q$ and in the same time $(D_1(\bar{Q}), D_2(\bar{Q})) = (D_1(Q), D_2(Q))$. Thus, on the uphill path from Q to σ a K-proturberance appears at least once and thus

$$H(\sigma) \geq H(Q) + 2K - h. \tag{3.23}$$

We can label the sides of the octagon Q in such a way that $L_2 = D_2 - l_1 - l_2 + 2 \leq l^* - 1$, which is equivalent to

$$l_1 + l_2 - 2 \geq D_2 - 1 - (l^* - 2). \tag{3.24}$$

The difference of the energy of the octagon Q and the standard octagon $Q(D_1, D_2)$ is at least

$$H(Q) - H(Q(D_1, D_2)) \geq \sum_{a=1}^{4} (F(l_a) - F(l^*)). \tag{3.25}$$

If $\min(D_1, D_2) \geq 3l^* - 1$, we use the bound (3.20) to evaluate, using (3.24), the right hand side

$$\begin{aligned} \sum_{a=1}^{4} (F(l_a) - F(l^*)) &\geq F(l_1) - F(l^*) + F(l_2) - F(l^*) \geq \\ &\geq h(l_1 + l_2 - 2) - 4K \geq (D_2 - 1)h - 4K - (l^* - 2)h. \end{aligned} \tag{3.26}$$

With the help of the equality (2.18), $\eta h = 2K - h - (l^* - 2)h$, we get from (3.23) and the above inequality the sought lower bound with 1 in place of $3l^* - \min(D_1, D_2)$ in the definition (3.14).

If $\min(D_1, D_2) = D_2 = 3l^* - 2$, the bound (3.24) asserts that

$$l_1 + l_2 - 2 \geq 2(l^* - 1) + 1. \tag{3.27}$$

Supposing, say, $\max(l_1, l_2) = l_1$, we can infer that

$$l_1 \geq l^* + 1 \tag{3.28}$$

and thus

$$\sum_{a=1}^{4} (F(l_a) - F(l^*)) \geq F(l_1) - F(l^*) \geq hl^* - 2K. \tag{3.29}$$

Using now the bound (3.23) and then the bound (2.17) and the definition (2.18), we get

$$\begin{aligned} H(\sigma) &\geq H(Q(D_1, D_2)) + hl^* - 2K + 2K - h = H(Q(D_1, D_2)) + h(l^* - 1) = \\ &= H(Q(D_1, D_2)) + h(3l^* - 3) - 4K + 2\eta h. \end{aligned} \tag{3.30}$$

Thus, we finished the proof for $\sigma \in \partial\mathcal{D}(D_1, D_2) \setminus \partial\widehat{\mathcal{D}}(D_1, D_2)$.

Let us now turn to the case $\sigma \in \partial\mathcal{D}(D_1, D_2) \cap \partial\widehat{\mathcal{D}}(D_1, D_2)$. If $\min(D_1, D_2) \geq 3l^*$, the needed bound is the already proved inequality (3.15).

In the case $(D_2 =) \min(D_1, D_2) = 3l^* - 1$, one can consider a path leading to Q with $L_2(Q) \geq l^*$. In the case leading to (3.21) the needed bound is amply satisfied since

$$D_2 h - 4K + 2h < 2J - 4K - h. \tag{3.31}$$

In the case leading to (3.22) we get

$$H(\sigma) - H(Q) \geq h(L_2 - 1) \geq h(l^* - 1) \geq h(3l^* - 2) - 4K + \eta h \tag{3.32}$$

using (2.17).

If $(D_2 =) \min(D_1, D_2) = 3l^* - 2$, we get again either the sufficient bound (3.21) or

$$H(\sigma) - H(Q) \geq h(L_2 - 1) \geq h(l^* - 1) = h(3l^* - 3) - 4K + 2\eta h. \tag{3.33}$$

Finally, to prove that

$$\min_{\omega:\sigma_0\to\mathcal{D}^c}\sup_{\sigma\in\omega} H(\sigma) = H(Q(D_1,D_2)) + E(D_1,D_2), \tag{3.34}$$

we have to find a path ω from the standard octagon $Q(D_1,D_2)$ to a saddle configuration σ with

$$H(\sigma) = H(Q(D_1,D_2)) + E(D_1,D_2),$$

such that for every $\zeta\in\omega$ one has

$$H(\zeta) \leq H(Q(D_1,D_2)) + E(D_1,D_2).$$

If $\min(D_1,D_2)\geq 3l^*$, the saddle point σ is actually on the boundary $\partial\widehat{\mathcal{D}}(D_1,D_2)$.

A path satisfying the above condition can be taken, for example, by first cutting one layer along two oblique sides of the standard octagon and then along the coordinate side between them. The highest point on this path, before the final steady growth when cutting coordinate side, is

$$H(Q(D_1,D_2)) + 2(l^*-1)h - (2K-h) < H(Q(D_1,D_2)) + (\min(D_1,D_2)-1)h - 4K$$

once $\min(D_1,D_2)\geq 3l^*$.

If $\min(D_1,D_2) = 3l^*-1$, corresponding to $\min(L_1,L_2) = l^*+1$, one reaches the saddle point of $\partial\mathcal{D}(D_1,D_2)$ already when finishing the cut of two oblique sides attached to the shorter coordinate side (yielding the coordinate side of the length l^*-1) with the energy

$$H(Q(D_1,D_2)) + 2(l^*-1)h - (2K-h) = H(Q(D_1,D_2)) + \widehat{E}(D_1,D_2) + \eta h.$$

Finally, if $\min(D_1,D_2) = 3l^*-2$, corresponding to $\min(L_1,L_2) = l^*$, one reaches, already when cutting only one oblique side, the saddle point with the energy

$$H(Q(D_1,D_2)) + (l^*-1)h = H(Q(D_1,D_2)) + \widehat{E}(D_1,D_2) + 2\eta h.$$

■

In the following Lemma we evaluate, using the reversibility of the process, the hitting time τ_η to a configuration η, starting from a configuration σ, in terms of the difference of energies of the concerned configurations.

Lemma 3.4 For every $\epsilon > 0$ and all $\sigma \neq \eta \in \Gamma$, $H(\sigma) < H(\eta)$, one has

$$P_\sigma(\tau_\eta \leq \exp\{\beta(H(\eta) - H(\sigma) - \epsilon)\}) \xrightarrow[\beta\to\infty]{} 0.$$

Proof Given $T \in \mathrm{N}$, one has

$$P_\sigma(\tau_\eta < T) = \sum_{0<s<T} \sum_{\overline{\sigma}_1,\dots,\overline{\sigma}_{s-1}\in\Gamma\setminus\eta} P_\sigma(\sigma_0 = \sigma, \sigma_1 = \overline{\sigma}_1, \dots, \sigma_{s-1} = \overline{\sigma}_{s-1}, \sigma_s = \eta) =$$

$$= \exp\{-\beta(H(\eta) - H(\sigma))\}\times$$

$$\times \sum_{0<s<T} \sum_{\overline{\sigma}_1,\dots,\overline{\sigma}_{s-1}\in\Gamma\setminus\eta} P(\sigma_0 = \eta, \sigma_1 = \overline{\sigma}_{s-1}, \dots, \sigma_{s-1} = \overline{\sigma}_1, \sigma_s = \sigma) \leq$$

$$\leq T \exp[-\beta(H(\eta) - H(\sigma))].$$

■

Our next task is to characterize octagons with respect to their tendency to grow or shrink. This will be done in Propositions 1–4. Before presenting them, however, we introduce some tools that will be repeatedly used.

First, we define a set of Markov chains obtained by observing our stochastic trajectories only when they pass through a particular set of configurations.

Let Q be a local minimum (octagon) of the energy and B any connected set of configurations containing Q. Here connected means that, for all $\sigma, \eta \in B$, there exists a sequence $\sigma_1 \dots \sigma_k \in B$ such that $\sigma_1 = \sigma$, $\sigma_k = \eta$, and (σ_i, σ_{i+1}) are "nearest neighbour configurations" for $i = 1, \dots, k-1$, with σ and τ considered to be nearest neighbours iff there exists $x \in \Lambda$ such that $\tau = \sigma^{(x)}$.

The *exterior boundary* ∂B of B (cf. (3.11)) is the set

$$\partial B = \{\eta = \sigma^{(x)};\ \eta \notin B, \sigma \in B, x \in \Lambda\}. \tag{3.35}$$

For a particular Q and B we consider the following Markov chain, $\{\xi_n\}$, with the space of states

$$X = Q \cup \partial B.$$

Introducing the sequence of times

$$v_0 < u_0 \leq v_1 < u_1 \leq v_2 < \dots$$

with $v_0 = 0$, $u_i, v_i \in \mathrm{N}$,

$$\begin{aligned} u_n &= \inf\{t > v_n : \sigma_t \neq \sigma_{t-1}\}, \\ v_n &= \inf\{t \geq u_n : \sigma_t \in \partial B \cup Q\}, \end{aligned} \tag{3.36}$$

we set

$$\xi_n = \sigma_{v_n}, \qquad \sigma_0 \in Q \cup \partial B, \tag{3.37}$$

and

$$\begin{aligned} \theta &= \inf\{n : \xi_n = Q\} \\ \nu &= \inf\{n \geq \theta : \xi_n \in \partial B\}. \end{aligned} \tag{3.38}$$

For every $s \in \mathrm{N}$ one has

$$P_Q(\tau_{\partial B} > s) \geq P_Q(\nu > s) = P(Q \to Q)^s = (1 - P(Q \to \partial B))^s,$$

where

$$\begin{aligned} P(Q \to Q) &= P(\xi_1 = Q | \xi_0 = Q), \\ P(Q \to \partial B) &= \sum_{\eta \in \partial B} P(\xi_1 = \eta | \xi_0 = Q) = P(\xi_1 \in \partial B | \xi_0 = Q). \end{aligned} \tag{3.39}$$

We have

$$
\begin{aligned}
(3.40)\quad & P(\xi_1 \in \partial B | \xi_0 = Q) = \\
&= \sum_{s_0=0}^{\infty} \sum_{s=1}^{\infty} \sum_{\bar\sigma_1,\dots,\bar\sigma_{s-1} \in B\setminus\{Q\}} \sum_{\bar\sigma_s \in \partial B} P(\sigma_0 = Q, \dots, \sigma_{s_0} = Q, \sigma_{s_0+1} = \bar\sigma_1, \dots, \sigma_{s_0+s} = \bar\sigma_s) = \\
&= \sum_{s_0=1}^{\infty} \sum_{s=1}^{\infty} \sum_{\bar\sigma_1,\dots,\bar\sigma_{s-1} \in B\setminus\{Q\}} \sum_{\bar\sigma_s \in \partial B} e^{-\beta[H(\bar\sigma_s)-H(Q)]} \times \\
&\qquad \times P(\sigma_0 = \bar\sigma_s, \sigma_1 = \bar\sigma_{s-1}, \dots, \sigma_{s-1} = \bar\sigma_1, \sigma_s = Q, \dots, \sigma_{s+s_0} = Q) \le \\
&\le e^{-\beta \min_{\sigma\in\partial B}[H(\sigma)-H(Q)]} \sum_{\bar\sigma \in \partial B} \sum_{s_0=1}^{\infty} \sum_{s=1}^{\infty} P_{\bar\sigma}(\sigma_t \in B \setminus Q, \forall t < s, \sigma_t = Q, \forall t \in [s, s+s_0]) \le \\
&\le |\partial B| e^{-\beta \min_{\sigma\in\partial B}[H(\sigma)-H(Q)]} P_\sigma(\tau_Q \ge 1).
\end{aligned}
$$

In conclusion, for all $\varepsilon > 0$ one has

$$
P(Q \to \partial B) < e^{-\beta\left[\min_{\sigma\in\partial B} (H(\sigma)-H(Q)) - \varepsilon\right]}, \tag{3.41}
$$

whenever β is large enough.

For a set of octagons, say Q_j, $j = 1, \dots, N$, we will consider suitable connected (mutually disjoint) sets B_j containing Q_j and, as above, we introduce simultaneously the corresponding Markov chains ξ_n^j (see (3.37)) with the hitting times ν_j, θ_j defined as in (3.38). To avoid ambiguities (the stochastic trajectory can come back to an octagon Q_j with $j > i$), the process ξ^j is defined considering the portion of the trajectory between the moment it first enters ∂B_j until, after visiting Q_j, it first enters ∂B_{j+1} (visiting, possibly, other B_i's with $i < j$ in the meantime).

Often we will combine two subsequent events with the help of the following composition operator. Namely, in general, consider two events $\mathcal{E}^1_{t_1}, \mathcal{E}^2_{t_2}$, taking place in the intervals of time $[0, t_1]$, $[0, t_2]$, respectively. This means that $\mathcal{E}^1_{t_1}$, $\mathcal{E}^2_{t_2}$ are subsets of Ω measurable with respect to the σ-algebra generated by $\{\sigma_t\}_{t\in[0,t_1]}$, $\{\sigma_t\}_{t\in[0,t_2]}$, respectively. We define the *composition* $(\mathcal{E}^1_{t_1}; \mathcal{E}^2_{t_2})$ of $\mathcal{E}^1_{t_1}$, $\mathcal{E}^2_{t_2}$ by taking

$$
(\mathcal{E}^1_{t_1}; \mathcal{E}^2_{t_2}) = \mathcal{E}^1_{t_1} \bigcap T_{t_1} \mathcal{E}^1_{t_2} \tag{3.42}
$$

with T_t denoting the translation operator shifting the events in a natural way by a time t.

More generally, given the events $\mathcal{E}^1_{t_1}, \dots, \mathcal{E}^n_{t_n}$ taking place in the intervals $[0, t_1]$,…, $[0, t_n]$, we set

$$(\mathcal{E}^1_{t_1}; \mathcal{E}^2_{t_2}; \dots; \mathcal{E}^n_{t_n}) = \mathcal{E}^1_{t_1} \bigcap T_{t_1} \mathcal{E}^2_{t_2} \bigcap \cdots \bigcap T_{t_1+t_2+\cdots+t_{n-1}} \mathcal{E}^n_{t_n}. \tag{3.43}$$

Given $\mathcal{E}_t$ for every $t \in \mathrm{N}$ and a value $\bar{t} \in \mathrm{N}$, we set

$$\bar{\mathcal{E}}_{\bar{t}} = \bigcup_{t=1}^{\bar{t}} \mathcal{E}_t. \tag{3.44}$$

We call $\bar{\mathcal{E}}_{\bar{t}}$ the *extension* of $\mathcal{E}_t$ up to $\bar{t}$.

Finally, we use $\bar{\mathcal{E}}$ to denote the extension $\bar{\mathcal{E}}_\infty$. Namely, we simply write

$$\bar{\mathcal{E}} = \bigcup_{t=1}^{\infty} \mathcal{E}_t. \tag{3.45}$$

Given $\mathcal{E}^1_{t_1}, \dots, \mathcal{E}^n_{t_n}$, that we suppose to be defined for any n-ple $t_1, \dots, t_n \in \mathrm{N}$, we define the *composition of the extensions* $\bar{\mathcal{E}}^i_{\bar{t}_i}$, $i = 1, \dots, n$, as

$$\begin{aligned}(\bar{\mathcal{E}}^1_{\bar{t}_1}; \bar{\mathcal{E}}^2_{\bar{t}_2}; \dots; \bar{\mathcal{E}}^n_{\bar{t}_n}) &= \bigcup_{t_1=1}^{\bar{t}_1} \cdots \bigcup_{t_n=1}^{\bar{t}_n} (\mathcal{E}^1_{t_1}; \dots; \mathcal{E}^n_{t_n}) = \\ &= \bigcup_{t_1=1}^{\bar{t}_1} \cdots \bigcup_{t_n=1}^{\bar{t}_n} \Big[\mathcal{E}^1_{t_1} \bigcap T_{t_1} \mathcal{E}^2_{t_2} \bigcap \cdots \bigcap T_{t_1+t_2+\cdots+t_{n-1}} \mathcal{E}^n_{t_n}\Big].\end{aligned} \tag{3.46}$$

As mentioned above, we will be interested in a process passing through a series of local energetical minima characterized by octagons $Q_j, j = 1, \dots, N$. In a typical situation, the boundaries of subsequent connected sets $B_1, \dots, B_N$ intersect. Namely, we suppose that $\partial B_i \cap \partial B_{i+1} \neq 0$ for $i = 1, \dots, N-1$ and that a sequence of configurations $S_1, S_2, \dots, S_{N+1}$ is given such that $S_{i+1} \in \partial B_i \cap \partial B_{i+1}$ for $i = 1, \dots, N-1$, $S_1 \in \partial B_1$, and $S_{N+1} \in \partial B_N$. One can think about the configuration S_{i+1} as about a saddle point for the passage *up* from Q_i to S_{i+1} and then *down* from S_{i+1} to Q_{i+1}. Supposing further that a sequence

$t_u^1, t_d^1, \ldots, t_d^{N-1}, t_u^N \in \mathrm{N}$ is given, with u and d standing for *up* and *down*, we define, for $j = 1, \ldots, N$, the event

$$\mathcal{F}^{j,u}_{t_u^j} = \{\omega \in \Omega\colon \sigma_0 = Q_j, \tau_{\partial B_j} = t_u^j, \sigma_{\tau_{\partial B_j} = S_{j+1}}\} = \{\xi_0^j = Q_j, \xi^j_{\nu_j} = S_{j+1}, v_{\nu_j} = t_u^j\} \quad (3.47)$$

("ascent up to the saddle point")Notice that, since $\xi^j_{\nu_j} = S_{j+1} \in \partial B_j \cap \partial B_{j+1}$, the definition of the event $\mathcal{F}^{j,u}_{t_u^j}$ contains a condition that the stochastic trajectory is *not* revisiting any set B_i with $i < j$ before reaching ∂B_{j+1}.a nd, for $j = 1, \ldots, N-1$, the event

$$\mathcal{F}^{j,d}_{t_d^j} = \{\omega \in \Omega\colon \sigma_0 = S_{j+1}, \tau_{\partial Q_{j+1}} = t_d^j\} = \{\xi_0^{j+1} = S_{j+1}, v_{\theta_{j+1}} = t_d^j\} \quad (3.48)$$

("descent from the saddle point"). Further, we can consider the composition

$$(\mathcal{F}^{1,u}_{t_u^1}; \mathcal{F}^{1,d}_{t_d^1}; \mathcal{F}^{2,u}_{t_u^2}; \mathcal{F}^{2,d}_{t_d^2}; \ldots; \mathcal{F}^{N-1,d}_{t_d^{N-1}}; \mathcal{F}^{N,u}_{t_d^N}) \quad (3.49)$$

defined by the equation (3.46). Given $\bar t_u^i$, $\bar t_d^i$, $i = 1, \ldots, N-1$, $\bar t_u^N \in \mathrm{N}$, we consider also the extensions up to $\bar t_u^i, \bar t_d^i$:

$$(\bar{\mathcal{F}}^{1,u}_{\bar t_u^1}; \ldots; \bar{\mathcal{F}}^{N-1,d}_{\bar t_d^{N-1}}; \bar{\mathcal{F}}^{N,u}_{\bar t_u^N}) \equiv \cup_{t_u^1=1}^{\bar t_u^1} \cdots \cup_{t_d^{N-1}=1}^{\bar t_d^{N-1}} \cup_{t_u^N=1}^{\bar t_u^N} (\mathcal{F}^{1,u}_{t_u^1}; \ldots; \mathcal{F}^{N-1,d}_{t_d^{N-1}}; \mathcal{F}^{N,u}_{t_u^N}). \quad (3.50)$$

Observing that the events $(\mathcal{F}^{1,u}_{t_u^1}; \ldots; \mathcal{F}^{N,u}_{t_u^N})$ in the right hand side of the equation (3.50) are manifestly disjoint, we have

$$\begin{aligned} P(\bar{\mathcal{F}}^{1,u}_{\bar t_u^1}; \ldots; \bar{\mathcal{F}}^{N-1,d}_{\bar t_d^{N-1}}; \bar{\mathcal{F}}^{N,u}_{\bar t_u^N}) &= \sum_{t_u^1=1}^{\bar t_u^1} \cdots \sum_{t_d^{N-1}=1}^{\bar t_d^{N-1}} \sum_{t_u^N=1}^{\bar t_u^N} P(\mathcal{F}^{1,u}_{t_u^1}; \ldots; \bar{\mathcal{F}}^{N-1,d}_{t_d^{N-1}}; \mathcal{F}^{N,u}_{t_u^N}) \equiv \\ &\equiv [\sum_{t_u^1=1}^{\bar t_u^1} P(\mathcal{F}^{1,u}_{t_u^1})] \cdots [\sum_{t_d^{N-1}=1}^{\bar t_d^{N-1}} P(\mathcal{F}^{N-1,d}_{t_d^{N-1}})][\sum_{t_u^N=1}^{\bar t_u^N} P(\mathcal{F}^{N,u}_{t_u^N})] \equiv \\ &\equiv P(\bar{\mathcal{F}}^{1,u}_{\bar t_u^1}) \cdots P(\bar{\mathcal{F}}^{N-1,d}_{\bar t_d^{N-1}}) P(\bar{\mathcal{F}}^{N,u}_{\bar t_u^{N-1}}). \end{aligned} \quad (3.51)$$

We want now to restrict ourselves to a sequence of ascent events that can be defined only in terms of the above Markov chains. More precisely, in terms of the events whose characteristic functions are measurable with regard to the σ-algebra generated by the random variable ξ^i_n. This will be achieved by using

$$\bar{\mathcal{F}}^{j,u}_{\bar{t}^j_u} = \bar{\mathcal{F}}^{j,u} \cap \{\sigma_0 = Q_j, \tau_{\partial B_j} \leq \bar{t}^j_u\}, \tag{3.52}$$

where the event

$$\bar{\mathcal{F}}^{j,u} = \cup_{t^j_u=1}^{\infty} \mathcal{F}^{j,u}_{t^j_u} = \{\xi^j_0 = Q_j, \xi^j_{\nu_j} = S_{j+1}\}, \quad j = 1, \ldots, N, \tag{3.53}$$

has the desired measurability property. Introducing now

$$\mathcal{G} = \cap_{j=1}^{N-1} \{\tilde{\tau}_{\partial B_j} \leq \bar{t}^j_u\}, \tag{3.54}$$

where $\tilde{\tau}_{\partial B_j}$ is defined by

$$\tilde{\tau}_{\partial B_j} = \inf\{t > \tau_{Q_j} : \sigma_t \in \partial B_j\}, \tag{3.55}$$

and putting

$$\bar{\mathcal{F}} = (\bar{\mathcal{F}}^{1,u}; \bar{\mathcal{F}}^{1,d}_{\bar{t}^1_d}; \bar{\mathcal{F}}^{2,u}; \ldots; \bar{\mathcal{F}}^{N-1,d}_{\bar{t}^{N-1}_d}; \bar{\mathcal{F}}^{N,u}), \tag{3.56}$$

we get

$$\begin{aligned}
(\bar{\mathcal{F}}^{1,u}_{\bar{t}^1_u}; \ldots; \bar{\mathcal{F}}^{N-1,d}_{\bar{t}^{N-1}_d}; \bar{\mathcal{F}}^{N,u}_{\bar{t}^N_u}) &= (\bar{\mathcal{F}}^{1,u}; \bar{\mathcal{F}}^{1,d}_{\bar{t}^1_d}; \bar{\mathcal{F}}^{2,u}; \ldots; \bar{\mathcal{F}}^{N-1,d}_{\bar{t}^{N-1}_d}; \bar{\mathcal{F}}^{N,u}) \cap \mathcal{G} = \\
&= [\cup_{t^1_u=1}^{\infty} \cup_{t^1_d=1}^{\bar{t}^1_d} \cup_{t^2_u=1}^{\infty} \cdots \cup_{t^{N-1}_d=1}^{\bar{t}^{N-1}_d} \cup_{t^N_u=1}^{\infty} (\mathcal{F}^{1,u}_{t^1_u}; \ldots; \mathcal{F}^{N,u}_{t^N_u})] \cap \mathcal{G} \equiv \\
&\equiv \bar{\mathcal{F}} \cap \mathcal{G}.
\end{aligned} \tag{3.57}$$

To get a lower bound on the probability of $\bar{\mathcal{F}} \cap \mathcal{G}$ one can use the lower bound

$$P(\bar{\mathcal{F}} \cap \mathcal{G}) \geq P(\bar{\mathcal{F}}) - P(\mathcal{G}^c). \tag{3.58}$$

Thus, if we got a bound of the form

$$P(\mathcal{G}^c) < a \leq \frac{1}{2} P(\bar{\mathcal{F}}), \tag{3.59}$$

we would have

$$P(\bar{\mathcal{F}} \cap \mathcal{G}) > \frac{1}{2} P(\bar{\mathcal{F}}). \tag{3.60}$$

To evaluate the probability of the event $\bar{\mathcal{F}}$, we can use the following general strategy. First, we evaluate the probability $P(\xi^j_{\nu_j} = S_{j+1})$ relative to the Markov chain ξ^j (see the equations (3.37) and (3.38)). We have

$$P_{Q_j}(\xi^j_{\nu_j} = S_j) = \sum_{n=0}^{\infty}[1 - P(Q_j \to \partial B_j)]^n P(Q_j \to S_{j+1}). \tag{3.61}$$

¿From now on we suppose that the configurations $S_{i+1} \in \partial B_i \cap \partial B_{i+1}$, $i = 1, \ldots, N-1$, $S_1 \in \partial B_1$, and $S_{N+1} \in \partial B_N$ are chosen so that

$$\min_{\sigma \in \partial B_j} H(\sigma) = H(S_j), \quad j = 1, \ldots, N. \tag{3.62}$$

Using (3.41) we can thus infer that, for every $\epsilon > 0$ and β sufficiently large,

$$P(Q_j \to \partial B_j) \leq \exp\{-\beta[H(S_j) - H(Q_j) - \epsilon]\}. \tag{3.63}$$

Let us suppose further that

$$H(S_j) > H(S_{j-1}), \tag{3.64}$$

and that we have the lower bounds

$$P(Q_j \to S_{j+1}) \geq \exp\{-\beta[H(S_{j+1}) - H(Q_j) + \epsilon]\} \tag{3.65}$$

and

$$P_{S_{j+1}}(\theta_{j+1} \leq \bar{t}^j_d) \geq \exp(-\epsilon\beta) \tag{3.66}$$

for every $\epsilon > 0$ and β sufficiently large. Combining the bounds (3.63) – (3.65), we get a lower bound (3.61). Using now this bound together with the definitions (3.50), (3.53), and (3.56), the equation (3.51), and the bound (3.63), we conclude that

$$P(\bar{\mathcal{F}}) \geq \exp\{-\beta[H(S_N) - H(S_1) + \epsilon]\} \tag{3.67}$$

for every $\epsilon > 0$ and β sufficiently large. In our particular case, the equation (3.62) and the inequalities (3.64) and (3.65) will be satisfied and we will get estimates of the form (3.66).

On the other hand, we get upper bounds on $P(\mathcal{G}^c)$ that are superexponentially small in β (see the condition (C1) in the proof of Proposition 1) yielding (3.59) for β sufficiently large. Thus we will get

$$P(\bar{\mathcal{F}} \cap \mathcal{G}) > \exp\{-\beta[H(S_N) - H(S_1) + \epsilon]\} \tag{3.68}$$

for every $\epsilon > 0$ and β sufficiently large.

Finally, to state Proposition 1, we need the definititon of yet another stopping time. Namely, considering standard octagons $Q(D_1, D_2)$ with $D_i \geq 3l^* - 2$, $i = 1, 2$, we take $\bar{\tau}$ to be the first hitting time to a standard octagon different from $Q(D_1, D_2)$,

$$\bar{\tau} = \tau_{\mathrm{S}_Q \setminus Q(D_1, D_2)}. \tag{3.69}$$

Here

$$\mathrm{S}_Q = \bigcup_{D_1, D_2} Q(D_1, D_2) \tag{3.70}$$

is the set of all standard octagons. Supposing that $D_2 \leq D_1$ with $3l^* - 2 < D_2 < D^*$, we shall use $E(L_2)$ as a shorthand for $E(D_1, D_2)$ defined in (3.14):

$$E(L_2) = 2[h(l^* - 1) - (2K - h)] + h(L_2 - 3) \equiv h(D_2 - 1) - 4K = h(L_2 - 1) - 2h\eta \tag{3.71}$$

for $L_2 > l^* + 1$ and

$$E(l^* + 1) = 2h(l^* - 1) - (2K - h). \tag{3.72}$$

for $L_2 = l^* + 1$.

Proposition 1 Consider a standard octagon $Q(D_1, D_2)$ with $D_2 \leq D_1$ and let

$$3l^* - 2 < D_2 < D^*. \tag{3.73}$$

Let, further,

$$\tau = \tau_{Q(D_1, D_2 - 1) \cup Q(D_1 - 1, D_2)} \text{ if } D_1 = D_2 \tag{3.74}$$

and

$$\tau = \tau_{Q(D_1 - 1, D_2)} \text{ if } D_2 < D_1. \tag{3.75}$$

Then, for every $\epsilon > 0$, one has

$$\sup_{\sigma \in Q(D_1,D_2)} P_\sigma(\tau < \exp\{(\beta(E(L_2)+\epsilon)\} \text{ and } \tau = \bar{\tau}) \xrightarrow[\beta\to\infty]{} 1. \tag{3.76}$$

Proof We suppose that $D_1 > D_2$. The case $D_1 = D_2$ does not present any supplementary difficulty and is left to the reader.

First, let us consider the case $L_2 > l^* + 1$. The main step in the proof will be to show that

$$P_\sigma(\tau_{\partial\mathcal{D}(D_1,D_2)} > \exp\{\beta(E(L_2)+\epsilon)\}) \xrightarrow[\beta\to\infty]{} 0 \tag{3.77}$$

and

$$P_\sigma(\tau_Q > \exp\{\beta(2K - h + \epsilon)\}) \xrightarrow[\beta\to\infty]{} 0 \tag{3.78}$$

for every $\epsilon > 0$ and $\sigma \in \mathcal{D}(D_1, D_2)$. Once (3.77) and (3.78) will be proven, we can reason in the following way.

¿From (3.77) we know, in particular, that starting from Q one reaches with high probability the set $\partial\mathcal{D}(D_1, D_2)$ before the time $\exp\{\beta(E(L_2)+\epsilon)\}$. Moreover, it is unprobable to reach $\partial\mathcal{D}(D_1, D_2)$ outside the set $\mathcal{S}(D_1, D_2)$ of configurations yielding minima of H on $\partial\mathcal{D}(D_1, D_2)$ (see the definitions (3.10) and (3.11)). Indeed, one clearly has

$$P_Q(\tau_{\partial\mathcal{D}\setminus\mathcal{S}} < \tau_{\partial\mathcal{D}}) \le P_Q(\tau_{\partial\mathcal{D}\setminus\mathcal{S}} < \exp\{\beta(E(L_2)+\epsilon)\}) + P_Q(\tau_{\partial\mathcal{D}} \ge \exp\{\beta(E(L_2)+\epsilon)\}) \tag{3.79}$$

for every $\epsilon > 0$ (we have dropped the explicit dependence on D_1, D_2). The first term on the right hand side can be bounded with the help of Lemma 3.4, taking into account that according to Lemma 3.3, for every $\bar{\sigma} \in \mathcal{S}(D_1, D_2)$ one has

$$H(\bar{\sigma}) = \min_{\sigma \in \partial\mathcal{D}(D_1,D_2)} H(\sigma) = H(Q(D_1, D_2)) + E(L_2). \tag{3.80}$$

Once reaching $\mathcal{S}(D_1, D_2)$, with high probability one of the two possibilities occurs. Either we descent to $Q(D_1 - 1, D_2)$ with a fixed nonvanishing probability, or we return back to

$\mathcal{D}(D_1, D_2)$. The saddles in $\mathcal{S}$ consist just from a contracted octagon united with a unit square protuberance and, with probability approaching 1 as $\beta \to \infty$, this protuberance is either cut off, or it is made stable by a flip of a minus spin adjacent to it. Actually, following the argument of the proof of Lemma 3.3 we can show that for every $\bar{\sigma} \in \mathcal{S}(D_1, D_2)$ one has

$$P_{\bar{\sigma}}(\tau = 1) > \frac{1}{|\Lambda|} \text{ and } P_{\bar{\sigma}}(\tau_{\mathcal{D}(D_1,D_2)} < \tau_{\mathcal{D}^c(D_1,D_2)\setminus\{\bar{\sigma}\}} \mid \tau > 1) \xrightarrow[\beta\to\infty]{} 1. \tag{3.81}$$

After returning to $\mathcal{D}(D_1, D_2)$ and reaching Q, according to (3.78), we can repeat the attempt and prove finally, with the help of the strong Markov property, that for every $\sigma \in \mathcal{D}(D_1, D_2)$ one has

$$P_\sigma(\tau < \exp\{\beta(E(L_2) + \epsilon)\}) \xrightarrow[\beta\to\infty]{} 1. \tag{3.82}$$

To prove that

$$P_Q(\tau = \bar{\tau}) \xrightarrow[\beta\to\infty]{} 1, \tag{3.83}$$

for $\bar{\tau}$ defined in (3.69), or, in other terms, that

$$P_Q(\tau < \bar{\tau}) \xrightarrow[\beta\to\infty]{} 0, \tag{3.84}$$

we observe that, since $L_2 < L^*$, one has

$$\min_{\omega:Q(D_1,D_2)\to \mathrm{S}_Q\setminus\{Q(D_1,D_2),Q(D_1-1,D_2)\}} \max_{\sigma\in\omega}[H(\sigma) - H(Q(D_1, D_2))] > E(L_2). \tag{3.85}$$

Thus, for all sufficiently small $\epsilon > 0$, the set

$$\bigcup_{D_1',D_2' \neq (D_1,D_2),(D_1-1,D_2)} \mathcal{D}(D_1', D_2')$$

cannot be reached in a time $\exp\{\beta(E(L_2) + \epsilon)\}$ with probability approaching one as $\beta \to \infty$.

For $L_2 = l^* + 1$, similar reasoning is slightly more complicated. Instead of (3.81) we introduce the set

$$\mathcal{C} = \mathcal{D}(D_1 - 1, D_2) \cup \{Q : (D_1(Q), D_2(Q)) = (D_1, D_2) \text{ and } \min(L_1(Q), L_2(Q)) < l^*\}, \tag{3.86}$$

for which, for every $\bar{\sigma} \in \mathcal{S}(D_1, D_2)$, one has

$$P_{\bar{\sigma}}(\tau_{\mathcal{D}(D_1,D_2)} < \tau_{\mathcal{D}^c(D_1,D_2)\setminus\{\bar{\sigma}\}} \mid \tau_{\mathcal{C}} > 1) \xrightarrow[\beta\to\infty]{} 1. \tag{3.87}$$

Moreover, (still $L_2 = l^* + 1$) we will show that

$$P_{\bar{\sigma}}(\tau < \exp\{\beta h(l^* - 2 + \epsilon)\} \mid \tau_{\mathcal{C}} = 1) \xrightarrow[\beta\to\infty]{} 1 \tag{3.88}$$

for any $\bar{\sigma} \in \mathcal{S}(D_1, D_2)$ and $\epsilon > 0$. Thus, noticing that

$$h(l^* - 2) < E(l^* + 1).$$

we can conclude that (3.82) holds true.

Hence, to prove the Proposition we are left with the proof of (3.77), (3.78), and for $L_2 = l^* + 1$ also (3.88). Actually, the crucial point is to get (3.77). The claim (3.78) will be a byproduct of the proof of (3.77) (see the condition (C3) below).

To derive the bound (3.77), we will look more carefuly on a possible way of reaching the set $\partial\mathcal{D}$ from a configuration $\sigma \in \mathcal{D}$. The trajectory will be characterized by visiting several special octagons (see Fig. 3.8): First, we use $\widetilde{Q}_0 \in Q(D_1, D_2, l^*, l^*, l^*, l^*)$ to denote the standard octagon representing our initial condition. We can suppose, without loss of generality, that it is centered — its upper left corner is the point $(-1/2, 1/2)$ of the dual lattice. As usually, we use $R(D_1, D_2)$ to denote the rectangle circumscribed to $\widetilde{Q}_0$. The octagon $\widetilde{Q}_1$ is contained in $\widetilde{Q}_0$; it has the same rectangular envelope $R(D_1, D_2)$ and it differs from $\widetilde{Q}_0$ only by the length of one of its oblique edges — namely, the first one in the clockwise numeration l_i, $i = 1, 2, 3, 4$ starting with the uppermost right one is of length $l^* + 1$, $l_1 = l^* + 1$. The length of all the remaining oblique edges is again l^*,

$$\widetilde{Q}_1 \in Q(D_1, D_2, (l_i)_{i=1,\dots,4}, l_1 = l^* + 1, l_2 = l_3 = l_4 = l^*).$$

Further, the octagon $\widetilde{Q}_2$ is obtained from $\widetilde{Q}_1$ by replacing the oblique edge l_2 (of length l^*) adjacent to the shortest coordinate edge by an edge of length $l^* + 1$. The octagon $\widetilde{Q}_2$ is thus again centered and has two oblique edges of length l^*+1 adjacent to the shortest coordinate edge. Finally, the octagon $\widetilde{Q}_3$ is the element of $Q(D_1, D_2-1, l^*, l^*, l^*, l^*)$ obtained from $\widetilde{Q}_2$ by erasing the shortest coordinate edge (the vertical one on the right hand side) adjacent to the oblique edges of length l^*+1. The saddle configurations $\widetilde{S}_i$, $i=1,2,3$, are obtained from $\widetilde{Q}_i$ by adding a unit square protuberance (the first one in lexicographic order — denoted by a dot in Fig. 3.8) to

1) the first (uppermost left) oblique edge in the case of $\widetilde{Q}_1$;

2) the second oblique edge (down on the right) in the case of $\widetilde{Q}_2$, and

3) one of the shortest coordinate edges — namely, to the vertical one on the right hand side — in the case of $\widetilde{Q}_3$.

Let us first consider the case $L_2 > l^*+1$.

To prove (3.77), we follow [KO1] and introduce a event $\mathcal{E}^s_\sigma$ (of shrinking) starting from an arbitrary σ in $\mathcal{D}(D_1, D_2)$, taking place in an interval of time $T_1 = T_1(\delta_1)$,

$$T_1(\delta_1) = \exp\{\beta(2K - h + \delta_1)\}, \tag{3.89}$$

with sufficiently small $\delta_1 > 0$ to be specified later, and such that:

I f $\mathcal{E}^s_\sigma$ takes place then necessarily the set $\partial\mathcal{D}$ is reached (in a particular manner) before the time $T_1(\delta_1)$.

T he probability $P(\mathcal{E}^s_\sigma)$ satisfies the uniform lower bound

$$\inf_{\sigma\in\mathcal{D}(D_1,D_2)} P(\mathcal{E}^s_\sigma) \geq \alpha \tag{3.90}$$

with α chosen large enough to satisfy

$$\lim_{\beta\to\infty} (1-\alpha)^{\frac{T_1}{T_2}} = 0. \tag{3.91}$$

Here $T_2 = T_2(\delta_2)$ (again, the constant $\delta_2 > 0$ will be specified later) is given by

$$T_2(\delta_2) = \exp\{\beta(E(L_2) + \delta_2)\}. \tag{3.92}$$

Fig. 3.8

Splitting now the interval T_2 into T_2/T_1 intervals of length T_1, we can argue that the probability that, during any such interval, one has not reached $\partial\mathcal{D}$ is at most $1-\alpha$, since not reaching $\partial\mathcal{D}$ in time T_1 means that $\mathcal{E}^s_\sigma$ certainly did not take place. Using now the strong Markov property and taking into account (3.91), we see that an attempt to realize the event $\mathcal{E}^s_\sigma$ not later than T_2 will be succesful with high probability for large β — the equality (3.77) follows.

Now, let us first describe the event $\mathcal{E}^s_\sigma$ of shrinking in words — a formal definition will follow.

Starting from any σ in $\mathcal{D}(D_1, D_2)$, we first descent to $\widetilde{Q}_0$. Then we stay for a time of the order $\exp\{\beta(2K-h)\}$ inside $\mathcal{B}(\widetilde{Q}_0)$, the basin of attraction of $\widetilde{Q}_0$, leaving it afterwards through the saddle $\widetilde{S}_1$. After reaching $\widetilde{S}_1$ we descent to $\widetilde{Q}_1$ in a unique step. Staying inside $\mathcal{B}(\widetilde{Q}_1)$, for a time of order $\exp\{\beta(2K-h)\}$, we leave through $\widetilde{S}_2$ from which, again in one step, we descent to $\widetilde{Q}_2$. Finally, again staying in $\mathcal{B}(\widetilde{Q}_2)$ for a time of order $\exp\{\beta(2K-h)\}$, we ascent to $\widetilde{S}_3 \in \mathcal{S}(D_1, D_2)$.

Let us turn to a formal definition of $\mathcal{E}^s_\sigma$.

For every σ in $\mathcal{D}(D_1, D_2)$ and every $t_0 \in \mathrm{N}$, let

$$\mathcal{E}^0_{\sigma,t_0} = \{\sigma_0 = \sigma,\, \tau_{\widetilde{Q}_0} = t_0\}. \tag{3.93}$$

Namely, $\mathcal{E}^0_{\sigma,t_0}$ describes the event, after starting from σ, of hitting in the moment t_0, for the first time, the octagon $\widetilde{Q}_0$. Now, consider the events $\mathcal{F}^{1,u}_{t^1_u}$, $\mathcal{F}^{1,d}_{t^1_d}$, $\mathcal{F}^{2,u}_{t^2_u}$, $\mathcal{F}^{2,d}_{t^2_d}$, and $\mathcal{F}^{3,u}_{t^3_u}$, defined by (3.47) – (3.50) with $N = 3$, $Q_1, Q_2, Q_3 = \widetilde{Q}_0, \widetilde{Q}_1, \widetilde{Q}_2$, $B_1, B_2, B_3 = \mathcal{B}(\widetilde{Q}_0), \mathcal{B}(\widetilde{Q}_1), \mathcal{B}(\widetilde{Q}_2)$, and $S_2, S_3, S_4 = \widetilde{S}_1, \widetilde{S}_2, \widetilde{S}_3$. Moreover, we take $\bar{t}^1_u = \bar{t}^2_u = \bar{t}^3_u = \exp\{\beta(2K-h+\delta)\}$ and $\bar{t}^1_d = \bar{t}^2_d = 1$. Further, we consider the time $\bar{t}_0 = \exp\{\beta(2K-h+\delta)\} \equiv T_1(\delta)$, ($\delta > 0$, sufficiently small, to be specified later) for the first descent from $\partial\mathcal{D}(D_1, D_2)$ to $\widetilde{Q}_0$. Finally, the configuration S_1 is one of the minimal saddles in $\partial\mathcal{B}(\widetilde{Q}_0)$ satisfying the equality

$$\min_{\sigma \in \partial\mathcal{B}(\widetilde{Q}_0)} H(\sigma) = H(S_1).$$

It is obtained by adding a unit square protuberance to one of the oblique edges of $\widetilde{Q}_0$. Clearly,

$$H(S_1) = H(\widetilde{Q}_0) + 2K - h. \tag{3.94}$$

Introducing now

$$\mathcal{E}^s_\sigma = (\bar{\mathcal{E}}^0_{\sigma,\bar{t}^0}; \bar{\mathcal{F}}^{1,u}; \mathcal{F}_1^{1,d}; \bar{\mathcal{F}}^{2,u}; \mathcal{F}_1^{2,d}\bar{\mathcal{F}}^{3,u}) \cap \mathcal{G}. \tag{3.95}$$

(cf. (3.57)), we will suppose that the following conditions are satisfied:

" (C1)" $P(\mathcal{G}^c) \le \exp(-e^{C\beta})$ for a constant $C > 0$.

" (C2)" $P(\bar{\mathcal{F}}_1^{j,d}) > \lambda$, $j = 1, 2$, for a constant $\lambda > 0$ independent of β.

" (C3)" $P(\bar{\mathcal{E}}^0_{\sigma,\bar{t}_0}) = \sum_{t_0=1}^{\bar{t}_0} P(\mathcal{E}^0_{\sigma,t_0}) \ge \exp(-\epsilon\beta)$ for every $\epsilon > 0$ and all β sufficiently large. Let us notice that the equality (3.62) and the inequality (3.64) are satisfied in our case, that the validity of (3.66) immediately follows from an explicit computation, based on the definition of Metropolis dynamics, taking into account that the transitions $\widetilde{S}_{j+1} \to \widetilde{Q}_j$, $j = 1, 2, 3$, are just single spin flip events, and, finally, that the validity of (3.65) is immediate by considering a particular event leading from Q_j to S_{j+1} consisting of a set of subsequent h-erosions in the sites adjacent from the interior to the concerned oblique or vertical edge of $\widetilde{Q}_0$, $\widetilde{Q}_1$, and $\widetilde{Q}_2$ (see Fig. 3.8). (The same argument actually proves also the stronger condition (C2).)

¿From the strong Markov property we get

$$P(\mathcal{E}^s_\sigma) = P(\bar{\mathcal{E}}^0_{\sigma,\bar{t}^0}) P[(\bar{\mathcal{F}}^{1,u}; \mathcal{F}_1^{1,d}; \mathcal{F}^{2,u}; \mathcal{F}_1^{2,d}\bar{\mathcal{F}}^{3,u}) \cap \mathcal{G}]. \tag{3.96}$$

Using (3.66) together with (3.65), we can imply, as we have seen, the lower bound (3.67). From (C1) and (3.67), for β sufficiently large, we get (3.59), (3.60), and thus also (3.68). Finally from (C3), (3.68), (3.94), and (3.96) we deduce that

$$\inf_{\sigma \in \mathcal{D}(D_1,D_2)} P(\mathcal{E}^s_\sigma) \ge \alpha \equiv \exp\{-\beta(E(L_2) - 2K + h + \epsilon)\} \tag{3.97}$$

for every $\epsilon > 0$ and β sufficiently large. The validity of (3.91) then follows from the bound (3.97) — for any fixed δ_2 (see the equation (3.92)) once we take, say, $\delta = \frac{\delta_1}{2} = \frac{\delta_2}{4}$ for δ from the definition of times $\bar{t}^i_u$.

Hence, to conclude the proof of (3.77), we only need to verify the conditions (C1), (C2), and (C3) above.

To get the bound (C1), we suppose that for every $j = 0, 1, 2$ there exist a time $\tilde{t}_u^j$ and, for every $\sigma \in \mathcal{B}(\widetilde{Q}_j)$, an event $\widehat{\mathcal{E}}_\sigma^j$ of escape from $\mathcal{B}(\widetilde{Q}_j)$ such that:

" (i)" The occurrence of $\widehat{\mathcal{E}}_\sigma^j$ implies that

$$\tau_{\partial\mathcal{B}(\widetilde{Q}_j)} < \tilde{t}_u^j \tag{3.98}$$

" (ii)" For every $\epsilon_0 > 0$ sufficiently small and for every β sufficiently large,

$$\inf_{\sigma\in B_j} P(\widehat{\mathcal{E}}_\sigma^j) \geq \frac{\tilde{t}_u^j}{\bar{t}_u^j}\exp(\epsilon_0\beta). \tag{3.99}$$

The superexponential estimate (C1) then directly follows from (3.98) and (3.99) with the help of the strong Markov property.

To construct the event $\widehat{\mathcal{E}}_\sigma^j$, we repeat, in the present simpler situation of escape from $\mathcal{B} = \mathcal{B}(\widetilde{Q}_j)$, a construction similar to the one discussed above in the case of escape from $\mathcal{D}$. Namely, we chose time $\tilde{t}_u^j = T_0 + 1$ with

$$T_0 = (8\widetilde{J} + 8K + 2h)|\Lambda|/h, \tag{3.100}$$

the time that suffices, starting from any configuration, to reach a local minimum (see the equation (3.5)), and set

$$\widehat{\mathcal{E}}_\sigma^j = (\widehat{\mathcal{E}}_{\sigma,T_0}^0; \widehat{\mathcal{F}}^{j,u}), \tag{3.101}$$

where

$$\widehat{\mathcal{F}}^{j,u} = \{\xi_0^j = \widetilde{Q}_j, \quad \xi_{\nu_j}^j = \widetilde{S}_j, \quad v_{\nu_j} = 1\} \tag{3.102}$$

and

$$\widehat{\mathcal{E}}_{\sigma,T_0}^0 = \{\sigma_0 = \sigma,\, \tau_{\widetilde{Q}_j} < T_0,\, \sigma_t = \widetilde{Q}_j \text{ for all } t \in [\tau_{\widetilde{Q}_j}, T_0]\}. \tag{3.103}$$

The saddle configurations $\widetilde{S}_j$ for $j = 1, 2$ (and 3) have already been defined. The configuration $\widetilde{S}_0$ is identical to S_1 introduced above and it is obtained by adding a unit square protuberance to one of the oblique edges of $\widetilde{Q}_0$.

The event $\widehat{\mathcal{E}}^j_\sigma$ thus describes, starting from a generic $\sigma \in \mathcal{B}(\widetilde{Q}_j)$, a descent to $\widetilde{Q}_j$ in a time shorter than T_0 and staying in $\widetilde{Q}_j$ up to T_0. The event $\widehat{\mathcal{F}}^{j,u}$ simply consists of creating of a K-protuberance on an oblique edge of $\widetilde{Q}_j$.

It is easy see that, given δ (see definition of $\overline{t}^j_u$), the bound (3.99) is satisfied provided $\epsilon_0 < \delta/4$.

As we already mentioned, the condition (C2) follows from the definition of $\widetilde{S}_1$ and $\widetilde{S}_2$ with $\lambda = \frac{1}{|\Lambda|}$. With this probability the spin chosen for updating is the one on the unit square protuberance of $\widetilde{S}_1, \widetilde{S}_2$, respectively.

The idea of the proof of the condition (C3) is as follows. Starting from any $\sigma \in \mathcal{D}(D_1, D_2)$, after a time of order T_0 (3.100), we descend to octagon Q in $\mathcal{D}(D_1, D_2)$ with high probability. Then, in a time of order $\exp\{\beta(2K - h + \tilde{\epsilon})\}$, with $\tilde{\epsilon} > 0$ such that

$$2K - h + \tilde{\epsilon} < h(l^* - 1) - \tilde{\epsilon}, \qquad (3.104)$$

many K-protuberances occur with high probability, but no total erosion of any oblique edge of length $l \geq l^*$ takes place.

To provide a formal proof we introduce again a Markov chain — this time the chain $\{\eta_n\}$ obtained by looking at our process σ_t at the times of passing through octagons. The space of states of our chain is the set Y of all octagons Q in Λ,

$$Y = \{Q; Q \subset \Lambda\}.$$

Let

$$\overline{v}_0 = 0, \quad \overline{u}_n = \inf\{t > \overline{v}_n, \overline{\sigma}_t \notin Y\}, \text{ and}$$

$$\overline{v}_{n+1} = \inf\{t > \overline{u}_n : \overline{\sigma}_t \in Y\}.$$

We set

$$\eta_n = \sigma_{\overline{v}_n}$$

and, for every $A \subset Y$, introduce

$$\bar{\nu}_A = \inf\{n; \eta_n \in A\}. \tag{3.105}$$

Let

$$\bar{Y} = \{Q : (D_1(Q), D_2(Q)) = (D_1, D_2)\} \supset \{Q \in \mathcal{D}(D_1, D_2)\} \tag{3.106}$$

and

$$Y^< = \{Q \equiv Q(D_1, D_2, (l_i)_{i=1,\dots,4}) \in \bar{Y} : l_i \le l^*,\ i = 1, \dots, 4\}. \tag{3.107}$$

We will prove that for all $\epsilon > 0$ one has

$$\sup_{Q \in Y} P_Q(\bar{\nu}_{Y^<} > \exp\{\beta(2K - h + \epsilon)\}) \xrightarrow[\beta\to\infty]{} 0. \tag{3.108}$$

Postponing the proof of (3.108), let us first show that for every $\sigma \in \mathcal{D}(D_1, D_2), \epsilon > 0$, and β sufficiently large,

$$P_\sigma(\tau_Y < \exp(\epsilon\beta)) > 1/2. \tag{3.109}$$

It implies, with the help of the strong Markov property, that for every $\epsilon > 0$, $\sigma \in \mathcal{D}(D_1, D_2)$, and β sufficiently large, one has

$$P_\sigma(\tau_Y > \exp(2\epsilon\beta)) < (1/2)^{\exp(\epsilon\beta)}. \tag{3.110}$$

To get the lower bound (3.109), one again uses the property of time T_0 — with a strictly positive probability independent of β, starting from any configuration one reaches a local minimum in a time at most T_0.

¿From (3.110) one gets, for every $A \subset \bar{Y}$,

$$P_Q(\tau_A < \exp(\epsilon\beta)\nu_A) \xrightarrow[\beta\to\infty]{} 1. \tag{3.111}$$

Hence, by (3.111) and (3.108) we get

$$P_Q(\tau_{Y^<} < \exp\{\beta(2K - h + \epsilon)\}) \xrightarrow[\beta\to\infty]{} 1 \tag{3.112}$$

for every $\epsilon > 0$. On the other hand, with the same reasoning as that one leading to (3.110), we obtain

$$\inf_{\sigma\in\mathcal{D}(D_1,D_2)} P_\sigma(\tau_{\bar{Y}} < T_0) > \exp(-\epsilon\beta) \tag{3.113}$$

for every $\epsilon > 0$ and β sufficiently large.

Now, let us prove (3.108). This can be obtained very easily, once we take ϵ in (3.108) sufficiently small — namely, $\epsilon = \tilde{\epsilon}$ satisfying the inequality (3.104). For every $Q = Q(D_1, D_2, (l_i)_{i=1,\dots,4})$ such that for a nonempty subset of indices, $J \subset \{1, \dots, 4\}$, one has $l_j \geq l^*$, $j \in J$, let $Y^<(Q)$ be the set

$$Y^<(Q) = \{Q' \equiv Q'(D_1, D_2, (l'_i)_{i=1,\dots,4}) : l'_j \leq l_j \text{ for all } j \in J\}. \tag{3.114}$$

We have

$$P(Q \to Y \setminus (Y^<(Q) \cup \{Q\})) = P_Q(\sigma_{\bar{v}_1} \notin Y^<(Q)) \leq \exp\{-\beta(h(l^* - 1) - \epsilon)\} \tag{3.115}$$

for every $\epsilon > 0$ and β sufficiently large. Indeed, the bound (3.115) follows easily by reversibility, in a similar way as the equation (3.41), since for every $Q \in \mathcal{D}(D_1, D_2)$ one has

$$\min_{\omega:Q\to Y\setminus(Y^<(Q)\cup\{Q\})} \max_{\sigma\in\omega} H(\sigma) \geq H(Q) + h(l^* - 1)\beta. \tag{3.116}$$

On the other hand, one easily sees that for every $\epsilon > 0$ and β sufficiently large,

$$P(Q \to Y \setminus \{Q\}) \equiv P_Q(\sigma_{\bar{v}_1} \neq Q) \geq \exp\{-\beta(2K - h + \epsilon)\}$$

and, moreover,

$$P(Q \to Y^<(Q) \setminus \{Q\}) \geq \exp -(2K - h + \epsilon)\beta.$$

It is easy to see that, starting from any configuration $\sigma \in Y^<$ we reach the minimum $\widetilde{Q}_0$, with high probability, in a time of order $\exp\{\beta h(l^* - 2)\}$. Indeed, in a time of the order

$$T_1 = \exp\{\beta(h(l^* - 2) + \epsilon_1)\}, \tag{3.117}$$

if ϵ_1 is sufficiently small, no K-protuberances (or K-erosions) or *a fortiori* elementary events involving bigger increments of energy, take place with probability larger than $\exp(-\epsilon_1\beta)$ for β sufficiently large. Hence, with the same probability, the only possible elementary events are h-erosions and recoveries. Taking this fact into account, for all $\sigma \in Y^{<}$, we get

$$P_\sigma(\tau_{\widetilde{Q}_0} < T_1) > e^{-\beta\epsilon}. \tag{3.118}$$

The proof of (3.118) can be obtained by adapting the argument of proof of Theorem 1 a) of NS.

We only present here the main idea leading to (3.118). Starting from any $\sigma \in \mathcal{B}(\widetilde{Q}_0)$, there is a probability larger than $\exp\{-\beta(h(l^*-2)-\epsilon)\}$ to completely erode, in a time l^*-1, any oblique edge of $\widetilde{Q}_0$ of the length $\leq l^*-1$. Hence, in a time of order T_1, we certainly have to reach $\widetilde{Q}_0$ before T_1 since the circumscribed octagon can not grow and since *a fortiori* we know from the inequality

$$h(l^*-1) > (2K-h)$$

that no oblique edge $l \geq l^*$ can be completely eroded.

¿From (3.112), (3.113), and(3.118) we get (C3). Namely, for every $\epsilon > 0$ one has

$$P(\bar{\mathcal{E}}^0_{\sigma,\bar{t}_0}) = \sum_{t_0=1}^{\bar{t}_0} P(\mathcal{E}^0_{\sigma,t_0}) \geq \exp(-\epsilon\beta). \tag{3.119}$$

This concludes the proof of Proposition 1 when $L_2 > l^*+1$.

Consider now the case $L_2 = l^*+1$ (supposing always $L_1 > L_2$). The proof of Proposition 1, in this case, is similar to the one in the case $L_2 > l^*+1$. For any $\sigma \in \mathcal{D}(D_1, D_2)$ we will again introduce an event $\mathcal{E}^s_\sigma$. The main differences can be summarized in the following 2 points.

1) It follows from Lemma 3.3 above that the minimum of H in $\partial\mathcal{D}(D_1, D_2)$ is not achieved in $\widetilde{S}_3$ but in $\widetilde{S}_2$ (cf. Fig. 3.8) and in $\hat{S}_2$ defined as the saddle configuration obtained from $\widetilde{Q}_1$

by eroding the last $L_2 - 2 = l^* - 1$ unit squares adjacent to the coordinate edge of length $L_2 - 1 = l^*$ (in other words, $\hat{S}_2$ is obtained by adding a unit square protuberance to the coordinate edge of length $l^* + 2$ of an octagon in $Q(D_1 - 1, D_2, l^*, l^* - 1, l^*, l^*)$ — see Fig. 3.9).

Fig. 3.9

2) The path starting from $\widetilde{S}_2$ and reaching $\widetilde{Q}_3$ is not a pure descent anymore, but it involves tunnelling. We have

$$H(\widetilde{S}_2) - H(\widetilde{Q}_0) = H(\hat{S}_2) - H(\widetilde{Q}_0) = 2h(l^* - 1) - (2K - h) = E(l^* + 1),$$

and

$$H(\widetilde{S}_2) = H(\hat{S}_2) > H(\widetilde{S}_3) = H(\widetilde{Q}_0) + 2h(l^* - 1) - (2K - h) + h(l^* - 2) - (2K - h).$$

Let $D_1 > D_2 = l^* + 1 + 2(l^* - 1)$, i.e. $L_2 = l^* + 1$. The configurations $\widetilde{Q}_0, \widetilde{Q}_1, \widetilde{Q}_2; \widetilde{S}_1, \widetilde{S}_2, \widetilde{S}_3$ are defined in exactly the same way as in the case $L_2 > l^* + 1$.

Again, we define for every $\sigma \in \mathcal{D}(D_1, D_2)$, the event $\mathcal{E}^0_{\sigma,t_0}$ (cf. (3.93)) in the same manner like before.

Now, let

$$Q_1, Q_2 = \widetilde{Q}_0, \widetilde{Q}_1; \quad S_1, S_2 = \widetilde{S}_1, \widetilde{S}_2$$

and let $\bar{t}_0 = \exp\{\beta(2K - h + \delta)\}$, $\bar{t}^1_u = \bar{t}^2_u = \exp\{\beta(2K - h + \delta)\}$, and $t^1_d = 1$. We define

$$\mathcal{E}^s_\sigma = (\bar{\mathcal{E}}^s_{\sigma,\bar{t}^0}; \bar{\mathcal{F}}^{1,u}; \mathcal{F}^{1,d}_1; \bar{\mathcal{F}}^{2,u}) \cap \mathcal{G}. \tag{3.120}$$

(cf. (3.57) and (3.94)). We get

$$\inf_{\sigma\in\mathcal{D}(D_1,D_2)} P(\mathcal{E}^s_\sigma) \geq \alpha \equiv \exp\{-\beta(E(l^*+1)-2K+h+\epsilon)\}. \tag{3.121}$$

One verifies (3.121) in the exactly same way as in the case $L_2 > l^*+1$. We only have to check, in our case when $L_2 = l^*+1$, the validity of the bound (3.119).

However, the proof of (3.119) is even simpler since now the number of possible cases to be considered, namely the number of Q's in $\bar{Y}$, is much smaller; in fact $E(l^*+1) < hl^*$ and then no oblique edge of length bigger than l^*+1 can appear with high probability before the time $\exp\{\beta E(l^*+1)\}$.

Finally, to conclude the proof of Proposition 1 we have to verify (3.88). We remark that entering into the set $\mathcal{C}$ (see the equation (3.86)) from $\widetilde{S}_2$ in one step means to descent to $\widetilde{Q}_2$. The octagon $\widetilde{Q}_2$ contains a vertical edge of length l^*-1. Thus, to get (3.88), we reason in exactly the same way as when proving the equation (3.118). We leave the details to the reader. This concludes the proof of Proposition 1. ∎

In Proposition 1 we were considering the shrinking of a standard octagon with $D_1 \geq D_2 > 3l^*-2$. Now we turn to the case $D_1 > D_2 = 3l^*-2$. It means that $L_2 = l^*$ and

$$E(L_2) = E(l^*) = h(l^*-1) \tag{3.122}$$

according to (3.14) and (2.18).

Proposition 2 Consider a standard octagon $Q(D_1, D_2)$ with

$$D_1 > D_2 = 3l^*-2.$$

Then, for all $\epsilon > 0$, one has

$$P_{Q(D_1,D_2)}(\tau_{Q(D_1-1,D_2)} > \exp\{\beta(E(l^*)+\epsilon)\}) \xrightarrow[\beta\to\infty]{} 0. \tag{3.123}$$

Proof The proof is obtained along the same lines as in the case $L_2 = l^*+1$ above. The main difference (and simplification) is that typical paths to the saddle configurations in $\mathcal{S}(D_1, D_2)$

are now purely uphill, while the paths from $\mathcal{S}(D_1, D_2)$ to $Q(D_1-1, D_2)$ involve two, instead of one, tunnelling phenomena.

Let the octagons $\widetilde{Q}_0, \widetilde{Q}_1, \widetilde{Q}_3$, and the saddle configuration $\widetilde{S}_1$ be defined as for $L_2 \geq l^*+1$. In particular $\widetilde{S}_1$ is obtained from $\widetilde{Q}_1 \in Q(D_1, D_2, l^*+1, l^*, l^*, l^*)$ by adding a unit square to the oblique edge of length l^*+1. Moreover, let S_1^* be the saddle configuration obtained from the centered octagon $\widehat{Q}_1$ in $Q(D_1-1, D_2, l^*-1, l^*-1, l^*, l^*)$ by adding a unit square to its right hand vertical edge of length l^*+2.

We have $H(\widetilde{S}_1) = H(S_1^*) = H(\widetilde{Q}_0) + h(l^*-1)$.

The minimum of H in $\partial\mathcal{D}(D_1, D_2)$ is now reached, again acording to Lemma 3.3, in $\widetilde{S}_1$ and S_1^*. We define $\mathcal{C}$ as in (3.86) and the shrinking event $\mathcal{E}_\sigma^s$ simply by

$$\mathcal{E}_\sigma^s = (\bar{\mathcal{E}}^0_{\sigma,\bar{t}_0}; \bar{\mathcal{F}}^{1,u}) \bigcap \mathcal{G},$$

where the event $\bar{\mathcal{E}}^0_{\sigma,\bar{t}_0}$ is, for all $\sigma \in \mathcal{D}(D_1, D_2)$, defined in terms of events $\mathcal{E}^0_{\sigma,t_0}$, as in (3.44), while $\mathcal{E}^0_{\sigma,t_0}$ is defined like in (3.93); for $\bar{t}^0$ we take $\bar{t}^0 = \exp(h(l^*-2)+\delta)$ and

$$\mathcal{F}^{1,u} = \{\xi_0^1 = \widetilde{Q}_0,\ \xi_{\nu_1}^1 = \widetilde{S}_1\},$$

$$\mathcal{G} = \{\tilde{\tau}_{\partial\mathcal{B}(\widetilde{Q}_0)} \leq \bar{t}_u\}, \text{ and}$$

$$\bar{t}_u = \exp\{\beta(2K-h+\delta)\}.$$

Proposition 2 follows once we prove, in the present case, the bounds (3.88) and (3.119).

To get these two bounds we again argue as in the proof of (3.118) for the case $L_2 = l^*+1$. In particular, we notice that now, entering into $\mathcal{C}$ in one step actually means to descent to $\widetilde{Q}_1$. The octagon $\widetilde{Q}_1$ itself now has a coordinate edge (the vertical one on the righ hand side) of length l^*-1. Thus, in a time $t = \exp\{\beta(h(l^*-2)+\delta)\}$, with δ so small that

$$h(l^*-2)+\delta < (2K-h)-\delta,$$

no K-protuberance (or elementary events involving even higher increment in energy) takes place with high probability and, as a conseqence, the vertical edge of length l^*-1 will be completely eroded. In this way we reach the octagon

$$Q_2^* \in Q(D_1-1, D_2, l^*, l^*-1, l^*, l^*),$$

from which, in time $t = \exp\{\beta(h(l^*-2)+\delta)\}$, with the help of the same mechanism, we descent to $\widetilde{Q}_3$. We leave the details to the reader. ∎

Propositions 3 A–C describe subsequent shrinking, once we reached a regular octagon $Q(l^*)$. In the first step it shrinks by cutting an arbitrary one, coordinate or oblique, edge.

Proposition 3 A Consider a regular octagon $Q(l^*)$. Let G_1 be the set of (not standard) octagons

$$Q(D_1, D_2, l^*-1, l^*, l^*, l^*-1), \quad D_1 = 3l^*-2, \quad D_2 = 3l^*-3,$$

i.e.

$$L_1 = l^*+2, \quad L_2 = L_3 = L_4 = l^* \text{ (modulo rotations)} ,$$

and G_2 be the set of octagons

$$Q(D_1, D_2, l^*+1, l^*, l^*, l^*), \quad D_1 = D_2 = 3l^*-2,$$

$$L_1 = l^*-1 = L_2, \quad L_3 = L_4 = l^* \text{ (modulo rotations).}$$

Then, for any $\epsilon > 0$, one has

$$P_{Q(l^*)}(\tau_{G_1 \cup G_2} > \exp\{\beta(E(l^*)+\epsilon)\}) \xrightarrow[\beta\to\infty]{} 0. \tag{3.124}$$

Proof The proof is a straightforward adaptation of the proof of Proposition 2. ∎

Thus, starting from $Q(l^*)$ we reach an octagon with at least one side shorter than l^*. This condition is maintained also during the subsequent shrinking and we get:

Proposition 3 B Consider an octagon Q with the values d_1, d_2, D_1, D_2 such that

$$\begin{aligned} \text{either } \min(d_1, d_2) &< 4l^* - 2, \\ \text{or } \min(D_1, D_2) &< 3l^* - 2. \end{aligned} \tag{3.125}$$

Then, for every $\epsilon > 0$,

$$P_Q(\tau_{-\underline{1}} > \exp\{\beta(h(l^* - 2) + \epsilon)\}) \xrightarrow[\beta\to\infty]{} 0. \tag{3.126}$$

Proof We give only a sketch of the proof leaving the details to the reader. (See also the discussion of the growth event $\mathcal{E}^{(r)}$ introduced in Section 5.) We first observe that by the hypothesis, one has initialy at least one edge of length $l < l^*$. In a time of order $\exp[\beta(h(l^* - 2) + \delta]$, with

$$h(l^* - 2) + \delta < (2K - h) - \delta,$$

the only elementary processes taking place are h-erosions and recoveries (no K-protuberance or even more unlikely elementary events) with high probability, for large β. Hence, one of the minimal sides will be, with high probability for large β, eroded as it follows again from the arguments of proof of Theorem 1 a) in NS.

Now, after erosion of any of these edges, we pass to another octagon, with d_1, d_2, D_1 or D_2 decreased by one, in which there are still short ($\leq l^* - 1$) edges. Continuing this process, we finally reach the configuration $-\underline{1}$. ∎

Actually, the process of shrinking after reaching $Q(l^*)$ (resp. $G_1 \cup G_2$) may be described in greater detail. It will be useful to consider at once a collection of steps leading from a regular octagon to another regular octagon of smaller size. To describe it, we will introduce several simple geometrical notions. First, we want to define a *cut* operation of an oblique or coordinate edge.

We say that we pass from Q to $Q' \subset Q$ via an *oblique cut* if Q' is obtained from Q by eliminating all the unit squares adjacent from the interior to an oblique edge of Q. As a result of an oblique cut the length of the cut oblique edge increases by one whereas the lengths of the two coordinate adjacent edges decrease by one; d_1 or d_2 decrease by one.

For example, an oblique cut of a NW-SE edge with extremes y_1, x_2 in the octagon on Fig. 2.2, leads from $d_1, d_2, l_1, l_2, l_3, l_4$ to $d_1 - 1, d_2, l_1 + 1, l_2, l_3, l_4$. If we consider the corresponding values of the parameters $D_1, D_2, L_1, L_2, L_3, L_4$ characterizing the same octagon we pass to $D_1, D_2, L_1 - 1, L_2 - 1, L_3, L_4$.

We define also a coordinate (horizontal or vertical) cut on Q as the elimination of all the unit squares adjacent from the interior to a coordinate edge. As a result of a coordinate cut the corresponding length of the cut coordinate edge increases by two whereas the two oblique adjacent edges decrease by one, the length D_1 or D_2 decreases by one.

For example, by a horizontal cut of the edge with extremes x_1, y_1 (see Fig.2.2) we pass from $D_1, D_2, L_1, L_2, L_3, L_4$ to $D_1, D_2 - 1, L_1 + 2, L_2, L_3, L_4$, or, equivalently, from $d_1, d_2, l_1, l_2, l_3, l_4$ to $d_1, d_2, l_1 - 1, l_2, l_3, l_4 - 1$.

Since cuts of all sides will participate in subsequent shrinking, it is useful to label all sides of an octagon Q in a united way. Namely, we use $\mathcal{L}_i,\ i = 1, \ldots, 8$, to denote the eight edges of Q by taking $\mathcal{L}_i = l_i$, $\mathcal{L}_{i+4} = L_i$, $i = 1, \ldots, 4$.

We say that a cut of Q (oblique or coordinate) is *canonical* if it acts on an edge $\mathcal{L}_j$ of *minimal* length,

$$\mathcal{L}_j = \min_{i=1,\ldots,8} \mathcal{L}_i$$

Lemma 3.5 Consider a regular octagon in $Q(l),\ l > 2$. Suppose, we apply to Q sequentially a series of arbitrary canonical cuts (i.e., we always arbitrarily choose the edge to cut among the ones of minimal length). Then, for every l and for any such sequence

" i)" After exactly 14 canonical cuts we always reach an element of $Q(l-1)$. Namely, a regular octagon with the edge length decreased by one. We use $\mathcal{M}(l)$ to denote the set of all sequences of canonical contractions from $Q(l)$ to $Q(l-1)$.

" ii)" Any sequence $M = \{Q(l), Q^{(1)}, \dots, Q^{(13)}, Q^{(14)} \in Q(l-1)\} \in \mathcal{M}(l)$ contains only "almost regular" octagons in the sense that the differences in the lengths of any two oblique or coordinate edges in any $Q \in M$ is, for every $M \in \mathcal{M}(l)$, at most 3, $\max \mathcal{L}_i - \min \mathcal{L}_i \leq 3$. Moreover, the minimal possible length of an edge during any canonical sequence M is $l-2$, while the maximal one is $l+2$.

" iii)" In any sequence $M \in \mathcal{M}(l)$ we perform exactly 4 oblique cuts in the NE-SW direction, 4 cuts in NW-SE direction, 3 horizontal cuts, and 3 vertical cuts. There is 1 cut of an edge of length l, 9 cuts of edges of length $l-1$ and 4 cuts of edges of length $l-2$.

Proof The proof is a straightforward exercise. ∎

We will now introduce some notions that will be useful to describe the time dependence during a typical shrinking of a subcritical droplet.

Let $\tau_0, \tau_1, \dots, \tau_n, \dots$ be random times in which our process σ_t visits (after a change) the set $\mathcal{Q}$ of configurations containing a unique octagon:

$$\tau_0 = \inf\{t \geq 0 : \ \sigma_t \in \mathcal{Q}\},$$

$$\tau_{n+1} = \inf\{t > \tau_n : \ \sigma_t \in \mathcal{Q}\}, \quad n = 0, 1, 2, \dots.$$

Given l, $2 \leq l \leq l^*$, and $\epsilon > 0$, we say that σ_t is an *ϵ-canonical contraction path* from $Q(l)$ to $Q(l-1)$ if

$$\tau_0 = 0, \ \sigma_0 \in Q(l), \ \sigma_{\tau_1} \in Q^{(1)}, \dots, \sigma_{\tau_{14}} \in Q^{(14)},$$

where

" i)" $(Q^{(1)}, \dots, Q^{(14)} \equiv Q(l-1))$ is an element of the set $\mathcal{M}(l)$ of canonical contractions,

" ii)" $\exp\{\beta(h(\hat{\mathcal{L}}^{(i)} - 1) - \epsilon)\} < \tau_{i+1} - \tau_i < \exp\{\beta(h(\hat{\mathcal{L}}^{(i)} - 1) + \epsilon)\}, \ i = 1, \dots, 14$, where $\hat{\mathcal{L}}^{(i)} = \min_{j=i,\dots,8} \hat{\mathcal{L}}_j^{(i)}$ and $\hat{\mathcal{L}}_j^{(i)}$ are the lengths of the edges of $Q^{(i)}$.

Proposition 3 C Let G_1, G_2 be the set of octagons defined in Proposition 3 A and let $\epsilon > 0$. Then

$$P_{G_1 \cup G_2}(\tau_{-\underline{1}} > \exp\{\beta(h(l^* - 2) + \epsilon)\} \xrightarrow[\beta \to \infty]{} 0. \tag{3.127}$$

Moreover, with probability tending to one as $\beta \to \infty$,

" i)" starting from $G_1 \cup G_2$ our process will follow the remaining 13 steps of an ϵ-canonical contraction path up to $Q(l^* - 1)$,

" ii)" starting from $Q(l^* - 1)$ it will follow an ϵ-canonical contraction path up to $Q(l^* - 2)$ and so on up to $Q(2)$,

" iii)" finally, it will persist in $Q(2)$ for a time $t < \exp\{\beta(h + \epsilon)\}$ and then, after an h-erosion, it will proceed downhill to $-\underline{1}$.

Proof The validity of (3.127) is a corollary of Proposition 3 A.

The statements i) and ii) follow from Lemma 3.5 and the fact that in a time less than $\exp\{\beta(h(l^* - 2) + \epsilon)\}$, for ϵ sufficiently small, the only elementary events are, with high probability for β large, h-erosions and recoveries.

The statement iii) is immediate. ∎

Remark By Lemma 3.5 we get a lot of additional geometrical information about an ϵ-canonical contraction path that could have been, actually, included in the statement of Proposition 3 C.

Finally, we consider the much simpler case of growth of supercritical octagons.

Proposition 4 Let $Q(D_1, D_2)$ be a standard octagon with $D_2 \leq D_1$ such that

$$L_2 = D_2 - 2(l^* - 1) \geq L^*.$$

Let

$$\hat{\tau} = \tau_{Q(D_1, D_2+1) \cup Q(D_1+1, D_2)}. \tag{3.128}$$

Then, for every $\epsilon > 0$, one has

$$\sup_{\sigma \in Q(D_1,D_2)} P_\sigma(\hat{\tau} < \exp\{\beta(2J - 4K - h + \epsilon)\} \text{ and } \hat{\tau} = \bar{\tau}) \xrightarrow[\beta\to\infty]{} 1, \tag{3.129}$$

where $\bar{\tau}$ has been defined in (3.69). Proof According to Lemma 3.3, we have

$$\inf_{\omega:\sigma_0 \to \mathcal{D}^c(D_1,D_2)} \sup_{\sigma\in\omega} H(\sigma) = \min_{\sigma \in \partial\mathcal{D}(D_1,D_2)} H(\sigma) = E(D_1, D_2) = H(Q)+2J-4K-h. \tag{3.130}$$

In the present case, $L_2 > L^*$, the configurations in $\mathcal{S}(D_1, D_2)$, minimizing H on $\partial\mathcal{D}(D_1, D_2)$, are obtained by adding a unit square protuberance to one of the coordinate edges of an octagon in $Q(D_1, D_2)$. Consider the time $t(\delta) = \exp\{\beta(2J - 4K - h + \delta)\}$. It follows from (3.130) and Lemma 3.4 that if δ is sufficiently small, any $\sigma \in \partial\mathcal{D}\setminus\mathcal{S}$ cannot be reached before $t(\delta)$ with a probability approaching one as $\beta \to \infty$. Let us use $\widetilde{Q}_0$ to denote the octagon in $Q(D_1, D_2)$, corresponding to our initial condition and $\hat{S}_1$, the saddle in $\mathcal{S}(D_1, D_2)$ obtained by adding to $\widetilde{Q}_0$ the first unit square protuberance to its vertical right side.

For every $\sigma \in \mathcal{D}(D_1, D_2)$, let $\mathcal{E}^0_{\sigma,t_0}$ and $\bar{\mathcal{E}}^0_{\sigma,\bar{t}_0}$ be defined as in (3.93) and (3.44) with $\bar{t}_0 = \exp\{\beta(2K - h + \bar{\delta})\}$ with $\bar{\delta}$ sufficiently small. We notice that, by Lemma 3.4, before the time $t(\delta)$ one cannot see, with probability approaching one as $\beta \to \infty$, any J-protuberance occuring on a configuration $\bar{\sigma} \in \mathcal{D}(D_1, D_2)$ that would differ from $\widetilde{Q}_0$. Otherwise we would touch the configuration

$$\sigma' = \bar{\sigma} + J\text{-protuberance}$$

with $H(\sigma') - H(\widetilde{Q}_0) \geq 2J - 4K - h + \Delta$, for some positive Δ. Thus we can apply the same argument as that one used in the case $L_2 < L^*$. In this way it is easy to get (3.119).

Now, consider the set up for introducing our auxiliary Markov chains with $N = 1$, $Q_1 = \widetilde{Q}_0$, $B_1 = \mathcal{B}(\widetilde{Q}_0)$, $S_2 = \hat{S}_1$, and S_1 — the saddle obtained from $\widetilde{Q}_0$ by adding a K-protuberance to one of its oblique edge,

$$\min_{\sigma\in\partial\mathcal{B}(\widetilde{Q}_0)} H(\sigma) = H(S_1).$$

For every $\sigma \in \mathcal{D}(D_1, D_2)$ consider the event (of growth)

$$\mathcal{E}^g_\sigma = (\bar{\mathcal{E}}^0_{\sigma,\bar{t}_0}; \bar{\mathcal{F}}^{1,u}) \bigcap \mathcal{G},$$

where

$$\mathcal{F}^{1,u} = \{\xi^1_0 = \widetilde{Q}_0 \quad \xi^1_{\nu_1} = \hat{S}_1\},$$

$$\mathcal{G} = \{\tilde{\tau}_{\partial B(\widetilde{Q}_0)} \leq \bar{t}_u\},$$

and

$$\bar{t}_u = \exp\{\beta(2K - h + \tilde{\delta})\}.$$

As in the case $L_2 < L^*$, we deduce the analogue of (3.97). Namely, for every $\epsilon > 0$ and β sufficiently large,

$$\inf_{\sigma \in \mathcal{D}(D_1,D_2)} P(\mathcal{E}^g_\sigma) \geq \alpha \equiv \exp\{-\beta[(2J - 4K - h) - (2K - h) + \epsilon]\}. \tag{3.131}$$

Hence, by a recurrence argument similar to the one given in estimates (3.90), (3.91), with the help of the strong Markov property, we get

$$P_\sigma(\tau_{\partial\mathcal{D}} > \exp\{\beta(2J - 4K - h + \epsilon)\}) \underset{\beta\to\infty}{\longrightarrow} 0 \tag{3.132}$$

for every $\epsilon > 0$ and any $\sigma \in \mathcal{D}(D_1, D_2)$. Then, again with the help of the strong Markov property, we get the desired results in the same way as in the proof of Proposition 1 since, again,

$$P_{\bar{\sigma}}(\hat{\tau} = 1) > \frac{1}{|\Lambda|}, \quad P_{\bar{\sigma}}(\tau_{\mathcal{D}(D_1,D_2)} < \tau_{\mathcal{D}^c(D_1,D_2)\setminus\{\bar{\sigma}\}} | \hat{\tau} > 1) \underset{\beta\to\infty}{\longrightarrow} 1. \tag{3.133}$$

Once more, we leave the details to the reader. ∎

. Global saddle point

Similarly as in KO 1, the proof of our Theorems about "escape time and optimal route" are based on the existence of a set $\mathcal{A}$ with the following properties:

" i)" For every $\sigma \in \mathcal{A}$ and any $\epsilon > 0$ one has

$$\lim_{\beta \to \infty} P_\sigma(\tau_{-\underline{1}} < \tau_{+\underline{1}}) = 1$$

and

$$\lim_{\beta \to \infty} P_\sigma(\tau_{-\underline{1}} < \exp\{\beta(h(D^* - 1) + \epsilon)\}) = 1 \ .$$

" ii)" Every path ω starting in $-\underline{1}$ and ending in $+\underline{1}$ has to pass through the boundary $\partial\mathcal{A}$ of $\mathcal{A}$ defined by

$$\partial\mathcal{A} = \{\sigma \notin \mathcal{A}, \text{ there exists } x \text{ such that } \sigma^{(x)} \in \mathcal{A}\} \ .$$

" iii)" The minimal energy in $\partial\mathcal{A}$ is attained for "protocritical" (global saddle) configurations $\sigma \in \mathcal{P}$ — a single unit square attached to the longer coordinate side of a standard octagon $Q(D^* - 1, D^*)$ — with the energy $E^* \equiv H(Q(D^* - 1, D^*)) + 2J - 4K - h$. All configurations in $\partial\mathcal{A}$ that are not of this form have the energy by at least h higher.

As a first step toward the construction of the set $\mathcal{A}$ we construct for every configuration σ (from certain class) a configuration $\overline{\sigma}$ such that

$$S\sigma \prec \overline{\sigma}$$

for every standard S. Here we use the natural order on the set of configurations:

$$\sigma_1 \prec \sigma_2 \text{ iff } \{x : \sigma_1(x) = +1\} \subset \{x : \sigma_2(x) = +1\} \ .$$

In other words, the configuration $\overline{\sigma}$, to be called *the blown up envelope of* σ, is the "maximal" one (and actually even larger) to which we can arrive applying a standard sequence.

We begin by constructing $\overline{\sigma}$ in the case when σ corresponds to a single droplet $C = C(\sigma)$ (i.e. the set $C(\sigma)$ is connected). Suppose also that the rectangular envelope $R(C)$ does not wind around the torus and consider the monotone envelope $M = M(C)$. The set M consists of monotonous blocks connected in corners, of the form shown on Fig. 2.1, where four edges meet. We say that such point is a *bottleneck* of the set M, once the intersection of the boundary ∂M with each of the two outside right angles touching in this point has at least along one side the length one (Fig. 4.1).

Fig. 4.1 A set M with bottlenecks denoted by a thick dot. The points denoted by an arrow cease to be bottlenecks after "enveloping the components".

Disconnecting now the set M in bottlenecks, consider the union $M^{(1)}$ of octagonal envelopes of corresponding components. Some of the resulting points that were bottlenecks for M may not be such any more for the resulting set $M^{(1)}$. Considering only its bottlenecks we repeat the procedure and iterate until the set of bottlenecks does not change. We use $N(C)$ to denote the droplet constructed above from the set C and call it a *string* (i.e. a string is a monotonous droplet whose components, after disconnecting its bottlenecks, are octagons with oblique sides $l_i \geq 1$ — not necessarily larger or equal 2).

The configuration $\overline{\sigma}$ corresponding to the droplet $N(C(\sigma))$ is the sought blown up envelope with the desired properties.

Lemma 4.1a Let σ be a configuration with a single droplet $C(\sigma)$ contained in a rectangle that is not wrapped around the torus. Then

" i)" $H(\overline{\sigma}) \leq H(\sigma)$,

" ii)" $\sigma \prec \overline{\sigma}$ and $S\sigma \prec \overline{\sigma}$ for every standard sequence. Proof Since taking octagonal envelopes of components decreases the energy, for our proof of (i) it is sufficient to show that

$$H(\sigma) \geq H(M)$$

(we identify the droplet M with the corresponding configuration). We will show that there exists a sequence of configurations with decreasing energy and leading from σ to M.

First, we show that the holes inside $C = C(\sigma)$ (minus spins inside $\partial_{\text{out}} C$ — see Fig. 3.2) can be filled up. Consider the boundary of the holes, $\partial C \setminus \partial_{\text{out}} C$, and consider the lowermost horizontal line touching it. It touches it along a certain number of segments, with the spins around each of them necessarily taking the values shown in Fig. 4.2.

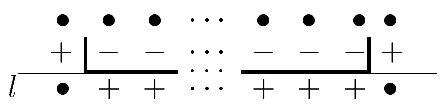

Fig. 4.2

Notice, in particular, the + spins below the line. If any of them were replaced by $-$, the line l would be pushed lower. The spins denoted by dots are arbitrary. Labelling the spins in the first row as $\sigma_{\mathrm{I}}, \sigma_1, \ldots, \sigma_k, \sigma_{\mathrm{II}}$ and those in the third line that are not fixed as $\sigma_{\mathrm{III}}, \sigma_{\mathrm{IV}}$, we get an energy decrease after flipping simultaneously all minus spins in the second row.

Namely,

$$-\Delta H = 2\widetilde{J} + \widetilde{J}\sum_{i=1}^{k}(1+\sigma_i) + K(1+\sigma_{\mathrm{I}}+\sigma_{\mathrm{III}}+\sigma_2) +$$
$$+ K(1+\sigma_{\mathrm{II}}+\sigma_{\mathrm{IV}}+\sigma_{k-1}) + K\sum_{i=2}^{k-1}(2++\sigma_{i-1}+\sigma_{i+1}) + hk$$

whenever $k \geq 2$ and

$$-\Delta H = 2\widetilde{J} + \widetilde{J}(1+\sigma_1) + K(\sigma_{\mathrm{I}}+\sigma_{\mathrm{II}}+\sigma_{\mathrm{III}}+\sigma_{\mathrm{IV}}) + h$$

for $k=1$. In both cases its minimum is attained if all spins denoted by dots are minus, $\sigma_i = -1, i = 1, \ldots, k,\ \sigma_{\mathrm{I}} = \sigma_{\mathrm{II}} = \sigma_{\mathrm{III}} = \sigma_{\mathrm{IV}} = -1$, and we get

$$-\Delta H \geq 2\widetilde{J} - 4K + h > 0$$

acording to our assumption $K \leq \frac{1}{2}(\widetilde{J} - h)$ (cf. Lemma 3.1). Flipping thus all considered minus spins above the line l, it will be pushed higher. Iterating the process we finally erase all holes in C, decreasing in the same time the energy of the configuration. Hence, we can suppose that $\partial_{\mathrm{out}} C = \partial C$.

In a similar way we can prove that filling the droplet up to M we further decrease the energy. Namely, consider a right angle, say $\{(x,y) \in \mathrm{R}^2; x \geq x_0, y \geq y_0\}$, such that C does not intersect its interior and such that it touches ∂C in at least two distinct points. Consider those two such points A and B whose distance is maximal (see Fig. 4.3 — even though we show here A and B belonging one to the horizontal and one to the vertical side of the angle, nothing prevents them to belong to the same side).

Considering now an order on the path winding around ∂C (as in the proof of Lemma 2.1), take the portion γ between A and B and the lowermost horizontal line l touching γ. If l does not contain the point (x_0, y_0), we can decrease the energy by flipping intervals of minuses just above the line in the same way as we did when erasing the holes of C. Iterating the procedure

Fig. 4.3

unless l passes through (x_0, y_0) and repeating then all with leftmost vertical line, we finally replace γ by a corresponding segment of the boundary of the angle. Repeating the same for all relevant angles, we finally obtain the configuration M and prove thus (i).

To prove (ii) we assume the contrary and consider a standard sequence S for which $S\sigma \prec \overline{\sigma}$ does not hold. Take the first site $x \in S$ outside $C(\overline{\sigma})$, $x \notin C(\overline{\sigma})$. Considering an arbitrary configuration ξ inside $C(\overline{\sigma})$ and $\xi(y) = -1$ for all $y \notin C(\overline{\sigma})$, we will show that the spin flip $\xi(x) = -1 \to +1$ always increases energy. This is clear if x is not attached to the boundary of $C(\overline{\sigma})$. Taking into account that $C(\overline{\sigma})$ is a string, the local situation around x attached to $\partial C(\overline{\sigma})$ is among the following ones (with the site x in the center of the square and up to rotations and reflections):

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ - & - & - \\ - & - & - \\ & \text{A)} & \end{array} \qquad \begin{array}{ccc} - & \bullet & - \\ - & - & \bullet \\ - & - & \bullet \\ & \text{B)} & \end{array} \qquad \begin{array}{ccc} - & \bullet & \bullet \\ - & - & \bullet \\ - & - & - \\ & \text{C)} & \end{array}$$

(Notice that the case A) covers for example the situations

$$\begin{array}{ccc} \bullet & - & - \\ - & - & - \\ - & - & - \end{array} \; , \quad \begin{array}{ccc} - & \bullet & - \\ - & - & - \\ - & - & - \end{array} \; , \text{ etc.,}$$

while B) e.g.

$$\begin{array}{ccc} - & \bullet & - \\ - & - & \bullet \\ - & - & - \end{array} \; , \quad \begin{array}{ccc} - & \bullet & - \\ - & - & - \\ - & - & \bullet \end{array} \quad \text{etc.)}$$

A glance into the catalogue of stable situations (page 12) insures us that in all three cases the minus spin in the center is stable irrespectively of the values of spins at sites denoted by dots, in contradiction with the assumption that the spin flip $\xi(x) = -1 \to +1$ decreases the energy. The stability in the case A) follows from a) of the catalogue, B) from a), e), and g), and C) from a), d), and g). ∎

Notice that the string $\overline{\sigma}$ might still contain unstable lattice sites — namely those plus spins that are surrounded by 4 nearest neighbour minuses. When applying a standard sequence S, one will eventually erase them obtaining a union of octagons contained in $C(\overline{\sigma})$.

Next we proceed to a case of a configuration σ consisting of several components. To treat this case we shall repeatedly use the following lemma to evaluate the sum of energies of two configurations by the energy of the configuration whose area of pluses is the union of areas of pluses of the two concerned configurations. For configurations σ_1 and σ_2 we define the minimum $\sigma_1 \wedge \sigma_2$ and the maximum $\sigma_1 \vee \sigma_2$ with respect to the order $\prec$. Namely, these configurations are given by taking minimum and maximum, respectively, of σ_1 and σ_2 site by site. Then we have the standard inequality (a base of the FKG inequality): Lemma 4.2 For any σ_1, σ_2 and the Hamiltonian (2.2) one has

$$H(\sigma_1) + H(\sigma_2) \geq H(\sigma_1 \vee \sigma_2) + H(\sigma_1 \wedge \sigma_2). \tag{4.1}$$

Proof Using the equality

$$a + b = \max(a, b) + \min(a, b), \tag{4.2}$$

we get

$$h \sum \sigma_1(x) + h \sum \sigma_2(x) = h \sum (\sigma_1 \vee \sigma_2)(x) + h \sum (\sigma_1 \wedge \sigma_2)(x).$$

The remaining two sums in the definition (2.2) of the Hamiltonian consist of terms of the form $\sigma(x)\sigma(y)$ and the inequality (4.1) will be verified once we show that

$$\sigma_1(x)\sigma_1(y) + \sigma_2(x)\sigma_2(y) \leq (\sigma_1 \vee \sigma_2)(x)(\sigma_1 \vee \sigma_2)(y) + (\sigma_1 \wedge \sigma_2)(x)(\sigma_1 \wedge \sigma_2)(y). \tag{4.3}$$

If $\sigma_1(x) = \sigma_2(x)$ we have an equality in (4.3) by (4.2).

If $\sigma_1(x) \neq \sigma_2(x)$, say $\sigma_1(x) = +1$, $\sigma_2(x) = -1$, the inequality (4.3) follows from

$$\sigma_1(y) - \sigma_2(y) \leq \max(\sigma_1(y), \sigma_2(y)) - \min(\sigma_1(y), \sigma_2(y)) \ .$$

■

Consider now a configuration σ whose set $C(\sigma)$ has several components $C_1, \ldots, C_k$. Let us suppose that the rectangles $R(C_1), \ldots, R(C_k)$ are not wrapped around the torus and let $\overline{\sigma}_1, \ldots, \overline{\sigma}_k$ be the strings obtained as blown up envelopes of the configurations $\sigma_1, \ldots, \sigma_k$ corresponding to droplets $C_1, \ldots, C_k$. Using $\overline{H}$ to denote the relative energy with respect to the energy of $-\underline{1}$, i.e. $\overline{H}(\sigma) = H(\sigma) - H(-\underline{1})$, by Lemma 4.1 we have

$$\overline{H}(\sigma) \geq \overline{H}(\overline{\sigma}_1) + \cdots + \overline{H}(\overline{\sigma}_k). \tag{4.2}$$

If the droplets $N_1, \ldots, N_k$ corresponding to configurations $\overline{\sigma}_1, \ldots, \overline{\sigma}_k$ were isolated, we would define $\overline{\sigma}$ by taking $C(\overline{\sigma}) = \cup_{i=1}^k N_i$.

If the droplets $N_1, \ldots, N_k$ are not isolated, we first glue them together and repeat the procedure. Namely, if two droplets N, N', have a nonempty intersection, consider the droplet $N \cup N'$. By Lemma 4.2 we have

$$H(N) + H(N') \geq H(N \cup N') + H(N \cap N').$$

Let us suppose that the rectangular envelopes $R(N)$ and $R(N')$ are subcritical (i.e. not winding around the torus and with minimal sides not exceeding D^*). Then $\overline{H}(N \cap N') \geq 0$. Indeed, applying any standard sequence to $N \cap N'$, we get a union of subcritical octagons of even lower energy. And the energy of a subcritical octagon with respect to the energy of the configuration $-\underline{1}$ is positive. Thus

$$\overline{H}(N) + \overline{H}(N') \geq \overline{H}(N \cup N'). \tag{4.3}$$

Using this observation, we can consider the components $\widetilde{N}_1, \ldots, \widetilde{N}_k$ of $N_1 \cup N_2 \cup \cdots \cup N_k$ and show that

$$\overline{H}(\sigma) \geq \overline{H}(\widetilde{N}_1) + \cdots + \overline{H}(\widetilde{N}_l), \tag{4.4}$$

once we suppose that the circumscribed rectangles of $\widetilde{N}_1, \ldots, \widetilde{N}_l$ are subcritical.

Further, we say that two droplets $\widetilde{N}$ and $\widetilde{N}'$ *stick* together if there exists a site $x \notin \widetilde{N} \cup \widetilde{N}'$ (necessarily touching $\partial \widetilde{N}$ as well as $\partial \widetilde{N}'$) and a configuration It is easy to notice that if such configuration exists, one can take the configuration with pluses in $\widetilde{N} \cup \widetilde{N}'$, but we do not need this fact.i nside $\widetilde{N}$ and $\widetilde{N}'$ such that, flipping $-1 \to +1$ in x, we decrease the energy of the configuration corresponding to $\widetilde{N} \cup \widetilde{N}'$ and obtain a connected set $\widetilde{N} \cup \widetilde{N}' \cup q(x)$ (here $q(x)$ is the unit square around x). To avoid ambiguities, we choose the pairs of droplets sticking together as well as the particular site x to be flipped using some canonical order — say, lexicographic order of the uppermost left corner of the droplet and the same order for the site x. Flipping the concerned site x and iterating the procedure we obtain a set $C_1', \ldots, C_m'$ of disjoint droplets such that neither pair of them sticks together. Clearly,

$$C_1' \cup \cdots \cup C_m' \supset N_1 \cup \cdots \cup N_k$$

and

$$H(\sigma) \geq H(C_1') + \cdots + H(C_m').$$

Supposing again that the circumsribed rectangles $R(C_1'), \ldots, R(C_m')$ are subcritical (and in particular are not winding around the torus), we repeat the same procedure as when we started from σ. Iterating it we finally get a set of disjoint strings $A_1, \ldots, A_n$ such that no pair of them sticks together. If their circumscribed rectangles are not wrapped around the torus and are subcritical, we say that the original configuration σ is *acceptable* and define $\overline{\sigma}$ so that $C(\overline{\sigma}) = A_1 \cup \cdots \cup A_n$. Clearly,

$$H(\sigma) \geq H(\overline{\sigma}).$$

Thus we get the definitive Lemma 4.1 Let σ be an acceptable configuration. Then

" (i)" $H(\overline{\sigma}) \leq H(\sigma)$,

" (ii)" $S\sigma \prec \overline{\sigma}$ for every standard sequence,

" (iii)" $\sigma \prec \overline{\sigma}$ and the summary length of boundaries of all droplets of σ inside a component A_i of $C(\overline{\sigma})$ is at least as large as the length of the boundary of the circumscribed rectangle $R(A_i)$. Proof The statement (i) was already proven during the construction.

To prove (ii) we use the same argument as when proving (ii) in Lemma 4.1a. One has only to observe that, considering a site x attached to a component A_i of $C(\overline{\sigma})$ and supposing that another component A_j is touching the unit square $q(x)$, the spin flip $\sigma(x) = -1 \to +1$ necessarily increases energy irrespectively of the configuration inside A_i and A_j. Indeed, if a configuration inside $A_i \cup A_j$ existed so that the energy decreases, the strings A_i and A_j would stick together which is not the case.

To prove (iii) we first notice that in the first stages of the construction of $\overline{\sigma}$ we only decreased the number of bounds — the length of the boundary of $\cup \widetilde{N}_i$ is not larger than the boundary of the original $C(\sigma)$. Further, whenever gluing two sticking components we do not change the length of the boundary. ∎

We say that a string A is *ephemere* if $\min(D_1(R(A)), D_2(R(A))) < 3l^* - 3$. Strings that are not ephemere have energy larger or equal the energy of the standard octagon inscribed in their rectangular envelope.

Lemma 4.3 Let A be a string that is not ephemere and, R be its rectangular envelope, $R = R(A)$, and Q be the standard octagon with $R(Q) = R$. Then

$$H(A) \geq H(Q). \tag{4.5}$$

Proof The string A consists of certain number of octagonal blocks touching in bottlenecks. Actually, the inside blocks — those not touching the boundary of the rectangle $R(A)$ — are in general "hexagons", while the two corner blocks — those touching the boundary of $R(A)$ — are in general "heptagons" (cf. Fig. 4.1). The energy of a string does not change if we reorder the inside blocks. In the case when any corner block contains also the vertex of the rectangle $R(A)$, (it is a "hexagon"), it can be interchanged with any inside block without changing the energy of the configuration.

After a reordering some bottlenecks may disapear and one can decrease the energy of the corresponding configuration by replacing a group of blocks, attached at points that are not bottlenecks any more, by an octagonal envelope. Using this observation in an iterative manner, one can finally replace the string A by a string $\bar{A}$ such that $R(A) = R(\bar{A})$, $H(A) \geq H(\bar{A})$, and all the inside blocks of $\bar{A}$ are unit squares. Indeed, starting from an arbitrary string A, one easily gets a string whose all inside blocks are rectangles such that, say, the vertical side of all of them is of length one. We decrease further the energy and get $\bar{A}$ by shrinking all inside blocks to unit squares and expanding in the corresponding manner one of the corner blocks (see Fig. 4.4). This last step does not change the number of corners while increasing the area of the droplet.

Fig. 4.4

To evaluate now the energy of such a droplet $\bar{A}$ with d inside unit square blocks, we will consider two cases:

" •" $d \geq \frac{1}{2}(3+\sqrt{5})l^* + 3$, or

" •" $d < 3l^* - 2$. These two cases cover all values of d once $(3-\sqrt{5})l^* > 10$, i.e. if $K > 7h$.

In the first case, the energy decreases if we replace all inside (unit square) blocks by a single hexagon with two oblique sides of length l^*. Indeed, to prove this we have to show that (cf. (3.18) and (3.6))

$$-hd - (d-1)6K - 2K > -hd^2 + 2F(l^*),$$

or equivalently, using (3.6) and (2.18),

$$P(d) \equiv d^2 - d(3l^* - 2 + 3\eta) + 2(l^* - 1 + \eta) + [(l^* - 1)^2 + \eta(2l^* - 1)] > 0.$$

Taking into account that $\eta \in (0,1)$, the discriminant of the quadratic expression $P(d)$ is bounded from above by $5(l^* + 2)^2$. Observing further that the term d^2 in $P(d)$ has a positive coefficient, the sought inequality is fulfilled once d exceeds the larger one from the solutions of the equation $P(d) = 0$. This solution is bounded from above by

$$\frac{1}{2}[3l^* + 1 + \sqrt{5}(l^* + 2)] \leq \frac{3 + \sqrt{5}}{2} l^* + 3.$$

The resulting string contains a hexagon that is attached to (if any) two corner blocks in points that are not bottlenecks and thus the energy decreases if we replace this string by its octagonal envelope. Its energy can be further lowered by replacing it by the standard octagon Q with the same circumscribed rectangle $R(A)$ (cf. Lemma 3.2).

On the other side, if $d < 3l^* - 2$ we consider the monotonous set $\overline{M}$ constructed from $\overline{A}$ as the union of the $d \times d$ square $\overline{Q}$ circumscribed to the union of inside (unit square) blocks with the corner blocks and the union of all their shifts, up to the distance d, along those sides of the rectangle $R(A)$ they are touching (see Fig. 4.5).

Fig. 4.5

Using D to denote $\min(D_1(R(A)), D_2(R(A)))$, the area covered by $\overline{M} \setminus \overline{A}$ is at least

$$2Dd - d^2 + d.$$

On the other side, the droplet $\overline{M}$ has by $6(d-1)$ less corners and the replacement of $\overline{A}$ by $\overline{M}$ is favourable once

$$[(2D+1)d - d^2]h > 6(d-1)K.$$

This inequality is fulfilled if

$$[2D + 1 - d]h > 6K = 3(l^* - 1 + \eta)h.$$

Thus, it is enough to take d such that

$$d < 2D + 4 - 3l^*.$$

Since $D \geq 3l^* - 3$, the inequality is fulfilled once $d < 3l^* - 2$. The energy of the droplet $\overline{M}$ is lower than that of its octagonal envelope and this, in turn, is lower than the energy of standard octagon Q. ∎

Now, we are ready to define the set $\mathcal{A}$.

Consider an acceptable configuration σ and the corresponding $\bar{\sigma}$ with nonsticking components $A_1, \ldots, A_k$. Consider further the family of sets $R_1, \ldots, R_k$, where $R_i = A_i$ if A_i is ephemere; otherwise R_i is the rectangular envelope of A_i. Two strings A_i, A_j are said to *interact* if at least one of them is not ephemere and the sets R_i and R_j (but not necessarily A_i and A_j) stick together (or intersect)However, if A_i and A_j stick together, they necessarily interact..

A family of strings $A_1, \ldots, A_k$ is said to form a *chain* $\mathcal{C}$ if every pair (A_i, A_j) of them can be linked by a sequence $\{A_{i_1}, \ldots, A_{i_n}\}$ of pairwise interacting strings from $\mathcal{C}$; $A_{i_1} = A_i$, $A_{i_n} = A_j$, and A_{i_l} and $A_{i_{l+1}}$ are interacting for all $l = 1, \ldots, n-1$.

Given a collection of chains $\mathcal{C}_1, \ldots, \mathcal{C}_n$ corresponding to the family $A_1, \ldots, A_k$, we start the following iterative procedure:

T he chains $\mathcal{C}_j^{(1)}$ of the "first generation" are identical to $\mathcal{C}_j$, $j = 1, \ldots, n$.

H aving defined $\mathcal{C}_j^{(r)}$, we construct the sets $R_j^{(r)}$ as the rectangular envelopes of the unions

$$\bigcup_{R \in \mathcal{C}_j^{(r)}} R$$

whenever $\mathcal{C}_j^{(r)}$ contains at least two sets; we put $R_j^{(r)} = R$ if $\mathcal{C}_j^{(r)} = \{R\}$.

We define $\mathcal{C}_j^{(r+1)}$ as the maximal chains of the family of sets $\{R_j^{(r)}\}$.

Iterating this procedure we reach a family of chains, each consisting of a single set. We call them *complete sets of the configuration* σ. It is easy to observe that every complete set from the resulting family $\bar{R}_1, \ldots, \bar{R}_s$ is either a rectangle containing certain amount of original strings $A_1, \ldots, A_k$, or it is one of ephemere strings contained in the family $A_1, \ldots, A_k$.

We introduce $\mathcal{A}$ as the set of all those acceptable configurations for which all complete sets $\bar{R}_1, \ldots, \bar{R}_s$ are subcritical — they can be placed in a rectangle that is not wrapped around the torus and whose minimal side does not exceed $D^* - 1$.

In the remaining part of the present Section we shall verify the properties i) – iii) of the set $\mathcal{A}$. It is easy to see that the property ii) of $\mathcal{A}$ is obvious from the definition, while the property i) follows from Propositions 1–3.

To prove the property iii) we first consider the configuration $\hat{\sigma}$ obtained by placing plus spins at all lattice sites in $\bar{R}_i$ for ephemere $\bar{R}_i = \bar{A}_i$ and in the standard octagon Q_i inscribed in $\bar{R}_i$ for $\bar{R}_i$ that is not ephemere. We will repeatedly use the bound

$$H(\sigma) \geq H(\hat{\sigma}). \tag{4.6}$$

This follows, iterating with the help of Lemma 4.3, from the following Lemma.

Lemma 4.4 Let A' and A'' be a pair of interacting strings and let ξ be the configuration with the set of pluses coinciding with $A' \cup A''$. Suppose that the rectangular envelope R of the set $A' \cup A''$ is subcritical. Then

$$H(\xi) \geq H(\hat{\xi}), \tag{4.7}$$

where $\hat{\xi}$ is the configuration with pluses in the standard octagon Q inscribed into R. Proof Suppose first that both A' and A'' are not ephemere and consider the standard octagons Q' and Q'' inscribed into the rectangular envelopes R' and R'' of A' and A'', respectively. Clearly, by lemma 4.3, $\bar{H}(\xi) \geq \bar{H}(Q') + \bar{H}(Q'') \geq \bar{H}(Q' \cup Q'')$. If Q' and Q'' stick together (i.e. Q' and Q'' intersect each other or there exists a site x such that flipping the spin at x decreases the energy) the energy of the single droplet $Q' \cup Q'' \cup q(x)$ (resp. $Q' \cup Q''$ if Q' and Q'' intersect) is smaller than $H(\xi)$. It can be further lowered by taking its monotonous envelope and, finally, by replacing it with the octagon Q. If Q' and Q'' do not stick together, there does not exist a site x such that the unit square $q(x)$ intersects the boundaries of both Q' and Q'' along a bond and we can see that the area of Q is by at least

$$\min_{y \in [0, l^*-1]} [(2(l^* - 1) - y)(2(l^* - 1) + y) + (2(l^* - 1) + y)(2(l^* - 1) - y)] \geq$$
$$\geq \min_{y \in [0, l^*-1]} [8(l^* - 1)^2 - 2y^2] \geq 6(l^* - 1)^2$$

larger than the area of $Q' \cup Q''$ (see Fig. 4.6).

Fig. 4.6

Taking into account that $4F(l^*) \geq -2(l^*-1)h$, we get

$$6(l^*-1)h > -4F(l^*),$$

and thus the energy decreases once we replace the configuration ξ by Q. (We are even not using the fact that the overall number of bonds in the boundaries of Q' and Q'' might be larger than the number of bonds in the boundary of Q leading to an even larger decrease of energy.)

It remains to consider the case when one of the strings, say A'', is ephemere. (See Fig. 4.7.) The only nontrivial situation is when Q' and A'' do not stick together. When replacing the configuration ξ (with energy not lower than that of the union of Q' and A'') by Q, we have to compensate the loss of all corners in $\partial A''$ by the surplus area $|Q| - |Q' \cup A''|$ and surplus length of the boundary $|\partial Q'| + |\partial A''| - |\partial Q|$. Consider now the strips H and V obtained as the union of all horizontal and vertical shifts of Q', respectively. It is easy to see that the surplus length is at least the number N_h of horizontal bonds of $\partial A''$ in V plus the number N_v of vertical bonds of $\partial A''$ in H. Each corner of $\partial A''$ in V is linked with at least one horizontal bond of $\partial A''$ in V. The number of corners attached to N_h horizontal bonds in V is at most $6\frac{N_h}{2} + 6$. Their loss is compensated by the surplus of those N_h bonds since

$$(3N_h + 6)K \leq 6N_h K \leq N_h J.$$

Similarly for the vertical bonds in H.

Fig. 4.7

Thus it remains to compensate for corners not contained in the closed set $H \cup V$. Consider the four quadrants — components of $(H \cup V)^c$. The set A'' intersects at most two of them. We use U_1, U_2, to denote these two quadrants and $M_i = U_i \cap A''$, $i = 1, 2$. Moreover, there is at most one quadrant of $(H \cup V)^c$, say U_1, whose both sides intersect A''. Without loss of generality we can suppose that M_2 does not intersect H and the circumscribed rectangle $R(M_2)$ to M_2 is $m_2 \times n_2$. Then there is at least $\min(m_2, n_2)(3l^* - 2)$ of surplus area of $|Q| - |Q' \cup A'|$ in the component of $H \setminus Q'$ that touches U_2. On the other side, at most $\min(m_2, n_2)l^*$ of the area may be lost from the portion of M_2 sticking out of Q. Using b_2 to denote the number of bottlenecks in M_2, there is also at least $b_2(3l^* - 2)$ of surplus area of $|Q| - |Q' \cup A'|$ in the component of $V \setminus Q'$ that touches U_2. In the same time, the number of corners in M_2 (that are not at the boundary of U_2 and thus were not accounted for before) is at most $6b_2 + 4(\min(m_2, n_2) - 1 - b_2)$ and we have

$$6b_2 K + 4(\min(m_2, n_2) - 1 - b_2)K \leq [(3l^* - 2)\min(m_2, n_2) - \min(m_2, n_2)l^* + b_2(3l^* - 2)]h.$$

To analyze the quadrant U_1, we distinguish two possibilities. First, suppose that M_1 contains at least one bottleneck, consider the quadrant U', with vertex in this bottleneck, that contains the octagon Q', and use n_h and n_v to denote the number of horizontal and vertical bonds in M_1, respectively. The number of corners of M_1 is at most $3\min(n_h, n_v)$. In the same time the surplus of the area $|(Q \setminus (Q' \cup A'')) \cap U' \cap (H \cup V)|$ is at least $\frac{n_h + n_v}{2}(3l^* - 2)$. To get the sought bound we just notice that

$$3\min(n_h, n_v)K = \frac{3}{2}\min(n_h, n_v)(l^* - 1 + \eta)h < \min(n_h, n_v)\frac{3}{2}l^* h \leq \frac{n_h + n_v}{2}(3l^* - 2)h.$$

If there is no bottleneck of A'' in U_1, the portion of A'' between two bottlenecks containing the set M_1 is clearly contained in a halfplane P, the oblique boundary ∂P of which passes through the vertex of of U_1 and is orthogonal to the axis of symmetry of U_1. The number of corners of M_1, not accounted for before, is in this case at most $2\min(n_h, n_v)$. On the other side, the area $|(Q \setminus (Q' \cup A'')) \cap P^c \cap (H \cup V)|$ is at least $\frac{\min(n_h,n_v)}{2}(3l^*-2)$ and

$$2\min(n_h,n_v)K \leq \frac{\min(n_h,n_v)}{2}(3l^*-2)h.$$

■

To verify the property iii) of $\mathcal{A}$, let $\xi \in \partial\mathcal{A}$ and $\sigma \in \mathcal{A}$ be such that $\xi = \sigma^{(x)} \notin \mathcal{A}$. The mapping, assigning to a configuration ζ its complete sets, is monotonous in the ordering $\prec$ on configurations and the ordering by inclusion on complete sets. As a consequence, the value $\sigma(x)$ is necessarily -1; otherwise $\sigma \in \mathcal{A}$ would imply also $\xi \in \mathcal{A}$. Moreover, the site x lies outside of all complete sets $\bar{R}_1, \ldots, \bar{R}_s$ of σ. Among the complete sets of ξ there exists a rectangle $\bar{R}(D_1, D_2)$ with the following properties:

” i)” $\bar{R}$ is supercritical, $\min(D_1, D_2) \geq D^*$,

” ii)” it contains the site x and several complete sets $\bar{R}_i$, say, $\bar{R}_1, \ldots, \bar{R}_k$, of σ; the remaining complete sets $\bar{R}_{k+1}, \ldots, \bar{R}_s$ of σ are also complete sets of ξ.

Our aim now is to prove that

$$H(\xi) \geq H(Q(D^*,D^*)) + h(D^*-1) - 4K = H(Q(D^*-1,D^*)) + 2J - 4K - h. \qquad (4.8)$$

If $\bar{R}$ is winding around the torus, referring briefly to (4.6), the inequality (4.8) is clearly satisfied. Thus, let us suppose that $\bar{R}$ is not wrapped around the torus. Consider first the configuration $\tilde{\xi}$, $H(\xi) \geq H(\tilde{\xi})$, obtained by restricting the configuration ξ to the union $\cup_{i=1}^{k} \bar{R}_i \cup q(x)$, where $q(x)$ is the unit square centered at the site x (i.e. considering ξ to be plus on $\cup_{i=1}^{k} \bar{R}_i \cup q(x)$ and taking minuses outside; we will see in a moment (Eq. (4.9)) that the energy decreases when skipping the subcritical sets $\bar{R}_{k+1}, \ldots, \bar{R}_s$). Further, consider the set $C^{(0)}$ consisting of the union of the unit square $q(x)$ and those sets among $\bar{R}_1, \ldots, \bar{R}_k$ that have a common edge with $q(x)$. Let us take the rectangular envelope $\tilde{R}^{(1)}$ of $C^{(0)}$ if $C^{(0)}$ is not ephemere and $\tilde{R}^{(1)} = C^{(0)}$ for ephemere $C^{(0)}$, and distinguish two cases, either the set $\tilde{R}^{(1)}$ is supercritical or not. (Notice that both, $C^{(0)}$ and $\tilde{R}^{(1)}$ may actually coincide with $q(x)$.)

If $\tilde{R}^{(1)}$ is supercritical, we decrease the energy of $\tilde{\xi}$ further by erasing all the remaining sets from $\bar{R}_1, \dots, \bar{R}_k$ that were not contributing to the set $C^{(0)}$ and considering the configuration $\xi^{(0)}$ yielded by the restriction of ξ to the set $C^{(0)}$. To see this, notice first that for any subcritical $\bar{R}_i$, the energy of the restriction ξ_i of ξ to $\bar{R}_i$, $\xi_i = \xi \bar{R}_i$, can be bounded from below as

$$\bar{H}(\xi_i) \geq 2K. \tag{4.9}$$

Indeed, if $\bar{R}_i$ is subcritical and nonephemere and denoting $\bar{D}_2 = D_2(\bar{R}_i) \leq D_1 = D_1(\bar{R}_i)$, we have

$$\begin{aligned} \bar{H}(\xi_i) &\geq 2J(\bar{D}_1 + \bar{D}_2) - h\bar{D}_1\bar{D}_2 - 2h(l^* - 1)^2 = \\ &= 2J\bar{D}_1 - h\bar{D}_1\bar{D}_2 + 2J\bar{D}_2 - 2h(l^* - 1)^2 \geq \\ &\geq h(l^* - 1)[(D^* - 1)3 - 2(l^* - 1)] \geq 2K. \end{aligned} \tag{4.10}$$

If $\bar{R}_i$ is ephemere, we have

$$\bar{H}(\xi_i) \geq 2J(\bar{D}_1 + \bar{D}_2) - 6K\bar{D}_2 - h\bar{D}_1(3l^* - 3) + 2K \geq 2K \tag{4.11}$$

once $K \leq J/3$. Thus, the energy decreases if we erase the configuration in those $\bar{R}_i$ that are not contributing to $C^{(0)}$ — the bound $2K$ on the right hand side of (4.9) is needed in case $\bar{R}_i$ is touching $q(x)$ in a corner.

Observing that $\bar{R}_1, \dots, \bar{R}_k$ are not interacting, there can be at most two sets, say $\bar{R}_1, \bar{R}_2$, contributing to $C^{(0)}$ (i.e. intersecting $q(x)$ along its edge).

Let us suppose first that $\bar{R}_1$ is the only set from $\bar{R}_1, \dots,$ contributing to $C^{(0)}$. Recalling that $\bar{R}_1$ is subcritical and $\tilde{R}^{(1)}$ is supercritical in the same time, we infer that the set $\bar{R}_1$ must be nonephemere and, moreover, it is necessarily the rectangle $(D^* - 1) \times D$ (or $D \times (D^* - 1)$)

with $D \geq D^*$ and with the unit square $q(x)$ attached to its longer side. Consider the standard octagon $\bar{Q}_1$ inscribed into $\bar{R}_1$. The configuration $\xi^{(0)}$ restricted to $\bar{R}_1$ is actually the configuration $\sigma^{(0)}$ obtained as the restriction of σ to $\bar{R}_1$ with the energy not lower than the energy of $\bar{Q}_1$ (cf. (4.6)), $H(\sigma^{(0)}) \geq H(\bar{Q}_1)$. If $q(x)$ does not touch $C(\sigma^{(0)})$, then

$$\bar{H}(\xi^{(0)}) = \bar{H}(q(x)) + \bar{H}(\sigma^{(0)}) \geq \bar{H}(q(x)) + \bar{H}(\bar{Q}_1) \tag{4.12}$$

and the bound (4.8) follows since $\bar{H}(q(x)) = 4J - 4K - h > 2J - 4K - h$ and $H(\bar{Q}_1) \geq H(Q(D^*, D^* - 1))$. If $q(x)$ touches $C(\sigma^{(0)})$, then the monotone envelope of $\xi^{(0)}$ is a union of a monotone configuration M in $\bar{R}_1$ with the unit square $q(x)$ sharing a common edge with M. Hence

$$H(\xi^{(0)}) \geq H(M) + 2J - 4K - h. \tag{4.13}$$

Using the bound $H(M) \geq H(\bar{Q}_1)$ according to Lemma 3.2, we get the sought inequality.

If there are two sets, $\bar{R}_1$ and $\bar{R}_2$, contributing to $C^{(0)}$, then at least one of them must be ephemere (otherwise $\bar{R}_1$ and $\bar{R}_2$ would interact). Suppose first that $\bar{R}_1$ is nonephemere and $\bar{R}_2$ is ephemere. Considering the halfplane p containing $\bar{R}_1$, whose boundary contains the edge separating q from $\bar{R}_1$, then, necessarily, $\bar{R}_2 \subset \mathrm{R}^2 \setminus p$. Moreover, the first row along the boundary of the halfplane p (the difference $p' \setminus p$, where p' is the halfplane p shifted by the unit vector orthogonal to its boundary) does not intersect the interior of $\bar{R}_2$ (it contains only the square $q(x)$), the square $q(x)$ is attached to $\bar{R}_1$ in a corner, and the second row contains a single square, to be denoted $\bar{q}$, of $\bar{R}_2$ (cf. Fig. 4.8).

Fig. 4.8

Since $\tilde{R}^{(1)}$ is supercritical, there exists a $D \times D^*$ (or $D^* \times D$) rectangle R^* with $D \geq D^*$ such that

" i)" R^* contains $q \cup \bar{R}_2$,

" ii)" R^* is the circumbscribed rectangle of $R^* \cap (\bar{R}_1 \cup q \cup \bar{R}_2)$,

" iii)" $R^* \cap \bar{R}_1$ is not ephemere (here, we use the assumption $10K < J$ that implies $10(l^* - 1) < D^*$). Let us consider the restriction σ_1 and σ_2 of the configuration $\sigma^{(0)}$ to $\bar{R}_1$ and $\bar{R}_2$, respectively. Considering the site x and its three neighbour and nearest neighbour sites in the halfplane p, one can easily convince oneself that

$$\bar{H}(\sigma_1) + \bar{H}(q \cup \sigma_2) \geq \bar{H}(\xi^{(0)}) \geq \bar{H}(\sigma_1) + \bar{H}(q \cup \sigma_2) - 2J. \tag{4.14}$$

On the other side

$$H(\sigma_1) \geq H(\bar{Q}_1) \geq H(Q_1), \tag{4.15}$$

where $\bar{Q}_1$ and Q_1 are the standard octagons inscribed into $\bar{R}_1$ and $\bar{R}_1 \cap R^*$, respectively, and

$$H(q \cup \sigma_2) \geq H(q \cup \bar{R}_2). \tag{4.16}$$

In the inequality (4.15) we are using the fact that $\bar{R}_1$ is subcritical. Using the first inequality from (4.14) for $\sigma_1 = Q_1$, we get

$$\bar{H}(Q_1) + \bar{H}(q \cup \bar{R}_2) \geq \bar{H}(Q_1 \cup q \cup \bar{R}_2). \tag{4.17}$$

Our aim now is to find a lower bound on the difference $H(Q_1 \cup q \cup \bar{R}_2) - H(Q(D^*, D))$. Expanding the droplet $Q_1 \cup q \cup \bar{R}_2$ to $Q(D^*, D)$, we have to compensate for the loss of all corners in $\bar{R}_2$ by the surplus area (similarly as in the proof of Lemma 4.4). Using n_h and n_v to denote the horizontal and vertical sizes of $\bar{R}_2 \setminus \bar{q}$, respectively, it suffices to notice that the area of $R^* \setminus \bar{R}_1 \cup q \cup \bar{R}_2$ is at least $\min(n_h, n_v)(D^* - (3l^* - 3)) + 2(D^* - 1)$. The last term stems from the rows containing the squares q and $\bar{q}$. On the other side, any horizontal or vertical line through a corner of $\bar{R}_2 \setminus q'$ contains at most 6 corners of $\bar{R}_2$ (a bottleneck counts for 4 corners). As the result we get

$$
\begin{aligned}
(4.18)\quad & H(Q_1 \cup q \cup \bar{R}_2) - H(Q(D^*, D)) \geq \\
& \geq -6\min(n_h, n_v)K + h\min(n_h, n_v)(D^* - (3l^* - 3)) + 2h(D^* - 1) - 2K \geq \\
& \geq 2h(D^* - 1) - 2K.
\end{aligned}
$$

The term $-2K$ stands for two corners of $\bar{R}_2$ in the last row (on the boundary ∂R^*). In the last inequality we used the fact that $6K < h(D^* - (3l^* - 3))$. Combining this with (4.14) and (4.17), and the bound $2J < hD^* \leq hD$, we get

$$
H(\xi^{(0)}) - H(Q(D^*-1, D)) - (2J - hD) \geq H(\xi^{(0)}) - H(Q(D^*-1, D)) \geq 2(D^*-1)h - 2K - 2J. \tag{4.19}
$$

Thus, using the inequality

$$
2(D^* - 2)h + 2K > (2(D^* - 2) + l^* - 1)h \geq 2D^*h > 4J \tag{4.20}
$$

satisfied once $l^* \geq 3$ (cf. (3.2)), we get

$$
H(\xi^{(0)}) - H(Q(D^* - 1, D)) \geq (D^* - 2)h - 2K > 2J - 4K - h. \tag{4.21}
$$

To get the sought inequality it remains to realize that $H(Q(D^* - 1, D)) \geq H(Q(D^* - 1, D^*))$ since $(D^* - 1)h < 2J$.

Consider now the case when both $\bar{R}_1$ and $\bar{R}_2$ are ephemere. For the strings not to interact, at least one of them, say $\bar{R}_2$, must be attached to the rectangle q through a rectangle $\bar{q}$, in the similar manner as above. Using the assumption $10K < J$, one can show that

$$
10(l^* - 1) < D^*. \tag{4.22}
$$

Observing now that any ephemere string is contained in a strip of thicknes $3l^* - 3$, we can conclude that, to combine to a supercritical rectangle $\tilde{R}^{(1)}$, the strips containing $\bar{R}_1$ and

$\bar{R}_2 \cup q$ are necessarily orthogonal to each other. Further, there exists a $D \times D^*$ (or $D^* \times D$) rectangle R^* with $D \geq D^*$ such that R^* is the circumbscribed rectangle of $R^* \cap (\bar{R}_1 \cup q \cup \bar{R}_2)$. The surplus area in R^* with respect to $\bar{R}_1 \cup q \cup \bar{R}_2$ is at least $(D^* - 3l^* + 3)^2$, while the number of corners in the strings $\bar{R}_1$ and $\bar{R}_2$ is at most $2 \times 6 \times (3l^* - 3)$. Taking into account that $2K < l^* h$, $2(D^* - 3l^*) \geq D^*$, and $-4F(l^*) \geq 2(l^* - 1)^2 h$, we get the bound

$$
\begin{aligned}
(4.23) \quad H(\xi^{(0)}) - H(Q(D^* - 1, D)) &\geq -2J - 12(3l^* - 3)K + (D^* - 3l^* + 3)^2 h - 4F(l^*) \geq \\
&\geq -2J - 6(3l^* - 3)l^* h + 3D^* h + (D^* - 3l^*)^2 h + 2(l^* - 3)^2 h.
\end{aligned}
$$

The sought inequality then follows from

$$
-6(3l^* - 3)l^* h + 3D^* h + (D^* - 3l^*)^2 h + 2(l^* - 3)^2 h > 4J - 4K - h \tag{4.24}
$$

since

$$
(D^* - 3l^*)^2 + 2(l^* - 3)^2 + 2(l^* - 1) > (6l^*)^2 + (l^*)^2 > 18(l^*)^2 > 6(3l^* - 3)l^*. \tag{4.25}
$$

Next, consider the case when $\tilde{R}^{(1)}$ is subcritical. Let us introduce $\bar{R}^{(1)} = \tilde{R}^{(1)}$ if $\tilde{R}^{(1)}$ is nonephemere. Otherwise we take for $\bar{R}^{(1)}$ the string $N(C^{(0)})$ obtained as the blown up envelope of the configuration corresponding to $C^{(0)}$. Consider now $\bar{R}^{(1)}$ and all complete sets among $\bar{R}_1, \dots, \bar{R}_k$ that were not used for $C^{(0)}$ and construct from them the set of chains $\mathcal{C}_j^{(1)}$ of the first generation. A sequence $\mathcal{C}_j^{(r)}, r = 1, \dots, m$ of chains of following generations is obtained from it by iteration. Since the sets $\bar{R}_1, \dots, \bar{R}_k$ are mutually noninteracting, for every generation r we get a chain, say $\mathcal{C}_1^{(r)}$, consisting of a set $\bar{R}^{(r)}$ containing the site x and certain subset of $\bar{R}_1, \dots, \bar{R}_k$, each of them interacting directly with $\bar{R}^{(r)}$. The remaining chains $\mathcal{C}_j^{(r)}, j = 2, \dots$ contain each just one complete set from those among $\bar{R}_1, \dots, \bar{R}_k$ that have not appeared in $\mathcal{C}_1^{(p)}$, $p \leq r$, in the preceding steps. Clearly, there is only one chain in the last generation, $\mathcal{C}_1^{(m)} = \{\bar{R}^{(m)}\} \equiv \{\bar{R}\}$.

Let us consider the last set $\bar{R}^{(p)}$ among $\bar{R}^{(r)}$, $r = 1, \dots, m$, that is subcritical and take the chain $\mathcal{C}_1^{(p)}$ with the complete sets in $\mathcal{C}_1^{(p)} \setminus \bar{R}^{(p)}$ ordered in a particular way, say in lexicographic order of their left upper corner. Let us unite them, one by one in the given order, with the set $\bar{R}^{(p)}$ until the circumscribed rectangle is supercritical. Cutting off the remaining complete sets from the chain $\mathcal{C}_1^{(p)}$ we get the chain $\tilde{\mathcal{C}}_1^{(p)} \subset \mathcal{C}_1^{(p)}$. Let us use $\tilde{R}'$ to denote the last complete set that was attached to form the chain $\tilde{\mathcal{C}}_1^{(p)}$ and $\tilde{R}$ either the blown up envelope or the circumscribed rectangle of the union of complete sets from $\tilde{\mathcal{C}}_1^{(p)} \setminus \{\tilde{R}'\}$, in dependence on whether it is ephemere or not. Clearly, $\tilde{R}$ and $\tilde{R}'$ are subcritical interacting sets with a supercritical rectangular envelope of their union.

The energy of the configuration $\tilde{\xi}$ can be evaluated by the sum of the energies of its restriction to $\tilde{R}$ and $\tilde{R}'$, respectively, as

$$\bar{H}(\tilde{\xi}) \geq \bar{H}(\tilde{\xi}\tilde{R}) + \bar{H}(\tilde{\xi}\tilde{R}') - 2K. \tag{4.27}$$

Indeed, the pluses of the original configuration $\tilde{\xi}$ are inside of the noninteracting sets $\bar{R}_1, \dots, \bar{R}_k$, and q, and thus only when $\tilde{R}'$ is touching by its corner the square q (included in $\tilde{R}$), the additional $2K$ may appear when joining the configurations $\tilde{\xi}\tilde{R}$ and $\tilde{\xi}\tilde{R}'$. Consider further, depending on whether $\tilde{R}$ ($\tilde{R}'$) is ephemere or not, the set $\tilde{Q}$ ($\tilde{Q}'$) defined as the set itself or its inscribed standard octagons. According to (4.6) one has

$$\bar{H}(\tilde{\xi}\tilde{R}) + \bar{H}(\tilde{\xi}\tilde{R}') \geq \bar{H}(\tilde{Q}) + \bar{H}(\tilde{Q}'). \tag{4.28}$$

Let us observe now that there exists a $D^* \times D^*$ square R^* such that

" i)" it intersects both sets $\tilde{R}$ and $\tilde{R}'$ in nondegenerated sets R and R', $R = \tilde{R} \cap R^*$ and $R' = \tilde{R}' \cap R^*$,

" ii)" it is the rectangular envelope of $R \cup R'$,

" iii)" it contains the intersection $\tilde{R} \cap \tilde{R}'$ (if it is nonempty), and

" iv)" if $\tilde{R}$ ($\tilde{R}'$) is nonephemere, the same is true for R (R'). Since both sets $\tilde{R}$ and $\tilde{R}'$ are subcritical, we decrease the energy of $\tilde{Q}$ ($\tilde{Q}'$), to the energy of Q (Q') defined as $Q \cap R^*$

$(Q' \cap R^*)$ for ephemere $\tilde{Q}$ $(\tilde{Q}')$ and as the inscribed standard octagon for nonephemere $\tilde{Q}$ $(\tilde{Q}')$. Hence

$$\bar{H}(\tilde{\xi}) \geq \bar{H}(Q) + \bar{H}(Q') - 2K. \tag{4.29}$$

Thus our task is to evaluate the sum of energies of two interacting sets Q and Q' (nonephemere standard octagons or ephemere string) whose union has R^* as the rectangular envelope. Consider first the case when both Q and Q' are nonephemere. Replacing Q and Q' by the standard octagon Q^* inscribed in R^*, the sum on the right hand side of (4.29) decreases by at least $h(D^* - 1)$, yielding thus (4.8). Indeed, if the rectangles R and R' just touch in the corner, the boundary has the same number of bonds as in Q^*, while the surplus area of R^*, as compared with that of $R \cup R'$, is at least $D^* + 2(3l^* - 4)^2$. The sought bound follows, once we realize that

$$4F(l^*) \geq -2(l^* - 1)^2 h - 2\eta(2l^* - 1) \geq -2(3l^* - 4)^2 h. \tag{4.30}$$

If the rectangles R and R' are intersecting, the surplus area shrinks. However, each line of at most D^* surplus sites lost in this way is compensated by surplus 2 bonds in the joint boundary of Q and Q', as compared with the boundary of Q^*. If, on the other side, the distance of the rectangles R and R', say in the vertical direction, is 1, there must exist at least two surplus horizontal bonds for them to interact, compensating thus the lack of two vertical bonds.

It remains to consider the case with, say, nonephemere R and ephemere R'. If R and R' just touch in the corner, the reasoning is the same as when discussing the case of supercritical $\tilde{R}^{(1)}$ above (cf. the bound (4.21)). The cases of intersecting R and R' or of R and R' whose distance is 1 are then treated with the same modifications as when both R and R' are nonephemere.

It is easy to observe that, except the case when a single $\bar{R}_1$ is contributing to $C^{(0)}$, the lower bounds are always sharp — at least once during the process of getting a lower bound we use a sharp bound. See e.g. the lower bounds (4.21), (4.24), (4.25). As a result, we can conclude that the only configurations from $\partial\mathcal{A}$ on which the bound can be achieved are those from $\mathcal{P}$, the remaining ones having strictly higher energy by at least the minimal amount h.

$\blacksquare$

5

. Proofs of Theorems

Similarly as when proving the statement of Proposition 1 from the bound (3.77), we will get Theorems 1 and 2 from

$$\lim_{\beta\to\infty} P_{-\underline{1}}(\tau_{\partial\mathcal{A}} \geq T(\tilde{\varepsilon})) = 0 \tag{5.1}$$

that will be shown to be valid for all $\tilde{\varepsilon} > 0$ with $T(\tilde{\epsilon}) = e^{\beta(E^*+\tilde{\epsilon})}$ (cf. (2.24), (2.28)). To prove (5.1), we follow KO 1 and define, in a similar manner as in the proof of Proposition 3.1, an event $\mathcal{E}_\sigma$ starting from an arbitrary σ in $\mathcal{A}$, taking place over an interval of time $T_1 = \exp\{\beta(E(l^*-1)+\delta)\}$ (cf. (3.71)) and such that:

(1) If $\mathcal{E}_\sigma$ takes place then necessarily the set $\partial\mathcal{A}$ is reached (in a particular manner) before the time T_1.

(2) For the probability $P(\mathcal{E}_\sigma)$ the uniform lower bound

$$\inf_{\sigma\in\mathcal{A}} P(\mathcal{E}_\sigma) \geq \alpha \tag{5.2}$$

holds with α such that

$$\lim_{\beta\to\infty}(1-\alpha)^{\frac{T_2}{T_1}} = 0, \tag{5.3}$$

where $T_2 = T(\tilde{\epsilon})$.

Using the equations (5.2) and (5.3), it follows by the strong Markov property that an attempt to reach $\partial\mathcal{A}$ not later then T_2 will be successful with high probability for large β. Indeed, it is enough to observe that after splitting T_2 into T_2/T_1 intervals of length T_1, staying inside of $\mathcal{A}$ for any of these intervals means that the event $\mathcal{E}_\sigma$ did not take place.

Once we have (5.1), we use the reversibility (Lemma 3.4) and refer to the property iii) of the set $\mathcal{A}$, to get an upper bound on the probability of reaching $\partial\mathcal{A}$ in a configuration outside $\mathcal{P}$. Noticing that, after starting from $\mathcal{P}$, there is a finite probability to go to $+\underline{1}$ before returning to $\mathcal{A}$ (cf. Proposition 4), we get Theorems 1 and 2.

Thus, our aim is to construct an event $\mathcal{E}_\sigma$ such that (5.2) and (5.3) hold true. First, we present the idea for the construction of $\mathcal{E}_\sigma$; formal definitions will follow. We begin by recalling that $\mathcal{A}$ is defined in such a way that for all $\sigma \in \mathcal{A}$ one descents (energy decreases) to a set of non-interacting subcritical octagons in time of order T_0 (see the equation (3.100)). Then, with high probability, in a time shorter than $T_1 = \exp\{\beta E(L^* - 1)\}$ we reach the configuration $-\underline{1}$. The first part of $\mathcal{E}_\sigma$ refers to this shrinking phenomenon. This stage of $\mathcal{E}_\sigma$ is called *contraction* and is denoted by $\mathcal{E}^{(c)}$.

The subsequent stage consists just in staying in $-\underline{1}$ for a time of order $\exp\{\beta E(L^* - 1)\}$; it is called *waiting* and is denoted by $\mathcal{E}^{(w)}$. This (random) time spent in $-\underline{1}$ before the growth of a droplet up to the critical nucleus can be considered as a "global resistence time" as it will be clear later; its introduction will lead to a gain of a factor $\exp\{\beta E(L^* - 1)\}$, due to the "temporal entropy", in our lower bound for the probability $P(\mathcal{E}_\sigma)$. (Remember that $E(L^* - 1) < 2\tilde{J} = 2J - 4K$ and the energy for creating a plus spin in a sea of minuses is $4\tilde{J} + 4K = 4J - 4K$; hence, the time for the creation of a plus spin is, with high probabiltity, much longer than $\exp\{\beta E(L^* - 1)\}$).

Fig. 5.1

Next, during the third stage, that we call *embryonal* and denote by $\mathcal{E}^{(e)}$, we create, sequentially, the first stable regular octagonRecall that we identify an octagon Q with the spin configuration where the pluses are precisely the spins inside Q.(of edge 2) $Q(l=2)$ in 12 elementary (single spin flip) steps with increase of energy during the first 11, and with a loss of energy during the 12th one. At each step the configuration will consist of a unique closed contour $\bar{\gamma}_i$, $i = 1, \ldots, 12$ as indicated in Fig. 5.1. Notice that in $\mathcal{E}_\sigma$, from $\bar{\gamma}_1$ up to the protocritical droplet, all the octagons will be centered — their upper left corner belonging to the dual lattice is located in the point $(-\frac{1}{2}, \frac{1}{2})$. The dot in Fig. 5.1 represents the origin.

Fig. 5.2

Subsequently, there is a stage called *regular*, $\mathcal{E}^{(r)}$, during which one passes through regular octagons $Q(l)$. The passage from an octagon $Q(l)$ to the following one, $Q(l+1)$, is by a sequence of canonical growth (reversed to sequence of canonical cuts (cf. Lemma 3.5)). Fig. 5.2 shows one particular sequence of octagons on the path from $Q(l)$ to $Q(l+1)$ (here $l=3$). Namely, the octagonal droplet $\gamma_l^{(i)}$, $i = 1, \ldots, 14$, is obtained from the preceeding one, $\gamma_l^{(i-1)}$, by adding the i-th edge as indicated. Notice that $\gamma_l^{(14)} \equiv \gamma_{l+1}^{(0)} \in Q(l+1)$. Further, use $S(l)$ to denote the saddle obtained by erasing from $Q(l+1)$ the last (in lexicographic order) l contiguous unit squares adjacent from the interior to its upper right oblique edge. More generally we denote by $S_l^{(i)}$ the sadle configuration obtained by adding to $\gamma_l^{(i-1)}$ the first unit square of the i-th edge.

The path, visiting 14 octagons indicated in Fig. 5.2, obtained in this way is *almost monotonous* path — it consists of a series of elementary transitions of the following form:

" •" first, starting from an octagon $Q^{(1)}$, say $Q^{(1)} = \gamma_l^{(i-1)}$, a monotonous ascent to $S^{(1)}$ (a saddle between $Q^{(1)}$ and $Q^{(2)}$);

" •" then a descent to a configuration $Q^{(2)}$ (again an octagon), higher then $Q^{(1)}$: $H(Q^{(2)}) > H(Q^{(1)})$;

" •" another ascent to $S^{(2)}$ with $H(S^{(2)}) > H(S^{(1)})$ and so on (see Fig. 5.3).

Fig. 5.3

To get a lower bound on probability of $\mathcal{E}^{(r)}$, we suppose that a path in $\mathcal{E}^{(r)}$ stays in the basin of attraction $\mathcal{B}(\gamma_l^{(i-1)})$ (see (3.7)) up to a random time shorter then $\exp\{\beta[H(S_l^{(i-1)}) - H(\gamma_l^{(i-1)}) + \varepsilon]\}$ with $\varepsilon > 0$ chosen sufficiently small; then it ascents to $S_l^{(i)}$ and afterwards descents to $\gamma_l^{(i)}$; in the next step it stays in $\mathcal{B}(\gamma_l^{(i)})$ up to a time of order at most $\exp\{\beta[H(S_l^{(i)}) - H(\gamma_l^{(i)}) + \varepsilon]\}$ then it ascents to $S_l^{(i)}$ and so on.

The above times are called *resistance times*. Their introduction in definition of $\mathcal{E}^{(r)}$ (and in the subsequent stages) is motivated by the necessity of exploiting the "temporal entropy" to get a correct lower bound. It turns out that, by the above choice of the resistance times, one gets exactly the needed factors. Notice that since the minimum of H on $\partial\mathcal{B}(\gamma_l^{(i)})$ is reached in $S_l^{(i)}$, the probability that, during the time interval of the order $\exp\{\beta[H(S_l^{(i)}) - H(\gamma_l^{(i)})]\}$, the process does not leave $\mathcal{B}(\gamma_l^{(i)})$ is almost one (here we are using the reversibility of the process (see S 1 and Lemma 3.4 above)). On the other side, by the particular choice of a sequence of elementary transitions in the definition of $\mathcal{E}^{(r)}$, the process enters the basin of attraction $\mathcal{B}(\gamma_l^{(i)})$ through the saddle point $S_l^{(i)}$. The combination of these two facts is crucial to get a lower bound that is sufficient for the event $\mathcal{E}$ to satisfy the condition (5.3).

Aa already mentioned in the Introduction, a "local" criterium can be formulated that allows to choose a sequence of elementary transitions to yield a correct probability estimate for the event $\mathcal{E}^{(r)}$: the passage to a successive minimum of H has to occur through a saddle whose height is exactly the one of the minimum of H on the boundary of the basin of attraction of the successive minimum. We stay in the basin of attraction of this new minimum for a "typical" resistance time and then pass to the next one.

As we will see later, such a criterion can be generalized to non-almost-monotonous sequences of steps appearing in the subsequent stages of $\mathcal{E}$, provided we substitute the basin of attraction $\mathcal{B}(Q)$ of a single octagon Q by its generalization $\mathcal{D}(D_1, D_2)$ (see (3.10)).

The "resistance" inside $\mathcal{D}(D_1, D_2)$ will be against a mechanism of escape that is not monotonous (in energy) any more. Actually, it is exactly the escape described in the shrinking event $\mathcal{E}_\sigma^s$ introduced in the proof of Proposition 1 (see the definition (3.95)). In addition, the descent from a saddle (in $\partial\mathcal{D}(D_1, D_2)$) to the corresponding minimum will involve some tunneling phenomena (passing through some local saddle points).

The stage $\mathcal{E}^{(r)}$ ends once we reach the octagon $Q(l^*)$. After it, there is a stage that we call *transient* and denote by $\mathcal{E}^{(t)}$ during which, following a very special (not almost monotone) sequence of octagons, we reach a standard octagon with $L_1 = L_2 = l^* + 2$. Namely, the first one with $\min(L_1, L_2) > l^* + 1$. To describe the "transient", which turns out to be a somehow complicated mechanism, we need to define several particular octagons. To this end we will make a repeated use of drawings.

The droplet $Q^{(0)}_{l^*,l^*}$ is the regular (and simultaneously the smallest standard) centered octagon with $l = l^*$. The configurations $\bar{Q}^{(i)}_{l^*,l^*}$, $i = 1, 2, 3$ are the octagons obtained by adding, sequentially, to $Q^{(0)}_{l^*,l^*}$ the edges 1, 2, and 3, as indicated in Fig. 5.4. We notice here

Fig. 5.4

that $\bar{Q}^{(1)}_{l^*,l^*}$, $\bar{Q}^{(2)}_{l^*,l^*}$, respectively, correspond to Q^*_2, $\widetilde{Q}_1$ introduced in the proof of Proposition 2.

The configurations $\bar{S}^{(i)}_{l^*,l^*}$, $i = 1, 2, 3$ are the saddles obtained by adding to $Q^{(i-1)}_{l^*,l^*}$ the first unit square of the i-th edge. Further, let $Q^{(0)}_{l^*+1,l^*} \equiv \bar{Q}^{(3)}_{l^*,l^*}$ (= standard octagon with $L_1 = l^* + 1$, $L_2 = l^*$) and let $Q^{(i)}_{l^*+1,l^*}$, $i = 1, 2, 3$ be the octagons obtained by adding, sequentially, to $Q^{(i-1)}_{l^*+1,l^*}$ the edges $1, 2$ and 3 as indicated in Fig. 5.5. Again, the configurations $S^{(i)}_{l^*+1,l^*}$ are the saddles obtained by adding to $Q^{(i-1)}_{l^*+1,l^*}$ the first unit square of the i-th edge. Similarly, let $Q^{(0)}_{l^*+1,l^*+1} \equiv Q^{(3)}_{l^*+1,l^*}$ (= standard octagon with $L_1 = L_2 = l^* + 1$) and $Q^{(i)}_{l^*+1,l^*+1}$, $i = 1, 2, 3$, be the octagons obtained by adding, sequentially, to $Q^{(i-1)}_{l^*+1,l^*+1}$ the edges $1, 2$ and 3 (cf. Fig. 5.5). Again, the configurations $S^{(i)}_{l^*+1,l^*+1}$ are the saddles defined in the same way as $S^{(i)}_{l^*+1,l^*}$.

In the first part of the event $\mathcal{E}^{(t)}$ our process will stay in $\mathcal{D}(D_1 = D_2 = l^*)$ — the generalized basin of attraction of the standard (and regular) octagon $Q(D_1 = D_2 = l^*)$ that has been defined in the equation (3.10) — for a time of order $\exp\{\beta h(l^* - 1)\}$. Then, after visiting for the last time $Q^{(0)}_{l^*,l^*}$, it will pass to $\bar{Q}^{(1)}_{l^*,l^*}$ through $\bar{S}^{(1)}_{l^*,l^*}$, staying subsequently in $\mathcal{B}(\bar{Q}^{(1)}_{l^*,l^*})$ (the true basin of attraction of $\bar{Q}^{(1)}_{l^*,l^*}$) for a time of order $\exp\{\beta h(l^* - 2)\}$. Then it jumps to $\bar{Q}^{(2)}_{l^*,l^*}$, passing through $\bar{S}^{(2)}_{l^*,l^*}$; it stays in $\mathcal{B}(\bar{Q}^{(2)}_{l^*,l^*})$ for a time of order $\exp\{\beta h(l^* - 2)\}$ and then it passes again to $Q^{(3)}_{l^*,l^*}$ through $\bar{S}^{(3)}_{l^*,l^*}$. (Notice that when we say "the process stays in certain set of configurations for a time of order $\exp\{\beta\Delta\}$", we actually mean "it stays there for a random time shorter than $\exp\{\beta(\Delta + \varepsilon)\}$ with a suitable, sufficiently small, positive

Fig. 5.5

ε").

This first part of $\mathcal{E}^{(t)}$ can be better understood by observing the landscape of the energy depicted in Fig. 5.6.

Fig. 5.6

The second and third parts of $\mathcal{E}^{(t)}$, consisting of the transitions from the octagons $Q^{(0)}_{l^*+1,l^*}$ to $Q^{(0)}_{l^*+1,l^*+1}$ and from $Q^{(0)}_{l^*+1,l^*+1}$ to $Q^{(0)}_{l^*+2,l^*+1}$, respectively, are very similar.

Namely, we stay in $\mathcal{B}(Q^{(0)}_{l^*+1,l^*})$ for a time of order $\exp\{\beta h(l^*-1)\}$ then pass to $Q^{(1)}_{l^*+1,l^*}$ through $S^{(1)}_{l^*+1,l^*}$ and stay in $\mathcal{B}(Q^{(1)}_{l^*+1,l^*})$ for a time $\exp\{\beta h(l^*-2)\}$; after that we go to

$Q^{(2)}_{l^*+1,l^*}$ passing through $S^{(2)}_{l^*+1,l^*}$ and after staying in $\mathcal{B}(Q^{(2)}_{l^*+1,l^*})$ for a time $\exp\{\beta(2K-h)\}$ we pass through $S^{(3)}_{l^*+1,l^*}$ to $Q^{(3)}_{l^*+1,l^*+1}$.

After reaching the octagon $Q^{(0)}_{l^*+2,l^*+1}$, the last part of $\mathcal{E}^{(t)}$ starts; it consists of a transition to $Q^{(0)}_{l^*+2,l^*+2}$ via a mechanism that will be repeated several times during the subsequent stage of $\mathcal{E}$ that we call *standard* and denote by $\mathcal{E}^{(s)}$. In other words, the last part of $\mathcal{E}^{(t)}$ can be considered also as the first part of $\mathcal{E}^{(s)}$.

Fig. 5.7

During $\mathcal{E}^{(s)}$ our process will visit a sequence of growing standard octagons inscribed in squares or almost squares of the form:

$L_1, L_2 \equiv L, L+1 \to L+1, L \to L+1, L+1 \to L+2, L+1, \dots$ from $L_1 = l^*+2$, $L_2 = l^*+1$ up to $L_1 = L^*$, $L_2 = L^*-1$ (cf. (2.17)). These transitions are performed via a canonical (not almost monotonous) sequence of (not standard) octagons. Consider a standard octagon with $L_1 = L_2 = L$, $l^*+2 \le L \le L^*-1$ and with upper left corner in $(-\frac{1}{2}, \frac{1}{2})$; call it $Q^0_{L,L}$. Let $Q^{(i)}_{L,L}$, $i = 1,2,3$ be the octagons obtained by adding, sequentially, to $Q^{(i-1)}_{L,L}$ the edges $1,2$ and 3 as indicated in Fig. 5.7. Similarly we call $Q^{(0)}_{L+1,L} \equiv Q^{(3)}_{L,L}$ the standard octagon with $L_1 = L+1, L_2 = L,\quad l^*+1 \le L \le L^*-1$, and with left upper corner in $(-\frac{1}{2}, \frac{1}{2})$; $Q^{(i)}_{L+1,L}$, $i = 1,2,3$, are the octagons obtained by adding, sequentially to $Q^{(i-1)}_{L+1,L}$ the edges

Fig. 5.8

$1, 2, 3$ as indicated in Fig. 5.7. The configurations $\bar{S}^{(i)}_{L_1,L_2}$, $i = 1, 2, 3$, are the saddles obtained from $Q^{(i-1)}_{L,L_2}$ by adding the first unit square to the i-th edge.

The canonical mechanism to pass from a standard octagon, say $Q^{(0)}_{L,L}$ to the following one in the sequence, $Q^{(0)}_{L+1,L}$, is as follows. We stay in $\mathcal{D}(D_1, D_2)$, $D_1 = L + 2(l^* - 1)$, $D_2 = L + 2(l^* - 1)$, for the time $\exp\{\beta E(L)\}$ (see the equations (3.71) and (3.72)) then we jump to $S^{(1)}_{L,L}$ and then in a time $\sim \exp\{(2K - h)\beta\}$ we pass to the next standard octagon $Q^{(0)}_{L+1,L}$. This transition from $S^{(1)}_{L,L}$ to $Q^{(0)}_{L+1,L}$ is not a purely downhill path but involves two tunnelings. The case of the transition $Q^{(0)}_{L+1,L} \to Q^{(0)}_{L+1,L+1}$ is completely analogous: in general the resistance time in $\mathcal{D}(D_1, D_2)$ with $D_1 = L_1 + 2(l^* - 1)$ and $D_2 = L_2 + 2(l^* - 1)$ is $\exp\{\beta E(L_1 \wedge L_2)\}$. A transition from $Q_{L,L}$ to $Q_{L+1,L}$ for $L \geq l^* + 2$ is represented in Fig. 5.8.

A similar picture describes the transition $Q^{(0)}_{L+1,L} \to Q^{(0)}_{L+1,L+1}$. In Fig. 5.9 we represent the transition from $Q^{(0)}_{l^*+2,l^*+1}$ to $Q^{(0)}_{l^*+2,l^*+2}$, namely the last part of the transient event.

We summarize the description of the transient event $\mathcal{E}^{(t)}$ in Figs. 5.10, 5.11.

Before passing to a formal description of the event $\mathcal{E}$, we would like to make several additional observations concerning the events $\mathcal{E}^{(t)}$ and $\mathcal{E}^{(s)}$.

1) The transition $\gamma^{(13)}_{l^*-1} \to S(l^*) \to Q^{(0)}_{l^*,l^*} \to \bar{S}^{(1)}_{l^*,l^*} \to \ldots$ is not almost monotonous.

Fig. 5.9

Fig. 5.10

2) The sequence $\bar{Q}^{(1)}_{l^*,l^*}$, $\bar{Q}^{(2)}_{l^*,l^*}$, $\bar{Q}^{(3)}_{l^*,l^*}$ is not analogue of $Q^{(1)}_{l^*+1,l^*}$, $Q^{(2)}_{l^*+1,l^*}$, $Q^{(3)}_{l^*+1,l^*}$ or, more generally of $Q^{(1)}_{L_1,L_2}$, $Q^{(2)}_{L_1,L_2}$, $Q^{(3)}_{L_1,L_2}$ for the subsequent values of L_1, L_2. This part of the event $\mathcal{E}^{(t)}$ provides a good example to clarify the local criterion we referred to before. In the sequence $Q^{(0)}_{l^*,l^*}$, $\bar{S}^{(1)}_{l^*,l^*}$, $\bar{Q}^{(1)}_{l^*,l^*}$, and $\bar{S}^{(2)}_{l^*,l^*}$, we always respect the condition of passing through a saddle S such that $H(S)$ is the minimum in the boundary of the (possibly generalized) subsequent basin of attraction.

Fig. 5.11

To clarify better this point let us suppose that we defined the first part of $\mathcal{E}^{(t)}$ concerning the transition $Q^{(0)}_{l^*,l^*} \to Q^{(0)}_{l^*+1,l^*}$ in a different way. Namely, instead of $\bar{Q}^{(i)}_{l^*,l^*}$ we take the sequence of octagons $Q^{(i)}_{l^*,l^*}$, $i = 1, 2$ and 3 defined, in the usual way, according to Fig. 5.12. Notice that $\bar{Q}^{(1)}_{l^*,l^*} \neq Q^{(1)}_{l^*,l^*}$, whereas $\bar{Q}^{(2)}_{l^*,l^*} = Q^{(2)}_{l^*,l^*}$, $\bar{Q}^{(3)}_{l^*,l^*} = Q^{(3)}_{l^*,l^*} \equiv Q^{(0)}_{l^*+1,l^*}$, and $Q^{(i)}_{l^*,l^*}$ would be the analogue for $L_1 = L_2 = l^*$ of our $Q^{(i)}_{L_1,L_2}$ as they are defined for the subsequent L_1, L_2's.

Fig. 5.12

If we define $S^{(i)}_{l^*,l^*}$ as the saddle obtained by adding to $Q^{(i-1)}_{l^*,l^*}$ the first unit square in the i-th edge, we can represent the landscape of energy for this different transition as indicated in Fig. 5.12.

Fig. 5.13

We observe that in the mechanism described in Fig. 5.13 what gets wrong with respect to our criterion is the transition $Q^{(1)}_{l^*,l^*} \to S^{(2)}_{l^*,l^*} \to Q^{(2)}_{l^*,l^*}$, since the minimum of H in $\partial\mathcal{D}(Q^{(2)}_{l^*,l^*})$ is reached in $\bar{S}^{(2)}_{l^*,l^*}$ and not in $S^{(2)}_{l^*,l^*}$.

3) In the sequence $Q^{(0)}_{l^*+1,l^*} \to Q^{(0)}_{l^*+1,l^*+1} \to Q^{(0)}_{l^*+2,l^*+2}$ we could have followed another path without violating our local criterion. This is a consequence of a degeneracy of the minimum of H in $\mathcal{D}(D_1 = 3l^* - 1, D_2 = 3l^* - 1)$ and in $\mathcal{D}(D_1 = 3l^*, D_2 = 3l^* - 1)$; as we already noticed in the proof of Proposition 2, this minimum is reached for $L_1 \wedge L_2 = l^* + 1$, both in $\widetilde{S}_2$ and in $\hat{S}_2$ where $\widetilde{S}_2$ and $\hat{S}_2$ are saddles defined during the proof of Proposition 1 and represented in Fig. 3.8, 3.9, respectively. To be more precise, consider the sequence $\bar{Q}^{(1)}_{l^*+1,l^*}$, $\bar{Q}^{(2)}_{l^*+1,l^*} \equiv Q^{(2)}_{l^*+1,l^*}$, $\bar{Q}^{(3)}_{l^*+1,l^*} = Q^{(3)}_{l^*+1,l^*}$, $\bar{S}^{(1)}_{l^*+1,l^*}$, $\bar{S}^{(2)}_{l^*+1,l^*}$, $\bar{S}^{(3)}_{l^*+1,l^*}$ similarly to $\bar{Q}^{(1)}_{l^*,l^*}$, $\bar{Q}^{(2)}_{l^*,l^*}$, $\bar{Q}^{(3)}_{l^*,l^*}$, $\bar{S}^{(1)}_{l^*,l^*}$, $\bar{S}^{(2)}_{l^*,l^*}$, and $\bar{S}^{(3)}_{l^*,l^*}$ (see Fig. 5.4). Then, for $D_1 = 3l^* - 1 = D_2$, the configuration $\bar{S}^{(2)}_{l^*,l^*}$ coincides with $\hat{S}_2$ (see Fig. 3.9) and

$$\min_{\sigma \in \partial B(D_1,D_2)} H(\sigma) = H(\bar{S}^{(2)}_{l^*+1,l^*}) = H(\hat{S}^{(2)}_{l^*+1,l^*}). \tag{5.4}$$

A similar statement is true for $D_1 = 3l^*, D_2 = 3l^* - 1$.

4) Our mechanism of growth is, in all the stages, exactly the reverse of the best mechanism of shrinking described in the proof of Propositions 1, 2, 3.

To clarify the relation between the notation we used in Propositions 1, 2, and 3, to describe the shrinking phenomenon (following the drift) and the present construction for describing the growth up to the critical size (against the drift), consider, for example, a standard octagon $\widetilde{Q}_0$ with $L_2 = L + 1$, $L_1 = L$. We recall that $\widetilde{Q}_3$ is again a standard octagon with $L_1 = L_2 = L$. We have

$$Q^{(3)}_{L,L-1} \equiv Q^{(0)}_{L,L} = \widetilde{Q}_3$$

$$Q^{(1)}_{L,L} = \widetilde{Q}_2, \quad Q^{(2)}_{L,L} = \widetilde{Q}_1, \quad Q^{(3)}_{L,L} = \widetilde{Q}_0.$$

5) Looking at Fig. 5.10 one easily realizes that the relative heights in energy of $S^{(1)}_{L_1,L_2}$, $S^{(2)}_{L_1,L_2}$, and $S^{(3)}_{L_1,L_2}$, in the transitions from $L_1, L_2 \to L'_1, L'_2$ in $\mathcal{E}^{(t)}$, namely, in the transitions $l^*, l^* \to l^*+1, l^* \to l^*+1, l^*+1 \to l^*+2, l^*+1 \to l^*+2, l^*+2$, change according to the value of $L'_1 \wedge L'_2$.

For $L'_1 \wedge L'_2 = l^*$, the saddle $S^{(3)}$ is the highest one.

For $L'_1 \wedge L'_2 = l^*+1$, the saddle $S^{(2)}$ is the highest one.

For $L'_1 \wedge L'_2 = l^*+2$ (or larger) the saddle $S^{(1)}$ is the highest one.

In any case the differences in height between the saddles (that change sign passing from $L'_1 \wedge L'_2 = l^*$ to $L'_1 \wedge L'_2 = l^*+2$) are, in absolute value, of order ηh with $\eta < 1$ defined in the equation (2.18).

6) The introduction of the generalized basin $\mathcal{D}(D_1, D_2)$ in place of the usual $\mathcal{B}(Q)$ (that was needed in $\mathcal{E}^{(t)}$ and $\mathcal{E}^{(s)}$) is strictly related to the presence of a non almost monotonous path.

The next to the last transition in the standard stage ends with the formation of a standard octagon with $L_1 = L^*$ and $L_2 = L^* - 1$.

The very last transition is then just a flip of a minus spin adjacent to a long coordinate edge in the standard octagon with $L_1 = L^*$ and $L_2 = L^* - 1$ (namely, the creation of a unit square protuberance adjacent to that edge). In this way we form a global saddle ("protocritical") droplet in $\mathcal{P}$ (see [NS]) and enter into $\partial\mathcal{A}$ (see Fig. 5.14).

To conclude our preliminary discussion we want to say that the property of $Q(D^*, D^*)$ of being the minimal supercritical standard octagon can be expressed by the fact that L^* is the minimal value of L for which

$$H(S^{(1)}_{L,L}) - H(Q_{L+1,L}) = E(L) > 2\tilde{J} - h.$$

Fig. 5.14

Remember that during the growth up to the protocritical droplet, described by $\mathcal{E}^{(s)}$ we had

$$H(S^{(1)}_{L,L}) - H(Q_{L+1,L}) < 2\tilde{J} - h$$

as indicated in Fig. 5.8.

Let us now start with the formal definition of $\mathcal{E}$. Given $\sigma \in \mathcal{A}$ and $t_c \in \mathrm{N}$ to be fixed later, we define

$$\mathcal{E}^{(c)}_{\sigma,t_c} = \{\sigma_0 = \sigma, \tau_{-\underline{1}} = t_c\}. \tag{5.5}$$

Given $t_w \in \mathrm{N}$ we set

$$\mathcal{E}^{(w)}_{t_w} = \{\sigma_t = -\underline{1}, 0 \le t \le t_w\}. \tag{5.6}$$

Now, consider the sequence of clusters $\bar{\gamma}_1, \ldots \bar{\gamma}_{12}$ specified in Fig. 5.1. We define

$$\mathcal{E}^{(e)} = \{\sigma_0 = -\underline{1}, \sigma_1 = \bar{\gamma}_1, \ldots, \sigma_{12} = \bar{\gamma}_{12}\}. \tag{5.7}$$

We recall that the energy is strictly increasing from -1 to $\bar{\gamma}_{12}$; indeed, we have

$$
\begin{aligned}
&H(\bar{\gamma}_1) - H(-\underline{1}) = 4J - 4K - h, && H(\bar{\gamma}_2) - H(\bar{\gamma}_1) = 2J - h,\\
&H(\bar{\gamma}_3) - H(\bar{\gamma}_2) = 2J - h - 2K, && H(\bar{\gamma}_4) - H(\bar{\gamma}_3) = -h + 2K,\\
&H(\bar{\gamma}_5) - H(\bar{\gamma}_4) = 2J - 2K - h, && H(\bar{\gamma}_6) - H(\bar{\gamma}_5) = 2J - 4K - h,\\
&H(\bar{\gamma}_7) - H(\bar{\gamma}_6) = 2K - h, && H(\bar{\gamma}_8) - H(\bar{\gamma}_7) = 2K - h,\\
&H(\bar{\gamma}_9) - H(\bar{\gamma}_8) = 2J - 4K - h, && H(\bar{\gamma}_{10}) - H(\bar{\gamma}_9) = 2K - h,\\
&H(\bar{\gamma}_{11}) - H(\bar{\gamma}_{10}) = 2J - 4K - h
\end{aligned}
\tag{5.8}
$$

To present explicit definitions for the subsequent stages $\mathcal{E}^{(r)}, \mathcal{E}^{(t)}, \mathcal{E}^{(s)}$ of the event $\mathcal{E}_\sigma$, we would like to use the general set-up, described in Section 3, based on the introduction of a set of auxiliary Markov chains.

Thus, we have to specify an integer N, a sequence of octagons $Q_1, \ldots, Q_N$, connected sets $B_1, \ldots, B_N$, saddles $S_2, \ldots, S_{N+1}$, with $S_i \in \partial B_i \cap B_{i+1}$, $i = 1, \ldots, N$, $S_{N+1} \in \partial B_N$, as well as resistance times $\bar{t}^i_u$ and "descent" times $\bar{t}^i_d$.

We begin by saying that part of the octagons $Q_1, \ldots, Q_N$ entering in our construction will be standard (see the equation (2.17)). The other ones will be non-standard (some l_i's will differ from l^*); further, the sets B_i, $i = 1, \ldots, N$ will be either

1) the basin of attraction $\mathcal{B}(Q_i)$ of Q_i (see the equation (3.7)), if Q_i is not standard or

2) the domain of attraction $\mathcal{D}(D_1, D_2)$ (see the equation (3.10)) if $Q = Q(D_1, D_2)$ is a standard octagon.

At the end of our construction we will consider the set $\bar{\mathcal{F}} \cap \mathcal{G}$ (cf. the equations (3.57), (3.54), (3.56), (3.48), and (3.53). It will contain the regular, the transient and the standard stages. We have, with obvious meaning of the symbols,

$$
\begin{aligned}
N =& N_r + N_t + N_s\\
N_r = 14(l^* - 2),\ N_t &= 7,\ N_r = 2[L^* - (l^* + 2)],
\end{aligned}
\tag{5.9}
$$

and

$$Q_1, S_2, Q_2, S_3, \ldots, Q_N, S_{N+1} \equiv \gamma_2^0, S_2^1, \ldots, \gamma_2^{13}, S_2^{14}, \gamma_3^0, \ldots, \gamma_{l^*-1}^{13}, S_{l^*-1}^{14}, Q_{l^*,l^*}^{(0)}, \bar{S}_{l^*,l^*}^{(2)}, Q_{l^*,l^*}^{(2)}, S_{l^*,l^*}^{(3)}, Q_{l^*+1,l^*}^{(0)}, S_{l^*+1,l^*}^{(1)}, Q_{l^*+1,l^*}^{(1)}, S_{l^*,l^*}^{(2)}, Q_{l^*+1,l^*+1}^{(0)}, S_{l^*+1,l^*+1}^{(1)}, Q_{l^*+1,l^*+1}^{(1)}, S_{l^*+1,l^*+1}^{(2)}, Q_{l^*+2,l^*+1}^{(0)}, S_{l^*+2,l^*+1}^{(1)}, Q_{l^*+2,l^*+2}^{(0)}, S_{l^*+2,l^*+2}^{(1)}, \ldots, S_{L^*-1,L^*-1}^{(1)}, Q_{L^*,L^*-1}^{(0)}, S_{L^*,L^*-1}^{(1)}. \tag{5.10}$$

Now we specify the times $\vec{t}_u^i, \vec{t}_d^i$ corresponding to the different stages: regular, transient and standard.

The times $\vec{t}_u^i, \vec{t}_d^i$ for the regular stages are denoted by $\vec{t}_u^{j,l}, \vec{t}_d^{j,l}$.

Thy correspond, respectively, to the transitions

$$\gamma_l^{(j)} \to S_l^{j+1},\ j = 0, \ldots, 13, S_l^{j+1} \to \gamma_l^{(j+1)}, j = 0, \ldots, 12, S_l^{14} \to \gamma_{l+1}^{(0)},$$

for $l : 2 \le l \le l^* - 1$. They are given by

$$\begin{aligned} \vec{t}_u^{j,l} &= \exp\{\beta[h(l-1)+\delta]\} \ \text{ for } j \neq 1,2,4,6, \\ \vec{t}_u^{j,l} &= \exp\{\beta[h(l-2)+\delta]\} \ \text{ for } j = 1,2,4,6, \\ \vec{t}_d^{j,l} &= \exp\{\beta\delta\}, \end{aligned} \tag{5.11}$$

with δ to be chosen later.

For the transient stage we have

$$\vec{t}_u^j = \exp\{\beta(h(l^*-2)+\delta)\}$$

for the transitions

$$Q_{l^*,l^*}^{(2)} \to S_{l^*,l^*}^{(3)}, Q_{l^*+1,l^*}^{(1)} \to S_{l^*+1,l^*}^{(2)}, Q_{l^*+1,l^*+1}^{(1)} \to S_{l^*+1,l^*+1}^{(2)},$$

further,

$$\vec{t}_u^j = \exp\{\beta(h(l^*-1)+\delta)\}$$

for the transitions

$$Q^{(0)}_{l^*,l^*} \to \bar{S}^{(2)}_{l^*,l^*}, Q^{(0)}_{l^*+1,l^*} \to S^{(1)}_{l^*+1,l^*},$$

as well as

$$\bar{t}^j_u = \exp\{\beta(2h(l^* - 1) - 2(K - h)) + \delta)\}$$

for

$$Q^{(0)}_{l^*+1,l^*+1} \to S^{(1)}_{l^*+1,l^*+1}, Q^{(0)}_{l^*+2,l^*+1} \to S^{(1)}_{l^*+2,l^*+1},$$

and

$$\bar{t}^j_u = \exp\{\beta(E(l^* + 2) + \delta)\}$$

for

$$Q^{(0)}_{l^*+2,l^*+1} \to S^{(1)}_{l^*+2,l^*+2}.$$

Moreover, we take

$$\bar{t}^j_d = \exp\{\beta\delta\}$$

for the transitions

$$S(l^*) \to Q^{(0)}_{l^*,l^*}, \bar{S}^{(2)}_{l^*,l^*} \to Q^{(2)}_{l^*,l^*}, S^{(3)}_{l^*,l^*} \to Q^{(0)}_{l^*+1,l^*}, S^{(1)}_{l^*,l^*} \to Q^{(1)}_{l^*+1,l^*}, S^{(1)}_{l^*,l^*+1} \to Q^{(1)}_{l^*+1,l^*+1}.$$

For all the other transitions of the transient stage we take

$$\bar{t}^j_d = \exp\{\beta(2K - h + \delta)\}$$

and for any transition

$$Q^{(0)}_{L_1,L_2} \to S^{(1)}_{L_1,L_2}$$

of the standard stage,

$$\bar{t}^j_u = \exp\{\beta(E(L_1 \wedge L_2) + \delta)\}.$$

Finally, for every *descent* time of the standard stage we take

$$\bar{t}^j_d = \exp\{\beta(2K - h + \delta)\}.$$

Now we are ready to present the definition of our event $\mathcal{E}_\sigma$,

$$(5.12)\quad \mathcal{E}_\sigma = (\bar{\mathcal{E}}^{(c)}_{\bar t_c}; \bar{\mathcal{E}}^{(w)}_{\bar t_w}; \mathcal{E}^{(e)}; (\mathcal{F}\bigcap\mathcal{G}) =$$
$$= \bigcup_{t_c=1}^{\bar t_c}\bigcup_{t_w=1}^{\bar t_w} \mathcal{E}^{(c)}_{t_c}\bigcap T_{t_c}\mathcal{E}^{(w)}_{t_w}\bigcap T_{t_c+t_w}\mathcal{E}^{(e)}\bigcap T_{t_c+t_w+12}(\bar{\mathcal{F}}\bigcap\mathcal{G}).$$

We repeat that $\bar{\mathcal{F}}\bigcap\mathcal{G}$ is the event, corresponding to the regular, transient and standard stages defined like in the equation (3.57) using the previously defined Q's, B's, S's (see the equation (5.10)) and the corresponding times $\bar t^u_i, \bar t^d_i$. It is immediate to verify inequality (3.62) in our case. Now, to get the basic estimate given by inequality (3.67), namely, in our case,

$$P(\bar{\mathcal{F}}) \geq \exp\{-\beta(H(\mathcal{P}) - H(S_1) + \epsilon)\}, \tag{5;13}$$

we first need to verify (3.66). For every *downhill* transition, namely for every transition of the regular case as well as for the *descent* transitions of the transient case corresponding to $\bar t^j_d = \exp\{\beta\delta\}$ this is an immediate consequence of the inequality

$$\Big(\frac{1}{|\Lambda|}\Big)^{T_o} > \exp(-\epsilon\beta),$$

valid for every $\epsilon > 0$ and β sufficiently large (see (3.100)).

For the transitions

$$S^{(2)}_{l^*+1,l^*} \to Q^{(0)}_{l^*+1,l^*+1}, S^{(2)}_{l^*+1,l^*+1} \to Q^{(0)}_{l^*+2,l^*+1}, S^{(2)}_{l^*+2,l^*+1} \to Q^{(0)}_{l^*+2,l^*+2}$$

of the transient stage, as well as for every transition of the standard case, we notice that from S_j in one step, with probability larger that $\frac{1}{|\Lambda|}$ we go to B_{j+1}. Then the equation (3.66) is an immediate consequence of the argument of proof of Proposition 1.

The inequality (3.65) is very easy to deduce by remarking that $Q_j \to S_{j+1}$ is always an uphill single spin flip transition with the exception of the first part of the transient stage, $Q^{(0)}_{l^*,l^*} \to \bar S^{(2)}_{l^*,l^*}$. In this last case it is easy to prove the equation (3.65) by using a trial event with the resistance time of the order $\exp\{\beta(h(l^*-2)+\delta)\}$ in $\bar{Q}^{(1)}_{l^*,l^*}$ exploited in the usual way. We leave the details to the reader.

For all the other cases that, we repeat, are single spin flip uphill transitions, the lower bound given by (3.65) is immediate.

To get the equation of the form (3.59) we proceed like in the proof of Proposition 1. Namely, we prove condition (C1)

$$\text{there exists } C>0 \text{ such that } \mathcal{P}(\mathcal{G}^c) < \exp(-e^{C\beta}), \tag{5;14}$$

by introducing, for every $j=1,\dots,N$ certain events $\widehat{\mathcal{E}}^j_\sigma$ and times $\tilde{t}^j_u$ satisfying conditions (3.98) and (3.99).

To this end, we observe that, for every non-standard (subcritical) octagon Q_j appearing in $\mathcal{G}$, the corresponding B_j is just the usual basin of attraction and we can introduce the event $\widehat{\mathcal{E}}^j_\sigma$ in the following way:

(1) First, we pass to Q_j via a descent path in a time of order T_0. (For every $\epsilon>0$ and β sufficiently large the corresponding probability is larger than $\exp(-\epsilon\beta)$.)

(2) Then we pass to the minimal saddle S in ∂B_j (corresponding to the shrinking mechanism since Q_j is subcritical) by a sequence of corner erosions.

The corresponding probability estimate is like (3.65) with a proper choice of ϵ.

Otherwise, for the cases of standard octagons (appearing both in the transient and in the standard stages) we take $B_j=\mathcal{D}(L_1+2(l^*-1),L_2+2(l^*-1))$ for the domain of attraction. The time $\tilde{t}^j_u$ can be taken as $\exp\{\beta(E(L_1\wedge L_2)+\epsilon)\}$ and the escape event $\widehat{\mathcal{E}}^j_\sigma$ is just the shrinking event $\mathcal{E}^s_\sigma$ constructed in the proof of Proposition 1.

Now, it is easy to see that, for every sufficiently small $\epsilon>0$ and β sufficiently large, if

$$\bar{t}_w=\exp\{\beta[E(L^*-1)+\delta]\},\quad \delta>0,$$

then

$$P(\bar{\mathcal{E}}^{(w)}_{\bar{t}_w};\mathcal{E}^e) \geq \exp\{\beta[E(L^*-1) - H(\bar{\gamma}_{11}) + H(-\underline{1}) - \epsilon]\}. \tag{5;15}$$

Moreover, from Propositions 1, 2, and 3, it follows that for every sufficiently small ϵ and β sufficiently large, once

$$\bar{t}_c = \bar{t}_w = \exp\{\beta[E(L^*-1)+\delta]\},$$

then

$$P(\bar{\mathcal{E}}_{\bar{t}_c}) \geq \exp\{-\beta\epsilon\} \tag{5;16}$$

From (5.12),(5;13), (5;14), (3.67), (5;15), and (5;16) it follows that

$$P(\mathcal{E}_\sigma) \geq \exp\{\beta[E(L^*-1) + H(\mathcal{P}) - H(-\underline{1}) - \epsilon]\}$$

for all sufficiently small ϵ and β sufficiently large.

Finally from (5.2),(5.3), and (5;15) we get (5.1).

Describing the event $\mathcal{E}$, we actually defined an *ϵ-typical path* appearing in the statement of Theorem 3.

The only difference is that while in the definition of $\mathcal{E}_\sigma$ we considered a very particular sequence of configurations (for example, all concerned octagons are centered), defining the set $\mathcal{U}_\epsilon$ of all ϵ-typical paths we can be slightly more flexible and allow also droplets of different positions and orientations.

Namely, an ϵ-typical path describes the typical way followed by our process starting from $-\underline{1}$ to form a critical nucleus and then to go to $+\underline{1}$. It contains, in particular, the stages that we have described when defining our trial event $\mathcal{E}_\sigma$ except for the initial *contraction* and *waiting* stages. In other words an ϵ-typical path will pass, initially, through the *embryonal, regular, transient,* and *standard* stages spending, in the appropriate basins or domains of attraction, suitable intervals of time (*resistance times*).

Then, after getting the set of global saddles (protocritical droplets in $\mathcal{P}$) a new stage starts that we call *supercritical*: we pass from $\mathcal{P}$ to $+\underline{1}$ through a suitable sequence of growing standard octagons with proper resistance times.

The embryonal stage is uphill; the regular, transient, and standard are uphill in average and, finally, the supercritical is downhill in average.

The first (subcritical) portions (embryonal, regular, transient, and standard) of an ϵ-typical path will involve notions generalizing the ones already seen in the definition of the event $\mathcal{E}_\sigma$. Embryonal, regular, transient, and standard stages in $\mathcal{E}_\sigma$ are strictly related to a particular example, with a particular choice of locations and orientations, of the corresponding ones in $\mathcal{U}_\epsilon$. Namely, the generalization in $\mathcal{U}_\epsilon$ with respect to $\mathcal{E}_\sigma$ is only related to geometrical trasformations like translations, rotations or reflections with respect to some lattice axes of the corresponding clusters, octagons, standard octagons involved in $\mathcal{E}_\sigma$. The family of sequences of eleven clusters taking part to the embryonal stage of an ϵ-typical path (sequentially visited in eleven steps) will be specified in great detail. Then, assigning the regular, transient and standard stages, similarly to what we did in the definition of $\mathcal{E}_\sigma$, will consist in specifying a set of sequences of *octagons* $Q_1, \dots, Q_{N+1}$ with $Q_1 = Q(2)$, $Q_{N+1} = Q(D^*, D^*) \equiv$ critical nucleus, connected regions $B_1, \dots, B_{N+1}$ with $B_i \ni Q_i$, saddles $S_2, \dots, S_{N+1}$ with $S_{i+1} \in B_i \cup B_{i+1}$, resistance times (see below) .

Further, $N = N_r + N_t + N_s$ with N_r, N_t and N_s given in the equation (5.9) (the numbers of octagons in *any* sequence, in the concerned stage are exactly the same as the numbers of the corresponding ones in $\mathcal{E}_\sigma$).

Then, after $Q(D^*, D^*)$, comes the supercritical stage $Q_{N+1}, \dots, Q_{N+2(M-D^*)}, +\underline{1}$, with the corresponding $B_{N+1}, \dots, B_{N+2(M-D^*)}$, $S_{N+2}, \dots, S_{N+2(M-D^*)}$ and resistance times.

Regular and part of the transient stage will involve non-standard octagons. Part of transient, standard and supercritical stages will involve standard octagons. In the definitions introducing $\mathcal{U}_\epsilon$, during the evolution along a typical path we are less and less specific in the following sense: at the very beginning we assign a class of sequences of clusters (no resistance times up to $Q(2)$), then, in the regular and part of transient stage, we specify a class of non-standard octagons (without specifying the sequences of non-octagonal clusters in between). Subsequently (in part of transient stage and in the standard stage) we will be able to specify only a class of sequences of standard octagons (without specifying the sequences of non-standard octagons in between). Finally, in the supercritical stage, we will not even be able to specify a precise class of sequences of standard octagons and much larger fluctuations have to be allowed. We can say that, as the time goes on our tube of trajectories becomes less and less narrow: it will correspond to the maximum possible specification compatible with an almost full probability estimate.

Now let us start with the detailed definitions.

The first part of the $\mathcal{U}_\epsilon$, called *embryonal*, is given by the set of all paths ($\sigma_1 = \gamma_1, \dots, \sigma_{11} = \gamma_{11}$) where $(\gamma_1, \dots, \gamma_{11})$ is a generic sequence of connected clusters with

" i)" γ_1 given by a unit square; γ_{11} given by the saddle configuration $\bar{\gamma}_{11}$ of Fig. 5.1 arbitrarily located and oriented (the droplet $\bar{\gamma}_{11}$ can go downhill to $Q(2)$ by a single spin-flip);

" ii)" γ_j obtained from γ_{j-1} by adding a unit square touching it and, in this way, increasing the energy without bypassing the energy level of $\bar{\gamma}_{11}$:$H(\gamma_j) > H(\gamma_{j-1})$,
$H(\gamma_j) < H(\bar{\gamma}_{11})$, $j = 2, \dots, 11$.

It is clear by inspection that, with the rule ii), starting from γ_1, after 11 steps we always end up in γ_{11}.

An example of a sequence implementing the characteristics of the embryonal stage is given in Figure 5.1.

Now, for the definition of the subsequent stages we have to define, preliminarily, the

Q_i's, B_i's, and S_i's.

Like in $\mathcal{E}_\sigma$, for all $i = 1, \ldots, N + 2(M - D^*)$ we take $B_i = \mathcal{B}(Q_i)$ = basin of attraction of Q_i if Q_i is a non-standard octagon and $B_i = \mathcal{D}(D_1, D_2)$ = extended domain of attraction of Q_i, if Q_i is a standard octagon. The saddles $S_i \in \partial B_i \cap B_{i+1},\ i = 1, \ldots, N + 2(M - D^*)$ and, for $i = 2, \ldots, N$, $S_i \in$ {set of minimal saddles in ∂B_i}; whereas for $i = N + 1, \ldots, N + 2(M - D^*)$, $S_i \in$ {set of minimal saddles in ∂B_{i+1}}.

Then the B_i's and the S_i's are determined once the Q_i's are given. Any sequence $Q_1, \ldots, Q_{N+2(M-D^*)}$ corresponding to a typical path will be called *typical sequence of octagons* (it will follow the embryonal stage). The set of all typical sequences will be called *typical tube* and will be denoted by $\mathcal{T}_\epsilon$.

Now let $V_i \equiv V(Q_i)$ be given by:

" i)" $V_i = h\hat{\mathcal{L}}_j$ with $\hat{\mathcal{L}}_j \equiv \min_{i=1,\ldots,8} \mathcal{L}_i$

(see definitions before Lemma 3.5), whenever Q_i is a non-standard octagon,

" ii)" $V_i = E(L)$, whenever Q_i is a standard octagon with $\min\{L_1(Q_i), L_2(Q_i)\} = L$, with $l^* \leq L \leq L^* - 1$,

" iii)" $V_i = 2J - 4K - h$, whenever Q_i is standard and supercritical,
$L = \min\{L_1(Q_i), L_2(Q_i)\} \geq L^*$.

A path σ_t will be an ϵ-typical path if, after the embryonal stage it will visit sequentially $Q_1 \in Q(2), \ldots, Q_{N+2(M-D^*)}$ in the following way:

Starting from Q_i it will spend some time t_i inside B_i.

Then, after passing through S_{i+1}, it will reach Q_{i+1} for the first time. Calling t_i the time interval between first arrivals in Q_i and Q_{i+1} we have:

$$\exp \beta(V_i - \epsilon) < t_i < \exp \beta(V_i + \epsilon).$$

So, to conclude the definition of $\mathcal{U}_\epsilon$ we only need to assign the typical tube of octagons. Now we start this definition by distinguishing the regular, transient, standard and supercritical portions.

The *regular portion* of the typical tube is denoted by $\mathcal{T}_\epsilon^{(r)}$. Starting from $Q(2)$, $\mathcal{T}_\epsilon^{(r)}$ contains the set of all the reverse of a sequence of canonical contractions (see Section 3) from $Q(3)$ to $Q(2)$ and so on up to $Q(l^*)$. The transient and standard portions $\mathcal{T}_\epsilon^{(t)}$ and $\mathcal{T}_\epsilon^{(s)}$ of the typical tube are simply given by the set of sequences of octagons obtained by the set of sequences appearing in the definitions given above when introducing $\mathcal{E}_\sigma$, modulo translations and the obvious rotations and reflections (we do not enter into a detailed classification leaving the easy exercise to the interested reader).

After getting $S_{N+1} \in \mathcal{P}$ the supercritical stage $\mathcal{T}_\epsilon^{(sc)}$ starts.

It will consist of the set of sequences of standard octagons $Q(D_1^{(j)}, D_2^{(j)})$, $j = 1, \dots, 2(M - D^*)$ with the following monotonicity property

$$\text{either } (D_1^{(j+1)}, D^{(j+1)}) = (D_1^{(j)} + 1, D^{(j)}) \text{ or } (D_1^{(j+1)}, D^{(j+1)}) = (D_1^{(j)}, D^{(j)} + 1).$$

Proof of Theorem 3

Theorem 3 follows directly by Propositions 1–4 and the easy observation that the embryonal path is the reverse of a downhill shrinking starting from γ_{11}, via a straightforward adaptation of the results of [S 1] based on reversibility of the process (Lemmas 2, 3, 4 and Theorem 1 therein).

CGOV

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