PREPRINT

More on Critical Phenomena in Growth Systems

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Abstract

This paper treats of interacting infinite or finite systems whose components' states are in the set $\{0, 1, 2, \ldots\}$. Components interact with each other in a local deterministic way, in addition to which every component's state grows by one with a constant probability θ at every moment of the discrete time.

Theorem 1 states conditions under which an infinite deterministic ($\theta = 0$) system is an 'eroder', that is every initial condition with a finite set of positive components turns into 'all zeroes' after a finite time.

Theorems 2,3,4 are about probabilistic systems with initial states 'all zeroes'. Our main question about infinite systems is whether the average value of components tends to infinity or remains bounded as $t \to \infty$. The analogous question about finite systems is how long the system's average remains less than a constant: this time may be bounded or tend to infinity as the system' size tends to infinity. Both in the infinite and finite cases sufficient conditions for the latter ways of behavior are given here, which are also necessary under natural assumptions.

1. Introduction

This paper's approach is similar to that of [1] (and we refer to [1] to save space), but the main results are stronger now. Note that some notations of our paper are different from those of [1] and some terms (e.g. polar) are given new definitions here. $\mathbf{R}$ stands for the set of real numbers, $\mathbf{R}_+$ is the set of non-negative real numbers, $\mathbf{Z}$ is the set of integers, $\mathbf{Z}_+$ is the set of non-negative integers, $\mathbf{Z}_{++}$ is the set of positive integers, $[1, k]$ is the segment $\{i \in \mathbf{Z} \; : \; 1 \le i \le k\}$, and $\mathbf{Z}_{\mathsf{size}}$ is the ring of residues modulo SIZE. As usual $[x]$ stands for the greatest integer that does not exceed a given real x. Also let $[x]^+$ stand for $\max([x], 0)$ and $\mathrm{ROUND}(x) = [x + 0.5]$ stand for the nearest integer. Expectation is denoted $\mathbf{E}(\cdot)$.

Let us compare random processes with the state space $\mathbf{R}_+^{\mathbf{Z}^2}$. To specify a state means to assign a real number $x(i,j) \ge 0$ to every $(i,j) \in \mathbf{Z}^2$. Take some initial state and assume that at every moment of the discrete time every $x(i,j)$ undergoes the following two steps:

Step 1 : $x(i,j)$ takes the value of some deterministic TRANSITION FUNCTION $\mathbf{trans}(\cdot)$ of its and its neighbors' previous states.

Step 2 : $x(i,j)$ is incremented by 1 with probability θ independently of the others' increments and of the prehistory.

If $\theta = 0$, our processes are deterministic; these are the subject of our Theorem 1 which gives a sufficient and necessary condition under which any initial state with a finite set of non-zero components turns into 'all zeroes' after several steps.

Our Theorems 2-4 refer to random processes with initial conditions 'all zeroes'. In this case our systems are space-uniform, so expectations of $x(i,j)$ at time t are the same for all $(i,j) \in \mathbf{Z}^2$ and we may call them $\mathbf{E}_\theta(t)$. We are interested in the behavior of $\mathbf{E}_\theta(t)$ as $t \to \infty$. Theorem 1 of [1] showed that in a fairly general case $\mathbf{E}_\theta(t)$ tends to infinity provided θ is close enough to 1. It is very easy to present transition functions for which $\mathbf{E}_\theta(t)$ tends to infinity with any $\theta > 0$, for example if **Step 1** does not change the value of $x(i,j)$ at all, or $\mathbf{trans}(\cdot)$ equals the arithmetical mean of its arguments. However, in the process in which

$$\mathbf{trans}(\cdot) = \text{ROUND}(\text{ARITHM}(x(i,j),\ x(i+1,j),\ x(i,j+1))), \tag{1}$$

where $\text{ARITHM}(\cdot)$ means the arithmetical mean, $\mathbf{E}_\theta(t)$ remains bounded by a constant, provided θ is small enough. This follows from our Theorem 2. (In fact, this process takes values in $\mathbf{Z}_+^{\mathbf{Z}^2}$.) Note that the function (1) does not favor smaller or greater values since

$$\forall\ x, y, z,\ a \in \mathbf{Z}\ :\ \text{ROUND}(\text{ARITHM}(a-x, a-y, a-z) = a - \text{ROUND}(\text{ARITHM}(x,y,z)).$$

Our Theorem 2 can be applied also to the following function which may seem to favor greater values:

$$\mathbf{trans}(\cdot) = \text{ALMOSTMAX}(x(i,j),\ x(i+1,j),\ x(i,j+1))), \tag{2}$$

where

$$\text{ALMOSTMAX}(x,y,z) = \begin{cases} \max\{x,y,z\} & \text{if at least two among } x,y,z \text{ equal } \max(x,y,z)\,, \\ \max\{x,y,z\} - 1 & \text{otherwise.} \end{cases}$$

Thus, for some functions $\mathbf{trans}(\cdot)$, including (1) and (2), there is a critical value of θ, which separates two modes of behavior – with bounded and growing $\mathbf{E}_\theta(t)$.

Theorem 3 proves the analogous phenomenon for finite systems in the same assumptions about the deterministic interaction. Of course, in finite systems $\mathbf{E}_\theta(t)$ can not remain bounded forever: it remains bounded during time which grows exponentially in the system's size.

Geometrical positions of neighbors are essential. For example, it is fairly evident that the processes which apply the same functions ROUND(ARITHM($\cdot$)) and ALMOSTMAX($\cdot$) to $x(i,j)$, $x(i+1,j)$, $x(i-1,j)$, have $\mathbf{E}_\theta(t) \to \infty$ for any $\theta > 0$. Theorem 4 generalizes this fact and states the class of transition functions with which the key condition (8) of Theorems 2 and 3 is not only sufficient, but also necessary for the statements of these theorems to be true.

2. Definitions and Formulations

Cardinality of a finite set S is denoted $|S|$. For any real function F on any non-empty set S we denote:

$$Max(F,S) = \max\{F(e) \ : \ e \in S\}, \qquad Min(F,S) = \min\{F(e) \ : \ e \in S\},$$
$$Gap(F,S) = Max(F,S) - Min(F,S).$$

A FAMILY means a set whose elements are sets. A family F given, UNION(F) stands for the union of elements of F. Say that any two elements of a set are CONNECTED by this set. Call a family F CONNECTED if for any $a, b \in \text{UNION}(F)$ there is a sequence $c_0 = a,\ c_1,\ \ldots,\ c_j = b$ in which every two next terms are connected by some element of F.

Let $\mathbf{Space} = \mathbf{Z}^{\mathsf{dim}}$ in the infinite and $\mathbf{Space} = \mathbf{Z}^{\mathsf{dim}}_{\mathsf{size}}$ in the finite case. Let $\mathbf{Time} = \{t \in \mathbf{Z} \ : \ t \geq -\mathsf{mem}\}$, where $\mathsf{mem} \in \mathbf{Z}_{++}$ is called MEMORY, and

$$\mathbf{V} = \mathbf{Volume} = \mathbf{Space} \times \mathbf{Time} = \{(s,t) : s \in \mathbf{Space},\ t \in \mathbf{Time}\}.$$

Elements of $\mathbf{V}$ are called POINTS and subsets of $\mathbf{V}$ are called POINT-SETS. If $v = (s,t) \in \mathbf{V}$, then $s(v) = s$ and $t(v) = t$. Points with $t < 0$ are called INITIAL, those with $t \geq 0$ are called INNER. $\mathbf{V}_{init}$ and $\mathbf{V}_{inner}$ stand for the initial and inner sets, i.e. sets of initial and inner points. Every inner point (s,t) has n NEIGHBORS

$$\mathbf{N}_i(s,t) = (s - \Delta s_i,\ t - \Delta t_i), \quad i \in [1,n],$$

where the sequence of NEIGHBOR VECTORS

$$\Delta = ((\Delta s_1,\ \Delta t_1), \ldots, (\Delta s_n, \Delta t_n)) \tag{3}$$

is one and the same for all inner points and $1 \leq \Delta t_i \leq \mathsf{mem}$ for all $i \in [1,n]$. For every inner point v its neighborhood $\mathbf{N}(v)$ is the set of its neighbors. Initial points have empty neighborhoods. To represent the system of neighborhoods, we shall use the directed graph V which has $\mathbf{V}$ as its set of vertices, and edges that go to every point from its neighbors. We shall also use the non-directed graph V' with $\mathbf{V}$ as its set of vertices, in which two vertices are connected if one is the the other's neighbor or they are different neighbors of one point.

Every point v has one and the same set $X_v \equiv X_0 = \mathbf{Z}_+$ of possible states. To every point $(s,t) \in \mathbf{V}$ there corresponds a variable $x(s,t) \in X_0$ which represent the state of this point, that is the state of component s at time t. The set $X = X_0^{\mathbf{V}}$ is the configuration space whose elements x are called CONFIGURATIONS. Generally, to any subvolume $S \subseteq \mathbf{V}$ there corresponds its configuration space $X_S = X_0^S$ whose elements $x(S)$ are called configurations on S. Given two configurations x and y on a subvolume S, $x \preceq y$ means that $x(v) \leq y(v)$ for all $v \in S$. For any configuration x its CARRIER $\mathbf{Carrier}(x)$ stands for the set of those points v where $x(v) > 0$. Call a configuration FINITARY if its carrier is finite.

For all inner points one and the same transition function is given

$$\text{(4)} \qquad \mathbf{trans} \ : \ X_{\mathbf{N}(v)} \mapsto X_v, \quad \text{where} \quad X_{\mathbf{N}(v)} = X_0^n \quad \text{and} \quad X_v = X_0.$$

Since our systems are uniform (i.e. neighbor vectors and the transition function are the same for all inner points), it is sufficient to speak about the neighborhood and transition function of the origin $\mathcal{O}$ when imposing conditions. Throughout the paper we assume

$$\text{(5)} \qquad \forall\, x \ : Min(x,\ \mathbf{N}(\mathcal{O})) \ \leq \ \mathbf{trans}(x(\mathbf{N}(\mathcal{O}))) \ \leq \ Max(x,\ \mathbf{N}(\mathcal{O})).$$

Sometimes we also assume monotony

$$\text{(6)} \qquad x \preceq y \implies \mathbf{trans}(x(\mathbf{N}(\mathcal{O})) \leq \mathbf{trans}(y(\mathbf{N}(\mathcal{O})),$$

and translation-invariance in the set of states

$$\text{(7)} \quad \forall\, a \in \mathbf{Z} \ : \ (\forall\, v \in \mathbf{N}(\mathcal{O}) \ : \ x(v) = y(v) + a) \implies \mathbf{trans}(x(\mathbf{N}(\mathcal{O}))) = \mathbf{trans}(y(\mathbf{N}(\mathcal{O}))) + a.$$

Note 1. Given (6), (7) and

$$(\forall\, v \in \mathbf{N}(\mathcal{O}) \ : \ x(v) = 0) \implies \mathbf{trans}(x) = 0.$$

Then (5) follows.

All of these assumptions seem not to be very restrictive; both our examples (1) and (2) satisfy them.

Definition 1. A non-empty subset $S \subseteq \mathbf{N}(\mathcal{O})$ is called a DRAG-SET (of $\mathcal{O}$) if for all x

$$\mathbf{trans}(x(\mathbf{N}(\mathcal{O}))) = Max(x, \mathbf{N}(\mathcal{O})) \implies Max(x, S) = Max(x, \mathbf{N}(\mathcal{O})).$$

Drag-sets of any inner point v are drag-sets shifted by v. Let $\mathsf{DRAG}(\Delta, \mathbf{trans})$ stand for the family of drag-sets. Since $\mathbf{N}(\mathcal{O})$ is always a drag-set, this family is non-empty, whence our next definition makes sense.

Since our volume $\mathbf{V}$ is a subset of the $(\dim+1)$-dimensional discrete space $\mathbf{Z}^{\dim+1}$, we can embed it into a continuous space $R^{\dim+1}$ with the same origin $\mathcal{O}$. Let $\textsc{conv}(S)$ stand for the convex hull of any $S \subset R^{\dim+1}$. For any number k and any set $S \subset R^{\dim+1}$ we define:

$$k \cdot S = \{k \cdot v \ : \ v \in S\}, \qquad \textsc{ray}(S) = \cup\{k \cdot S \ : \ k \geq 0\}.$$

Definition 2. For any sequence (3) of neighbor vectors and any transition function (4) denote

$$\sigma_{drag}(\Delta, \mathbf{trans}) = \cap\{\textsc{ray}(\textsc{conv}(S)) \ : \ S \in \mathsf{DRAG}(\Delta, \mathbf{trans})\}.$$

The following condition plays the key role in our paper:

$$\sigma_{drag}(\Delta, \mathbf{trans}) = \{\mathcal{O}\}. \tag{8}$$

In the deterministic case for every initial condition y the corresponding TRAJECTORY $\mathbf{TR}(y) = x \in X$ is defined by

$$x(v) = \begin{cases} y(v) & \text{if } t(v) < 0, \\ \mathbf{trans}(x(\mathbf{N}(v))) & \text{if } t(v) \geq 0. \end{cases} \tag{9}$$

Our first theorem generalizes some results of [2] and shows relevance of the condition (8) for the deterministic case. Call the deterministic map (9) an ERODER if for any finitary initial condition the resulting trajectory is also finitary.

Theorem 1. Consider the infinite case. Assume (5), (6) and (7). Then the map (9) is an eroder if and only if (8) holds.

Theorems 2,3,4 are about RANDOM PROCESSES, i.e. normed measures on X defined as follows. Let us use (every time the same) mutually independent HIDDEN VARIABLES $h(v)$ for all inner points v, everyone of which equals 1 with probability θ and 0 with probability $1-\theta$. Thus we have the hidden product-measure η on the configuration space $H = \{0,1\}^V$ of the hidden variables. To any sequence (3) of neighbor vectors, any transition function (4) and any value of θ there corresponds a process, i.e. the normed measure on X induced by η with the map from H to X defined in the following inductive way:

$$x(v) = \begin{cases} 0 & \text{if } t(v) < 0, \\ \mathbf{trans}(x(\mathbf{N}(v))) + h(v) & \text{if } t(v) \geq 0. \end{cases}$$

Theorem 2. Consider the infinite case. Assume (5) and (8). Then there is such a constant C that for all inner v and small enough θ

$$Prob(x(v) \geq k) \leq \begin{cases} \theta + (C\theta)^2 & \text{if } k = 1\,, \\ (C\theta)^{k+1} & \text{if } k > 1 \end{cases} \tag{10}$$

and

$$\mathbf{E}(x(v)) = \theta + O(\theta^2). \tag{11}$$

Theorem 3. Consider the finite case. Assume (5) and (8). Then for any T there is such a constant C that for any $s,\ t \le T^{\mathsf{size}}$, any large enough size and any small enough θ

$$Prob(x(s,t) \ge k) \le \begin{cases} \theta + (C\theta)^2 & \text{if } k = 1\,, \\ (C\theta)^{k+1} & \text{if } k > 1 \end{cases} \tag{12}$$

and

$$\mathbf{E}(x(s,t)) = \theta + O(\theta^2). \tag{13}$$

Theorem 4. Consider both infinite and finite case. Assume (5), (6), (7) and negation of (8). Then there is such a constant $C > 0$ that $\mathbf{E}(x(s,t)) \ge C \cdot \theta \cdot t$ for all θ and $(s,t) \in \mathbf{V}$.

Let us compare present results with those of [1]. The following are some definitions of [1] in present notations. A VETO-SET S is such a subset of $\mathbf{N}(\mathcal{O})$ that $\forall\ x\ :\ \mathbf{trans}(x(\mathbf{N}(\mathcal{O}))) \le Max(x,S)$. Let $\mathsf{VETO}(\Delta, \mathbf{trans})$ stands for the family of veto-sets and denote

$$\sigma_{veto}(\Delta, \mathbf{trans}) = \cap\{\text{RAY}(\text{CONV}(S))\ :\ S \in \mathsf{VETO}(\Delta, \mathbf{trans})\}.$$

It is evident that every veto-set is a drag-set, whence $\sigma_{drag}(\Delta, \mathbf{trans}) \subseteq \sigma_{veto}(\Delta, \mathbf{trans})$. Thus Theorems 2 and 3 of [1] follow from our Theorems 2 and 3. On the other hand, not every drag-set is a veto-set. For example, in our processes (1) and (2) every inner point has three neighbors, every two of which form a drag-set, but not a veto-set. Thus, in both examples $\sigma_{drag}(\Delta, \mathbf{trans}) = \{\mathcal{O}\}$, but $\sigma_{veto}(\Delta, \mathbf{trans}) \ne \{\mathcal{O}\}$. So our Theorems 2,3,4 can be applied to these examples while Theorems 2 and 3 of [1] can not. However, if $\mathbf{trans}(\cdot)$ can be expressed through minima and maxima of its arguments, every drag-set is a veto-set. So Theorem 4 of [1] follows from our Theorem 4.

3. Preliminaries

This section contains some facts and definitions which are used more than once in the proofs of our theorems.

For any set $S \subseteq \mathbf{N}(\mathcal{O})$ define a state $\mathsf{IND}_S \in X_{\mathbf{N}(\mathcal{O})}$ as follows:

$$\forall\ v \in \mathbf{N}(\mathcal{O})\ :\ \mathsf{IND}_S(v) = \begin{cases} 0 & \text{if } v \in S\,, \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 1. Given (5), (6) and (7). Then a set $S \subseteq \mathbf{N}(\mathcal{O})$ is a drag-set if and only if $\mathbf{trans}(\mathsf{IND}_S) = 0$.

Proof: If $S = \mathbf{N}(\mathcal{O})$, then S is evidently a drag-set and $\mathbf{trans}(\mathsf{IND}_S) = 0$ from (5).

Now let $S \subset \mathbf{N}(\mathcal{O})$. Then $Max(\mathsf{IND}_S,\ \mathbf{N}(\mathcal{O})) = 1$.

1) Assume $\mathbf{trans}(\mathsf{IND}_S) = 1$. Then $\mathbf{trans}(\mathsf{IND}_S) = Max(\mathsf{IND}_S,\ \mathbf{N}(\mathcal{O}))$, while $Max(\mathsf{IND}_S,\ S) = 0 < 1 = Max(\mathsf{IND}_S,\ \mathbf{N}(\mathcal{O}))$ whence S is not a drag-set.

2) Now assume that S is not a drag-set. So there is such $x \in X_{\mathbf{N}(\mathcal{O})}$ that $Max(x, S) < Max(x, \mathbf{N}(\mathcal{O}))$, but $\mathbf{trans}(x) = Max(x,\ \mathbf{N}(\mathcal{O}))$. Denote $M = Max(x,\ \mathbf{N}(\mathcal{O}))$ and define $y \in X_{\mathbf{N}(\mathcal{O})}$ as follows:

$$\forall\ v \in \mathbf{N}(\mathcal{O})\ :\ y(v) = \begin{cases} M-1 & \text{if } v \in S\ , \\ M & \text{otherwise.} \end{cases}$$

Note that $x(v) \leq y(v)$ for all v. Hence and from (6) $\mathbf{trans}(x) \leq \mathbf{trans}(y)$. But from (5) $Trans(y) \leq M$. Thus $\mathbf{trans}(y) = M$. Hence and from (7) $\mathbf{trans}(\mathsf{IND}_S) = 1$ □

Note 2. Assume (5), (6) and (7). In this case, Lemma 1 gives a rule to discern all the drag sets and check(8). This contrasts [3, 4] where some similar problems have been shown to be algorithmically unsolvable.

Lemma 2. Condition (8) is equivalent to the following: There are such a natural $r \leq \dim +1$ and such r linear functions $L_1, \ldots, L_r$ on $R^{\dim +1}$ that

(14) $$L_1(\mathcal{O}) = \ldots = L_r(\mathcal{O}) = 0,$$

(15) $$L_1 + \ldots + L_r \equiv -t$$

and

(16) For all $i \in [1, r]$ the set $\{e \in \mathbf{N}(\mathcal{O})\ :\ L_i(e)\ \leq\ 0\}$ is a drag-set.

Proof: is similar to that of Lemma 3 in [1] □

¿From now on, whenever (8) is given, we fix the number r and the r-tuple $L = (L_1, \ldots, L_r)$ which satisfy assertions of Lemma 2.

Choose any norm $\mathsf{norm}(v) = \mathsf{norm}(s,t)$ in $\mathbf{R}^{\dim +1}$ such that all the $(\dim +1)$ unit basis vectors have norm 1. Define distance $\mathsf{dist}(v,w) = \mathsf{norm}(v-w)$. Let distance $\mathsf{dist}(S_1, S_2)$ between two finite sets stand for the minimal distance between a point of S_1 and a point of S_2 and let

$\mathsf{diam}(S)$ stand for the diameter of any set S, that is the maximal distance between its points. Denote

(17) $\lambda = \max\{\mathsf{norm}(\Delta s_i, \Delta t_i) \ : \ i \in [1,n]\}$ and $\rho = \max\{L_i(v) \ : \ i \in [1,r], \ \mathsf{norm}(v) \leq 1\}$.

4. Proof of Theorem 1

Proof of Theorem 1 consists of the following two lemmas. For any finitary x on a subvolume $S \subseteq \mathbf{V}$ denote $\mathsf{TOP}(x) = Max(x, S)$.

Lemma 3. Consider the infinite case. Assume (5) and (8). Then for any finitary initial condition y the resulting trajectory $\mathbf{TR}(y)$ is also finitary and

$$\mathsf{diam}(\mathbf{Carrier}(\mathbf{TR}(y))) \ \leq \ \mathrm{const} \cdot (\mathsf{diam}(\mathbf{Carrier}(y)) + \mathsf{TOP}(y)) + \mathrm{const}.$$

Lemma 4. Consider the infinite case. Assume (5), (6), (7) and negation of (8). Then for any $T \in \mathbf{Z}_{++}$ there is a finitary initial condition x such that for every $t \in \mathbf{Time}$ there is s_t such that $y(s_t, t) = T$ where $y = \mathbf{TR}(x)$.

4.1. Proof of Lemma 3

¿From uniformity we may assume

(18) $$\exists \ v \in \mathbf{Carrier}(y) \ : \ s(v) = 0$$

whence

$$\forall \ w \in \mathbf{Carrier}(y) \ : \ \mathsf{norm}(w) \leq \mathsf{diam}(\mathbf{Carrier}(y)) + \mathsf{mem}.$$

Note 3. For any finitary initial y and $T \in \mathbf{Time}$

$$\mathsf{diam}(\{(s,t) \in \mathbf{Carrier}(\mathbf{TR}(y)) \ : \ t \ \leq \ T\}) \ \leq \ \mathsf{diam}(\mathbf{Carrier}(y)) + 2 \cdot \lambda \cdot T.$$

Call a configuration $x \in X$ SUPERIOR if $x(v) \ \geq \ \mathbf{trans}(x(\mathbf{N}(v)))$ for all inner v.

Note 4. Given (6). Given a trajectory x and a superior y such that $x(v) \ \leq \ y(v)$ for all initial v. Then $x \preceq y$.

Choose some $i \in [1, r]$ and let L_i stand for any of the functions provided by Lemma 2. Given any infinite increasing sequence K of real numbers $k_1 < k_2 < k_3 < \ldots$, let STAIRS stand for the following configuration $\mathbf{Stairs}_i(K) = x \in X$:

$$\forall\, v \in \mathbf{V} \;:\; x(v) = \begin{cases} 0 & \text{if } L_i(v) < k_1\,, \\ j & \text{if } k_j \le L_i(v) < k_{j+1}\,. \end{cases}$$

(A similar definition was used in [5] for the one-dimensional case.)

Lemma 5. Assume

(19) $$\forall j \;:\; k_{j+1} - k_j \ge 2 \cdot \lambda \cdot \rho.$$

Then $\mathbf{Stairs}_i(K)$ is a superior.

Proof: Denote $x = \mathbf{Stairs}_i(K)$. First prove that

(20) $$Gap(x, \mathbf{N}(v)) \;\le\; 1 \quad \text{for any inner} \quad v.$$

Assume the contrary: there are such $u_1, u_2 \in \mathbf{N}(v)$ that $x(u_2) - x(u_1) \ge 2$. Then there is such j that $L_i(u_1) < k_j$ and $L_i(u_2) \ge k_{j+1}$, whence

$$L_i(u_2) - L_i(u_1) > k_{j+1} - k_j \ge 2 \cdot \lambda \cdot \rho.$$

But this is impossible since

$$L_i(u_2) - L_i(u_1) = L_i(u_2 - u_1) \le \max\{L_i(w) \;:\; \mathsf{norm}(w) \le 2 \cdot \lambda\} \le 2 \cdot \lambda \cdot \rho.$$

This contradiction proves (20). Thus $Gap(x, \mathbf{N}(v))$ is 0 or 1. Now note that

(21) $$\mathbf{trans}(x(\mathbf{N}(v))) = Min(x, \mathbf{N}(v))$$

follows from (5) if $Gap(x, \mathbf{N}(v)) = 0$ and from Lemma 1 if $Gap(x, \mathbf{N}(v)) = 1$.

On the other hand, (16) assures that the set $\{e \in \mathbf{N}(v) \;:\; L_i(e) \le L_i(v)\}$ is a drag-set, whence this set is non-empty, whence $Min(L_i, \mathbf{N}(v)) \le L_i(v)$, whence $Min(x, \mathbf{N}(v)) \le x(v)$. This and (21) prove our lemma $\square$

Note 5. For any values of parameters $C > 0$ and $C_1, \ldots, C_r$ the configuration $[C \cdot L_i(v) + C_i]^+$ is a stairs. This stairs satisfies (19), provided

(22) $$C \le (2 \cdot \lambda \cdot \rho)^{-1}.$$

Let LIFETIME of any configuration x stand for $\max\{t(v) \;:\; x(v) > 0\}$.

Lemma 6. For any constants $C > 0$ and $C_1, \ldots, C_r$ the configuration $\min\{[C \cdot L_i + C_i]^+ \ : \ i \in [1,r]\}$ has a finite lifetime which does not exceed $(C_1 + \ldots + C_r)/C$.

Proof: Take any (s,t) where $y(s,t) > 0$. Then for all $i \in [1,r]$

$$[C \cdot L_i(s,t) + C_i]^+ > 0, \quad \text{whence} \quad [C \cdot L_i(s,t) + C_i] > 0, \quad \text{whence} \quad L_i(s,t) > -C_i/C.$$

Summing the last inequalities and using (15) proves our lemma $\square$

Lemma 7. Take any finitary initial y and assume (18). Take any $i \in [1,r]$ and denote $x = [C \cdot L_i + C']^+$ where $C > 0$ satisfies (22) and

$$C' = \mathsf{TOP}(y) + C \cdot \rho \cdot (\mathsf{diam}(\mathbf{Carrier}(y)) + \mathsf{mem}) + 1. \tag{23}$$

Then $\mathbf{TR}(y) \ \leq \ x$.

Proof: First let us prove that $y(v) \leq x(v)$ for every initial v. This is evident if $v \notin \mathbf{Carrier}(y)$. Now let $v \in \mathbf{Carrier}(y)$. Then using (18)

$$\begin{aligned} x(v) \ &\geq \ C \cdot L_i(v) + C' - 1 \ \geq \ C' - 1 - C \cdot \rho \cdot \mathsf{norm}(v) \quad \geq \\ & C' - 1 - C \cdot \rho \cdot (\mathsf{diam}(Carrier(y)) + \mathsf{mem}) \quad \geq \quad \mathsf{TOP}(y) \ \geq \ y(v). \end{aligned}$$

Now to prove Lemma 7 in general: Note 5 and Lemma 5 assure that x is a superior, which allows to apply Note 4 $\square$

Now to prove Lemma 3. Take any finitary initial y, assume (18) and denote $C = (2 \cdot \lambda \cdot \rho)^{-1}$. Then from Lemma 7

$$y \ \preceq \ \min\{[C \cdot L_i + C']^+ \ : \ i \in [1,r]\}.$$

where C' is defined by (23). ¿From Lemma 6 lifetime of the last expression does not exceed $\mathrm{const} \cdot C'$. Together with (23) and Note 3 this proves Lemma 3 $\square$

4.2. Proof of Lemma 4

Call a set $S \subset V$ STEADY if for every inner $v \in S$ every drag-set of v intersects S.

Lemma 8. Given assumptions of Theorem 1, there is such a steady set Tower, that

$$\forall \ t \in \mathbf{Time} \ : \ 1 \leq |\{v \in \mathsf{Tower} \cap \mathbf{Z}^{\dim+1} \ : \ t(v) = t\}| \leq \mathrm{const}. \tag{24}$$

Proof: is similar to that of Lemma 24 in [1] $\square$

Denote $\mathsf{STEP} = \{x \in X_{\mathbf{N}(\mathcal{O})} \ : \ Gap(x, \ \mathbf{N}(\mathcal{O})) = 1\}$.

Lemma 9. Given (5), (6) and (7). Then for any $x \in \mathsf{STEP}$

(25) $(\forall\ S \in \mathsf{DRAG}\ :\ Max(x,S) = Max(x,\mathbf{N}(\mathcal{O})) \implies \mathbf{trans}(x(\mathbf{N}(\mathcal{O})) = Max(x,\mathbf{N}(\mathcal{O})).$

Proof: Take any $x \in \mathsf{STEP}$, assume $Max(x,S) = Max(x,\mathbf{N}(\mathcal{O}))$ for all $S \in \mathsf{DRAG}$, but $\mathbf{trans}(x(\mathbf{N}(\mathcal{O}))) < Max(x,\mathbf{N}(\mathcal{O}))$ and come to a contradiction. Denote $M = Min(x,\mathbf{N}(\mathcal{O}))$ and $S = \{v \in \mathbf{N}(\mathcal{O})\ :\ x(v) = M\}$. Then $\mathsf{IND}_S = x - M$ and from (7) $\mathbf{trans}(\mathsf{IND}_S) = \mathbf{trans}(x) - M = 0$. Then S is a drag-set from Lemma 1. On the other hand $Max(x,S) = M < Max(x,\mathbf{N}(\mathcal{O}))$ which contradicts our assumption $\square$

Now to prove Lemma 4. Take any steady set Tower which satisfies (24) and define an initial configuration y as follows:

$$\forall\ v \in \mathbf{V}_{init}\ :\ y(v) = \begin{cases} T & \text{if } v \in \mathsf{Tower}\,, \\ 0 & \text{otherwise.} \end{cases}$$

¿From (24) y is finitary. Now let us prove by contradiction that $x(v) = T$ for all $v \in \mathsf{Tower}$ where $x = \mathbf{TR}(y)$. Let the set $\{v \in \mathsf{Tower}\ :\ x(v) < T\}$ be non-empty. Then there is a point v in this set, all of whose neighbors do not belong to this set. Since Tower is steady, every drag-set S of v intersects Tower, whence $Max(x,S) = Max(x,\mathbf{N}(v)) = T$. This and Lemma 9 assure $x(v) = T$. This contradiction proves Lemma 4 $\square$

5. Proof of Theorem 2: Family $\Phi(\,\cdot\,,\,\cdot\,)$

Given a hidden configuration $h \in H$, denote

$$W(h) = \{v \in \mathbf{V}_{inner}\ :\ h(v) = 1\} \quad \text{and} \quad U(h) = \{v \in \mathbf{V}_{inner}\ :\ x(h) = h(v) = 1\}.$$

Elements of $U(h)$ are called UPSTARTS.

5.1. Main Lemma

Lemma 10. Assume that for every inner point v and every $k \in \mathbf{Z}_{++}$ there is such a finite family $\Phi(v,k)$ of point-sets that

(26) $$\forall\ h \in H\ :\ (x(v) \geq k) \implies (\exists\ S \in \Phi(v,k)\ :\ S \subseteq W(h))$$

and there is such a constant C that for θ small enough

(27) $$\sum\{\theta^{|S|}\ :\ S \in \Phi(v,k)\} \leq \begin{cases} \theta + (C\theta)^2 & \text{if } k = 1\,, \\ (C\theta)^{k+1} & \text{if } k > 1\,. \end{cases}$$

Then Theorem 2 holds.

Proof: From (26), then (27):

$$Prob(x(v) \geq k) \leq \sum\{Prob(W(h) \supseteq S) \ : \ S \in \Phi(v,k)\} =$$
$$\sum\{\theta^{|S|} \ : \ S \in \Phi(v,k)\} \leq \begin{cases} \theta + (C\theta)^2 & \text{if } k = 1\,, \\ (C\theta)^{k+1} & \text{if } k > 1 \end{cases}$$

which proves (10). Then (11) follows from (10) and from the fact that $Prob(x(v) > 0) \geq \theta$ for any inner v $\square$

It remains to assure the assumptions of Lemma 10.

5.2. Polars and Trusses

Let a VECTOR FIELD ϕ on a set S stand for a map $\phi \ : \ S \to \mathbf{Z}^r$. A vector field ϕ given, $\phi(v)$ stands for its value on any $v \in S$; this value is a vector whose components are $\phi(v)_1, \ldots, \phi(v)_r$. For any $Q \subseteq S$ denote $\phi(Q) = \sum\{\phi(v) \ : \ v \in Q\}$. Call a vector field ϕ EVEN or p-EVEN on a set Q if all the components of $\phi(Q)$ equal p. Call a vector field on S OVERALL-EVEN if it is even on all subsets of S. A vector field ϕ on a set S given, its RANGE is RANGE$(\phi) = \{v \in S \ : \ \phi(v) \neq 0\}$. The zero vector field, which equals 0 on every element, has empty range. Call a vector field TRIVIAL if its range consists of one element.

Definition 3. Call a vector field π on S a POLAR if all of the following conditions hold:

- π is even on S.
- For any $v \in S$ and any $j \in [1,r]$ the value of $\pi(v)_j$ is either 0 or 1 or -1.
- For every $j \in [1,r]$ there is at most one point v where $\pi(v)_j = 1$; if it exists, it is called the j-th POSITIVE POLE of π and denoted $\pi^+(j)$.
- For every $j \in [1,r]$ there is at most one point v where $\pi(v)_j = -1$; if it exists, it is called the j-th NEGATIVE POLE of π and denoted $\pi^-(j)$.

Both positive and negative j-poles of a polar are called its j-poles. Call a polar a POSIPOLE if it has all the r positive poles (which may coincide) and none negative, and a NEGAPOLE if it has all the r negative poles (which may coincide) and none positive. $Trivial^+(v)$ and $Trivial^-(v)$ stand for the trivial posipole and negapole all of whose poles coincude with v.

Let us claim two sequences of polars equivalent if one turns into the other by some permutation, and call the resulting classes of equivalence TRUSSES. $\pi \in \mathcal{T}$ means that polar π is a member of truss $\mathcal{T}$, but different members of a truss may coincide. We designate a truss by any sequence $\mathcal{T} = \langle \pi_1, \ldots, \pi_k \rangle$ of its members. j-poles of members of a truss are called its j-poles. $|\mathcal{T}|$ stands for the number of members in the truss $\mathcal{T}$. For any two trusses $\mathcal{A}$ and $\mathcal{B}$ their concatenation $\mathcal{A} * \mathcal{B}$ results from writing one sequence after the other. Thus $\mathcal{T} * \langle \pi \rangle$ means truss $\mathcal{T}$ to which polar π is added. Conversely, $\pi \in \mathcal{T}$ given, $\mathcal{T} - \pi$ means $\mathcal{T}$ from which π is excluded. Of course,

$$|\mathcal{A} * \mathcal{B}| = |\mathcal{A}| + |\mathcal{B}|, \qquad |\mathcal{T} * \langle \pi \rangle| = |\mathcal{T}| + 1 \quad \text{and} \quad |\mathcal{T} - \pi| = |\mathcal{T}| - 1.$$

For any truss $\mathcal{T}$ its range $\textsc{range}(\mathcal{T})$ is defined as the union of ranges of its members. There is the empty truss whose number of members is 0 and whose range is empty.

To any truss $\mathcal{T}$ there corresponds its vector field $\textsc{vec}(\mathcal{T})$ which is the sum of its members as vector fields. A truss $\mathcal{T}$ given, $\textsc{vec}(\mathcal{T}, v)$ and $\textsc{vec}(\mathcal{T}, S)$ stand for the values of the corresponding vector field on the point v and set S. Call a truss p-even, even or overall-even if its vector field is. Call a truss CONNECTED if the family of ranges of its members is connected.

Note 6. Given a truss $\mathcal{T}$ which consists of trivial posipoles and negapoles, and a non-empty truss $\mathcal{U}$. Then:

1) $\mathcal{U}$ is overall-even iff $\mathcal{T} * \mathcal{U}$ is overall-even.

2) If $\mathcal{T} * \mathcal{U}$ is connected, then $\mathcal{U}$ is also connected and $\textsc{range}(\mathcal{U}) = \textsc{range}(\mathcal{T} * \mathcal{U})$.

For any r-tuple $F = (F_1, \ldots, F_r)$ of functions on a set S, any polar π and truss $\mathcal{T}$ on S and any $v \in S$ denote

$$Sum(F, \pi, v) = \sum\{F_j(v) \cdot \pi(v)_j \ : \ j \in [1, r]\} \qquad Sum(F, \pi) = \sum\{Sum(F, \pi, v) \ : \ v \in S\},$$
$$Sum(F, \mathcal{T}) = \sum\{Sum(F, \pi) \ : \ \pi \in \mathcal{T}\}.$$

Take the functions $L_1, \ldots, L_r$ which satisfy (14), (15) and (16), provided by Lemma 2 and define

$$L'_i = L_i + \frac{t}{r}, \quad i \in [1, r].$$

Note that

(28) $$L'_1 + \ldots + L'_r \equiv 0$$

and

(29) for all $i \in [1, r]$ the set $\{e \in \mathbf{N}(\mathcal{O}) \ : \ L'_i(e) \leq -\frac{1}{r}\}$ ia a drag-set .

Lemma 11. If a truss $\mathcal{T}$ on $\mathbf{V}$ is overall-even, then $Sum(L', \mathcal{T}) = 0$, where $L' = (L'_1, \ldots, L'_r)$.

Proof: By grouping together addends that pertain to a point and using (28) $\square$

5.3. Polars and Trusses on Graphs

All the graphs in this paper have no loops. In most cases (except Lemma 13) every two vertices a and b are connected with at most one edge, which is denoted $a - b$ if it is non-directed, and $a \to b$ if it is directed from a to b. If G and G' stand for graphs, $G' \subseteq G$ means that G' is a subgraph of G. A graph G given, $\textsc{ver}(G)$ stands for the set of its vertics.

A polar on a graph G means a polar on the set of vertices of G. A polar π on a graph G is called an EDGER on G if $\textsc{range}(\pi)$ consists of two vertices connected with an edge. We say that π lies on this edge and that this edge underlies π. Posi-edger stands for an edger which is a posipole.

An edger β on V is called a k-BOOST or a BOOST from u to v if $\textsc{range}(\beta) = \{u, v\}$ where $u \in \mathbf{N}(v)$ and β has only two poles: $\beta^+(k) = u$ and $\beta^-(k) = v$. A k-boost from u to v is called a k-ARROW from u to v if $L_k(u) - L_k(v) \leq 0$. Let $\textsc{boost}(u \to v)$ and $\textsc{arrow}(u \to v)$ stand for the boost and arrow from u to v.

Note 7. It follows from (15) and Lemma 11 that:

- $Sum(L', \pi) = 0$ for any trivial polar π.
- $Sum(L', \alpha) \geq 1$ for any arrow α.
- $|Sum(L, \varepsilon)| \leq \text{const}$ for any edger ε on V'. (One may take this $\text{const} = 2 \cdot \lambda \cdot \rho'$ where $\rho' = \max\{L'_i(v) \ :\ \text{norm}(v) \leq 1,\ \ i \in [1, r]\}$.)

Lemma 12. Given an overall-even truss $\mathcal{A} * \mathcal{E}$ on $\mathbf{V}$, where $\mathcal{A}$ consists of arrows and $\mathcal{E}$ consists of edgers on V'. Then $|\mathcal{A}| \leq |\mathcal{E}| \cdot \text{const}$.

Proof: From Lemma 11 and Note 7

$$0 = Sum(L', \mathcal{A} * \mathcal{E}) = Sum(L', \mathcal{A}) + Sum(L', \mathcal{E}) \geq |\mathcal{A}| - \text{const} \cdot |\mathcal{E}|,$$

whence $|\mathcal{A}| \leq \text{const} \cdot |\mathcal{E}|$ $\square$

5.4. Description of the family $\Phi(\ \cdot\ ,\ \cdot\ )$

For any inner point v and any $k \in \mathbf{Z}_{++}$ let $\Psi(v,k)$ stand for the set of triplets $(\mathcal{A},\ \mathcal{E},\ S)$ where $\mathcal{A}$ and $\mathcal{E}$ are trusses and S is a set, which satisfy the following **Conditions F:**

(F1) $\mathcal{A}$ consists of arrows, $\mathcal{E}$ consists of edgers on V'.

(F2) The truss $\mathcal{A} * \mathcal{E}$ is overall-even and connected.

(F3) $v \in \text{RANGE}(\mathcal{A} * \mathcal{E})$.

(F4) $S \subseteq \text{RANGE}(\mathcal{A} * \mathcal{E})$.

(F5) $|S| - k \geq |\mathcal{E}|/r$.

Now we define the family $\Phi(v,k)$ as follows: One element of this family is the set $\{v\}$. Otherwise a set S belongs to $\Phi(v,k)$ if there are such trusses $\mathcal{A}$ and $\mathcal{E}$ that $(\mathcal{A},\mathcal{E},S) \in \Psi(v,k)$.

Proof of (27) is based on the following lemma.

Lemma 13. Consider all triplets $(\mathcal{A},\mathcal{E},S)$ where $\mathcal{A},\mathcal{E}$ are trusses and S is a point-set, such that:

- $\mathcal{A}$ and $\mathcal{E}$ consist of edgers on V'.
- Truss $\mathcal{A} * \mathcal{E}$ is connected.
- $\text{RANGE}(\mathcal{A} * \mathcal{E})$ contains a given point.
- $S \subseteq \text{RANGE}(\mathcal{A} * \mathcal{E})$.
- $|\mathcal{A} * \mathcal{E}| = k$.

The number of these triplets does not exceed const^k.

Proof: It follows from Euler's theorem that to every triplet in question there corresponds a circuit on V' which starts and ends at the given point and makes $2k$ steps, every edger represented by two steps in the two opposite directions along the underlying edge. Let us encode the triplet by moving along this circuits step by step. At every step we must choose the edge to move along, choose the poles our edger has (among the $2r$ possible ones), distribute these poles between the two ends, and decide whether the current edger belongs to $\mathcal{A}$ or $\mathcal{E}$ and whether the current point belongs to S. Note that the degree of any vertex of V' does nor exceed n^2+2n. Therefore the number of triplets in question does not exceed const^k. (We may take this $\mathrm{const} = (2^{3r+2}(n^2+2n))^2$) $\square$

Now to prove (27). Evaluate the cardinalities of sets

$$\Phi_j(v,k) = \{S \in \Phi(v,k) \ : \ |S| = j\} \quad \text{where} \quad j \in \mathbf{Z}_+.$$

First $|\Phi_j(v,k)| = 0$ for all $j < k$, because of **(F5)**.

Now evaluate $|\Phi_j(v,k)|$ for $j \geq k$. Remember that $\Phi_1(v,1)$ contains the special element $\{v\}$ and leave it alone. Without this element, $|\Phi_j(v,k)|$ is no more than the number of different corresponding triplets $(\mathcal{A},\mathcal{E},S)$. If $S \in \Phi_j(v,k)$, then the corresponding trusses have (**(F5)**, then **(F1)** and **(F2)** allow to apply Lemma 12)

$$|\mathcal{E}| \leq \mathrm{const}\cdot(j-k) \implies |\mathcal{A}| \leq \mathrm{const}\cdot(j-k) \implies |\mathcal{A}*\mathcal{E}| \leq \mathrm{const}\cdot(j-k).$$

In the case $j = k$ this implies that $\mathcal{A}$ and $\mathcal{E}$ are empty, whence S is empty also, which is impossible since $|S| = j \geq k > 0$. Thus, including $\{v\}$

$$|\Phi_k(v,k)| = \begin{cases} 1 & \text{if } k=1\,, \\ 0 & \text{otherwise.} \end{cases}$$

In the case $j > k$, due to **(F1)**, **(F2)**, **(F3)** and **(F4)** we can apply Lemma 13 to obtain

$$|\Phi_j(v,k)| \leq \sum_{i=0}^{\mathrm{const}\cdot(j-k)} \mathrm{const}^i \leq \mathrm{const}^{j-k}.$$

Therefore for all θ small enough

$$\sum\{\theta^{|S|},\quad S \in \Phi(v,k)\} = \sum_{j=0}^{\infty} |\Phi_j(v,k)|\cdot\theta^j \leq \sum_{j=k+1}^{\infty} \mathrm{const}^{j-k}\cdot\theta^j \leq (\mathrm{const}\cdot\theta)^{k+1}$$

which proves (27).

6. Proof of Theorem 2: More Technicalities

6.1. Segmentation

Suppose that some directed graph $G \subseteq V$ contains an edge $u \to v$. To SEGMENT this edge means to delete this edge from G and substitute it by a 'chain'

(30) $$u = i_0 \to i_1 \to i_2 \to \cdots \to i_{\Delta t-1} \to i_{\Delta t} = v,$$

where i_0 and $i_{\Delta t}$ are our vertices u and v, and i_k, $\quad k = 1, \ldots, \Delta t - 1 = t(v) - t(u) - 1$, are new intermediate vertices connected by Δt new intermediate edges. We define the time function on the newly introduced vertices by the rule

$$t(i_k) = t(u) + k, \quad k \in [1, \Delta t - 1].$$

Note 8. Segmentation of $u \to v$ changes nothing if $t(v) - t(u) = 1$. Thus, segmentation is unnecessary in the special case $\mathsf{mem} = 1$.

6.2. Spanning Truss

Given a posipole π on a connected non-directed graph G, its (non-unique) SPANNING KIT consists of a SPANNING TREE $\textsf{span-tree}(\pi, G)$ and a SPANNING TRUSS $\textsf{span-truss}(\pi, G)$ which are defined as follows: First, $\textsf{span-tree}(\pi, G)$ is a minimal connected subgraph of G whose set of vertices contains $\mathrm{RANGE}(\pi)$. Of course, $\textsf{span-tree}(\pi, G)$ is a tree, whence every edge of it is a bridge. Based on this, for every edge $a - b$ of $\textsf{span-tree}(\pi, G)$ let us form a posi-edger ε_{a-b} on g whose k-th pole for any $k \in [1, r]$ coincides with a or b, namely with that one of the two, which does not remains connected with $\pi(k)$ if the edge $a - b$ is deleted from $\textsf{span-tree}(\pi, G)$. Now, $\textsf{span-truss}(\pi, G)$ consists of these posi-edgers for all edges of $\textsf{span-tree}(\pi, G)$.

Note 9.

1) $\mathrm{VER}(\textsf{span-tree}(\pi, G)) = \mathrm{RANGE}(\textsf{span-truss}(\pi, G)) \supseteq \mathrm{RANGE}(\pi)$.

2) $|\mathrm{RANGE}(\textsf{span-truss}(\pi, G))| = |\textsf{span-truss}(\pi, G)| + 1$.

3) Truss $\langle \pi \rangle * \textsf{span-truss}(\pi, G)$ is 1-even on every element of its range. Therefore no two k-th poles of it coincide.

4) Any truss $\mathcal{T}$ given, if $\mathcal{T} * \langle \pi \rangle$ is connected, then $\mathcal{T} * \textsf{span-truss}(\pi, G)$ is connected also.

Let us classify all the poles of $\textsf{span-truss}(\pi, G)$ into two categories: END POLES: the k-th end pole coincides with the k-th pole of π, and INNER POLES: all the others.

6.3. Transit Neighbors

Given any directed graph G, call those vertices whence edges go to a vertex, G-NEIGHBORS of this vertex. The set of G-neighbors of a vertex v is called its G-NEIGHBORHOOD and denoted

$\mathbf{N}_G(v)$. Neighborhoods $\mathbf{N}(\cdot)$ without indices, which we defined in Section 2 for all points, are V-neighborhoods, where the graph V was also defined in Section 2. G-NEIGHBORHOOD of a set S of vertices is the union of G-neighborhoods of elements of S. Further, $\mathbf{N}_G^k(S)$ is defined for every $k \in \mathbf{Z}_+$ and every set $S \subseteq \text{VER}(G)$ by the inductive rule:

$$\mathbf{N}_G^0(S) = S, \qquad \mathbf{N}_G^{k+1}(S) = \mathbf{N}_G(\mathbf{N}_G^k(S)).$$

The TRANSIT and the PROPER G-neighborhoods of S are

$$\mathbf{N}_G^{tran}(S) = \bigcup_{k=0}^{\infty} \mathbf{N}_G^k(S) \quad \text{and} \quad \mathbf{N}_G^{prop}(S) = \bigcup_{k=1}^{\infty} \mathbf{N}_G^k(S)$$

and their elements are called transit and proper G-neighbors of S respectively. Call two vertices G-COMPARABLE if one is a transit G-neighbor of the other. Call two sets G-comparable if some element of the first and some element of the second are G-comparable; otherwise these sets are G-uncomparable.

7. Proof of Theorem 2: Auxiliary Graphs

Throughout this and the next sections, an inner point **hub**, a positive integer **rub** and a hidden configuration h are fixed, such that

$$x(\mathbf{hub}) \geq \mathbf{rub}. \tag{31}$$

Our purpose is to assure (26), that is to present such a set $S \subseteq W(h)$ that $S \in \Phi(\mathbf{hub}, \mathbf{rub})$.

7.1. Graph G

A typed graph is a directed graph whose vertices have TYPES: every vertex v has $\text{TYPE}(v) \in \{X, A, B, C, D\}$. To obtain a typed graph G, we shall construct inductively a sequence of typed graphs G_0, G_1, G_2, $\ldots$ and at some step we stop. Some vertices of these graphs belong to $\mathbf{V}$, others will be newly introduced due to segmentation. Along with this, we shall prove the following

Conditions $\mathbf{G}_i$:

$(\mathbf{G1}_i)$ $x(\cdot) > 0$ for all those vertices of G_i which belong to $\mathbf{V}$.

$(\mathbf{G2}_i)$ A vertex of G has type D iff it does not belong to $\mathbf{V}$.

$(\mathbf{G3}_i)$ A vertex of G has empty G_i-neighborhood iff its type is X or C.

Initial condition: We start with a graph G_0 which has one vertex **hub** of type X and no edges. Note that **Conditions** $\mathbf{G}_0$ hold.

Stop-Rule: As soon as none vertex of G_i has type X, we stop and define $G = G_i$.

Induction Step: Suppose that the graph G_i has a vertex v whose type is X. By the induction assumption, $x(v) > 0$. Let us consider three cases.

Case 0. Let $\mathbf{trans}(x(\mathbf{N}(v))) = 0$. Then v is an upstart. In this case we turn G_i into G_{i+1} only by changing type of v from X to C.

Now let $\mathbf{trans}(x(\mathbf{N}(v))) > 0$. Then, due to Definition 1 at least one of the following two cases takes place:

Case 1. Every drag-set for v contains a point where $x(\cdot) \geq \mathbf{trans}(x(\mathbf{N}(v)))$. Then, from (16), for every $k \in [1, r]$ there exists $n_k(v) \in \mathbf{N}(v)$ where

$$x(n_k(v)) \geq \mathbf{trans}(x(\mathbf{N}(v))) \quad \text{and} \quad L_k(n_k(v)) - L_k(v) \leq 0. \tag{32}$$

In this case we obtain G_{i+1} from G_i by changing type of the vertex v from X to A and introducing the segmented edges $n_k(v) \to v, \quad k \in [1, r]$.

Case 2. There is a point $n(v) \in \mathbf{N}(v)$ where

$$x(n(v)) > \mathbf{trans}(x(\mathbf{N}(v))). \tag{33}$$

In this case we obtain G_{i+1} from G_i by changing type of the vertex v from X to B and introducing the segmented edge $n(v) \to v$.

In both cases all the newly introduced intermediate vertices, which do not belong to $\mathbf{V}$, are assigned type D. The end-points $n_k(v)$ and $n(v)$ may belong to G_i, in which case they keep the same type they had in G_i; otherwise they are assigned type X.

Note that **Conditions** $\mathbf{G}_i$ given, **Conditions** $\mathbf{G}_{i+1}$ are also true in all cases.

Lemma 14. G-induction stops.

Proof: The number of vertices of G_i plus the number of those vertices of G_i whose type is C, increases at every step. On the other hand, only elements of the finite set $\mathbf{N}^{tran}(\mathbf{hub})$ can be vertices of G_i $\square$

Note 10. A vertex of G has type C iff it is an upstart.

7.2. Tree T

Let us call vertices of G LEAVES and classify them into equivalence classes called BRANCHES by the following two rules:

- If two leaves have equal time values and a common transit G-neighbor, then they are equivalent.
- Only those leaves are equivalent for which it follows from the previous rule and transitivity.

Now define a directed graph T by the following two rules:

- $\text{VER}(T)$ is the set of branches.
- Branches A, B given, $A \in \mathbf{N}_T(B)$ iff $\exists\ a \in A,\ b \in B\ :\ a \in \mathbf{N}_G(b)$.

Note 11. The leaf **hub** is equivalent to no other leaf. Graph T is a tree in which all edges are directed towards the vertex $\{\mathbf{hub}\}$.

A branch B given, its BUSH, called $\text{BUSH}(B)$ stands for $\mathbf{N}_G^{tran}(B)$. A BUSH stands for a bush of some branch, which is called its ROOT.

Note 12. If two bushes are G-comparable, then their roots are T-comparable.

Say that a branch CUTS a set S of branches if $S \subseteq \mathbf{N}_T^{tran}(B)$. For any set S of branches there is just one branch that cuts S, all of whose T-neighbors do not cut S; call this branch ROOT of S and denote it $\text{ROOT}(S)$. For any polar π on G its ROOT or $\text{ROOT}(\pi)$ stands for the root of the set of branches which intersect its range, and its BUSH stands for $\text{BUSH}(\pi) = \text{BUSH}(\text{ROOT}(\pi))$.

7.3. Graphs $\nabla(\cdot)$

For any branch A whose T-neighborhood is non-empty, define a non-directed graph $\nabla(A)$ by the following two rules:

- $\text{VER}(\nabla(A)) = \mathbf{N}_T(A)$.
- Two different vertices B^1, B^2 of $\nabla(A)$ are connected with an edge in $\nabla(A)$ iff there exist $a \in A, \qquad b^1 \in \mathbf{N}_G^{tran}(B^1) \cap \mathbf{N}_G(a), \qquad b^2 \in \mathbf{N}_G^{tran}(B^2) \cap \mathbf{N}_G(a).$

Lemma 15. Every $\nabla(A)$ is connected.

Proof: is similar to that of Lemma 9 in [1] $\square$

8. Proof of Theorem 2: Q-Induction

8.1. Conditions $\mathbf{Q}_i$

Now i is an integer parameter which grows from 0 to some maximal value. For every value of i we shall form a $(4+r)$-tuple

$$\Omega_i = (\mathcal{A}_i,\ \mathcal{B}_i,\ \mathcal{C}_i,\ \mathcal{D}_i,\ S_i^1, \dots,\ S_i^r)$$

where $\mathcal{A}_i$, $\mathcal{B}_i$, $\mathcal{C}_i$ and $\mathcal{D}_i$ are trusses and $S_i^1, \dots, S_i^r$ are point-sets. Thereby we shall also define a truss $\mathcal{Q}_i = \mathcal{A}_i * \mathcal{B}_i * \mathcal{C}_i * \mathcal{D}_i$ and r functions $F_i^1, \dots, F_i^r$ on V

(34)
$$F_i^j(v) = \begin{cases} x(v) - h(v) & \text{if } v \in S_i^j\ , \\ x(v) & \text{otherwise.} \end{cases}$$

We shall construct Ω_i inductively, the integer parameter i increasing from zero. Along with constructing them, we shall prove by induction that they satisfy the following **Conditions** $\mathbf{Q}_i$:

- **($\mathbf{Q1}_i$)** $\mathcal{A}_i$ consists of arrows, $\mathcal{B}_i$ consists of boosts, $\mathcal{C}_i$ consists of posi-edgers on V', $\mathcal{D}_i$ consists of negapoles.

- **($\mathbf{Q2}_i$)** $\mathcal{Q}_i$ is overall-even and connected.

- **($\mathbf{Q3}_i$)** $\mathbf{hub} \in \text{RANGE}(\mathcal{Q}_i) \subseteq (\text{VER}(G) \cap \mathbf{V})$.

- **($\mathbf{Q4}_i$)** $\forall\ j\ :\ S_i^j \subseteq \text{RANGE}(\mathcal{Q}_i(j)) \cap W(h)$.

- **($\mathbf{Q5}_i$)** $\sum_{j=1}^r |S_i^j| + Sum(F_i, \mathcal{D}_i) \geq r \cdot x(\mathbf{hub}) + |\mathcal{B}_i| + r \cdot |\mathcal{C}_i|$.

- **($\mathbf{Q6}_i$)** $\text{VEC}(\mathcal{Q}_i - \delta,\ \text{BUSH}(\delta))$ is a posipole for every $\delta \in \mathcal{D}_i$.
- **($\mathbf{Q7}_i$)** For every $j \in [1, r]$ every j-pole of $\mathcal{Q}_i$ has a transit G-neighbor which is a j-pole of $\mathcal{D}_i$.

Initial Condition: $\mathcal{A}_0$, $\mathcal{B}_0$, $\mathcal{C}_0$ and $S_0^1, \dots, S_0^r$ are empty and $\mathcal{D}_0 = \langle Trivial^-(\mathbf{hub})\rangle$. Note that **Conditions** $\mathbf{Q}_0$ hold.

Stop-Rule: We stop as soon as all of the following **Conditions** $\mathbf{S}_i$ hold:

(S1$_i$) $\forall\, j\ :\ (\text{RANGE}(\mathcal{D}_i(j)) \cap W(h)) \subseteq S_i^j$.

(S2$_i$) $\text{RANGE}(\mathcal{D}_i) \subseteq U(h)$.

(S3$_i$) All the members of $\mathcal{D}_i$ are trivial.

Lemma 16. **Conditions Q$_i$** given, for every $j \in [1,r]$ no proper G-neighbor of an element of $\text{RANGE}(\mathcal{D}_i(j))$ can belong to S_i^j .

Proof: Assume the contrary: there are $v = \delta^-(j)$ and $u \in \mathbf{N}^{prop}(v) \cap S_i^j$. Then from **(Q4$_i$)** $u \in \text{RANGE}(Q_i(j))$. Then from **(Q7$_i$)** u has a transit G-neighbor $w \in D_i(j)$. Then from **(Q6$_i$)** $w = \delta^-(j)$ which is impossible since $w \neq v$ □

Lemma 17. **Conditions Q$_i$** given, bushes of members of $\mathcal{D}_i$ are pairwise G-uncomparable.

Proof: Assume the contrary: bushes of $\delta,\ \delta' \in \mathcal{D}_i$ are G-comparable. Then from Note 12

$$\text{ROOT}(\delta) \in \mathbf{N}_G^{tran}(\text{ROOT}(\delta'))$$

(or vice versa, which is analogous), whence $\text{BUSH}(\delta')$ contains at least $2{\cdot}r$ negative poles. Then, from **(Q2$_i$)**, it contain at least $2 \cdot r$ positive poles, which contradicts **(Q6$_i$)** □

Note 13. Condition **(S1$_i$)** given, $F_i^j(v) = x(v) - h(v)$ for all $j \in [1,r]$, $v \in \text{RANGE}(\mathcal{D}_i(j))$.

8.2. Induction step

Assume that Ω_i is constructed and **Conditions Q$_i$** proven. Assume also that at least one of **Conditions S$_i$** fails and define Ω_{i+1} .

Case 1: (S1$_i$) fails, i.e. there is $j \in [1,r]$ and $v = \delta^-(j) \in \text{RANGE}(\mathcal{D}_i)$, which belongs to $W(h)$ but not to S_i^j. In this case the only difference between Ω_{i+1} and Ω_i is that $S_{i+1}^j = S_i^j \cup \{v\}$.

Case 2: (S1$_i$) holds, but **(S2$_i$)** fails, i.e. $\text{RANGE}(\mathcal{D}_i(j)) \cap W \subseteq S_i^j$ for every j , but there is some $v = \delta^-(k) \in \text{RANGE}(D_i)$ which is not an upstart. Then, according to Stop-Rule of G-Induction, $\text{TYPE}(v)$ is A or B. Consider these two cases.

Case 2A: $\text{TYPE}(v) = A$. Then there exist $n_1(\delta^-(k)), \ldots, n_r(\delta^-(k))$. In this case the only difference between Ω_{i+1} and Ω_i is that $\mathcal{A}_{i+1} = \mathcal{A}_i * \langle \text{ARROW}(n_k(\delta^-(k)) \to \delta^-(k)) \rangle$ and $\mathcal{D}_{i+1}$ differs from $\mathcal{D}_i$ in only one respect: instead of δ it contains δ_{new} which has only one pole different from that of δ , namely $\delta_{new}^-(k) = n_k(\delta^-(k))$.

Case 2B: $\text{TYPE}(v) = B$. Then there exists $n(\delta^-(k))$. In this case the only difference between Ω_{i+1} and Ω_i is that $\mathcal{B}_{i+1} = \mathcal{B}_i * \langle \text{BOOST}(n(\delta^-(k)) \to \delta^-(k)) \rangle$ and $\mathcal{D}_{i+1}$ differs from $\mathcal{D}_i$ in only one respect: instead of δ it contains δ_{new} which has only one pole different from that of δ, namely $\delta^-_{new}(k) = n(\delta^-(k))$.

Case 3: $(\mathbf{S1}_i)$ and $(\mathbf{S2}_i)$ hold, but $(\mathbf{S3}_i)$ fails. So $\mathcal{D}_i$ contains a non-trivial negapole δ.

Let us prove that $\text{RANGE}(\delta) \cap \text{ROOT}(\delta) = \varnothing$ in this case. Assume the contrary: there is some $\delta^-(k) \in \text{ROOT}(\delta)$. Since all the elements of $\text{RANGE}(\delta)$ are upstarts, they have no G-neighbors (Note 10). Since $\delta^-(k)$ has no G-neighbors, $\text{ROOT}(\delta)$ is the branch that contains $\delta^-(k)$ and nothing else. This contradicts our assumption.

Thus, every pole of δ belongs to the transit G-neighborhood of some T-neighbor of $\text{ROOT}(\delta)$. This allows us to define a posipole π on $\mathbf{N}_T(\text{ROOT}(\delta))$ by the following rule applied to all $k \in [1, r]$: $\pi^+(k)$ is that T-neighbor of $\text{ROOT}(\delta)$ whose bush contains $\delta^-(k)$. ¿From Lemma 15, the graph $\nabla(\text{ROOT}(\delta))$ is connected. Therefore there exists a spanning kit which consists of $\mathsf{span\text{-}tree}(\pi, \nabla(\text{ROOT}(\delta))$ and $\mathsf{span\text{-}truss}(\pi, \nabla(\text{ROOT}(\delta))$.

For every member ε of this spanning truss (which is a posi-edger on the spanning tree) we form a posi-edger $\gamma(\varepsilon)$ on G as follows. Denote

$$\text{RANGE}(\varepsilon) = \{A, B\} \subset \mathbf{N}_T(\text{ROOT}(\gamma)).$$

Since vertices A and B of $\nabla(\text{ROOT}(\delta))$ are connected with an edge in this graph, there are leaves

$$c \in V, \qquad a \in \mathbf{N}_G^{tran}(A) \cap \mathbf{N}_G(c), \qquad b \in \mathbf{N}_G^{tran}(B) \cap \mathbf{N}_G(c).$$

Now define our posi-edger γ as follows:

$$\forall\ k \in [1, r]\ :\ \gamma^+(k) = \begin{cases} a & \text{if } \varepsilon(k) = A, \\ b & \text{if } \varepsilon(k) = B. \end{cases} \tag{35}$$

Truss Γ_i consists of these γ.

Note 14. $|\Gamma_i| = |\mathsf{span\text{-}truss}(\pi, \nabla(\text{ROOT}(\delta))|$.

Note 15. $j \in [1, r]$ given, all the j-poles of the truss $\langle \delta \rangle * \Gamma_i$ are mutually G-uncomparable, because they belong to mutually $\mathcal{T}$-uncomparable branches.

In **Case 3** the only difference between Ω_{i+1} and Ω_i is that $\mathcal{C}_{i+1} = \mathcal{C}_i * \Gamma_i$ and $\mathcal{D}_{i+1}$ differs from $\mathcal{D}_i$ as follows. Denote

$$\mathcal{T}_{i+1} = \mathcal{A}_{i+1} * \mathcal{B}_{i+1} * \mathcal{C}_{i+1} \quad \text{and} \quad \text{VER}_{span} = \text{VER}(\mathsf{span\text{-}tree}(\pi, \nabla(\text{ROOT}(\delta))).$$

¿From statement 3 of Note 9, $\mathcal{T}_{i+1}$ is 1-even on every $B \in \text{VER}_{span}$. Thus, for every $k \in [1, r]$ every branch $B \in \text{VER}_{span}$ serves as the k-th pole just for one member of $\langle \pi \rangle *\textsf{span-truss}(\pi, \nabla(\text{ROOT}(\delta))$. This means that for every $B \in \text{VER}_{span}$ and for every $k \in [1, r]$ the truss $\langle \delta \rangle * \Gamma_i$ has just one positive k-th pole in $\text{BUSH}(B)$, and let it be the negative k-th pole of the new negapole δ_B that corresponds to B. Now define

$$\mathcal{D}_{i+1} = (\mathcal{D}_i - \delta) * \langle \delta_B \ : \ B \in \text{VER}_{span} \rangle.$$

Thus Ω_{i+1} is defined in all cases. Now let us prove the implication **Conditions** $\mathbf{Q}_i \implies$ **Conditions** $\mathbf{Q}_{i+1}$. The only non-trivial proofs are the following ones.

$(\mathbf{Q2}_{i+1})$ in **Case 3** follows from statements 4) and 3) of Note 9.

Proof of $(\mathbf{Q5}_{i+1})$ in all cases involves estimation of the difference

$$Sum(F_{i+1}, \mathcal{D}_{i+1}) - Sum(F_i, \mathcal{D}_i). \tag{36}$$

Case 1: In this case the difference (36) equals -1 , because, due to Lemma 17, only one addend of $Sum(F_i, \mathcal{D}_i)$ may decrease, namely $F_i^j(\delta^-(j))$, and it decreases by 1 . This is compensated by the fact that $|S_{i+1}^j| - |S_i^j| = 1$ $\square$

Case 2A: In this case all we need to prove is that the difference (36) is non-negative, which boils down to $F_{i+1}^k(n_k(v)) \geq F_i^k(v)$. Since $S_{i+1^j} = S_i^j$ for all j , this can be rewritten as

$$F_i^k(n_k(v)) \geq F_i^k(v). \tag{37}$$

Let us prove it. Since $(\mathbf{S1}_i)$ holds,

$$F_i^k(v) = x(v) - h(v) = \mathbf{trans}(x(\mathbf{N}(v))).$$

On the other hand, from Note 13, $n_k(v)$ cannot belong to S_i^k , whence $F_i^k(n_k(v)) = x(n_k(v))$, and (37) follows from (32) $\square$

Case 2B: In this case all we need to prove is that the difference (36) is positive, which boils down to

$$F_i^k(n(v)) > F_i^k(v).$$

The right side equals $x(v) - h(v) = \mathbf{trans}(x(\mathbf{N}(v)))$ from Note 13 and the left side equals $x(n(v))$ from Lemma 16. Thus the inequality to prove follows from (33) $\square$

Case 3: In this case the proof boils down to the fact that the difference (36) is not less than $r \cdot |\text{VER}_{span}|$. Let us prove it.

Note 16. $x(\cdot) > 0$ at all inner poles, because they are vertices of G (see $(\mathbf{G1}_i)$).

Lemma 18. No inner j-pole belongs to S_i^j.

Proof: Assume that there is an inner j-pole $v \in S_i^j$. Then, from **(Q4$_i$)**, $v \in \text{RANGE}(\mathcal{Q}_i(j))$. Then, from **(Q7$_i$)**, v has a transit G-neighbor $w \in \mathcal{D}_i(j)$. Then, from Lemma 17 $w = \delta^-(j)$, which contradicts Note 15 □

Now let us estimate the difference (36). Since $S_{i+1}^j = S_i^j$ for all $j \in [1,r]$, this difference equals $Sum(F_i, \mathcal{D}_{i+1}) - Sum(F_i, \mathcal{D}_i)$ which is not less than $r \cdot |\text{VER}_{span}|$.

8.3. Result of Q-Induction

Lemma 19. **Q-induction** stops.

Proof: From **(Q4$_i$)** the sum $\sum |S_i^j|$ is limited, whence **Case 1** cannot occur infinitely. ¿From **(Q3$_i$)** and Lemma 17 $|\mathcal{D}_i|$ cannot be greater than $|G|$, whence **Case 3** also cannot occurs infinitely, because every its occurence increases $|\mathcal{D}_i|$. Now, $Sum(F_i, \mathcal{D}_i)$ is also limited because every value of functions F_i^j on vertices of G cannot exceed $t(\mathbf{hub})$. Thus, the left side of **(Q5$_i$)** is limited, whence the right side is also limited, whence $|\mathcal{B}_i|$ and $|\mathcal{C}_i|$ are limited, whence **Case 2** also cannot occur infinitely □

Lemma 20. When **Q-Induction** stops, $Sum(F_i, \mathcal{D}_i) = 0$.

Proof: It follows from (34) and **(S2$_i$)** □

When **Q-Induction** stops, we obtain the trusses $\mathcal{A}_i$, $\mathcal{B}_i, \mathcal{C}_i$, $\mathcal{D}_i$ and the sets $S_i^1, \ldots, S_i^r$ for which **Conditions Q$_i$** and **S$_i$** hold, and define

(38) $$\mathcal{A} = \mathcal{A}_i, \qquad \mathcal{E} = \mathcal{B}_i * \mathcal{C}_i, \qquad S = \text{RANGE}(\mathcal{A} * \mathcal{E}) \cap W(h).$$

Note 17. $S \supseteq \cup\{S_i^j \ : \ j \in [1,r]\}$.

Lemma 21. Let **Conditions Q$_i$** and **Conditions S$_i$** hold. Then either **hub** is an upstart, in which case $S = \{\mathbf{hub}\}$, or $\mathcal{A}$, $\mathcal{E}$, S defined by (38) satisfy **Conditions F**.

Proof: First let $\mathbf{hub} \in U(h)$. Then G-induction stops without making a step, and $\text{VER}(G) = \{\mathbf{hub}\}$. Q-induction makes r steps, at everyone of which **Case 1** takes place. In result we obtain

$$\mathcal{A}_r = \mathcal{B}_r = \mathcal{C}_r = \varnothing\,, \qquad \mathcal{D}_r = \langle Trivial^-(\mathbf{hub})\rangle, \qquad \forall\, j \,:\, S_r^j = \{\mathbf{hub}\}, \quad \text{whence} \quad S = \{\mathbf{hub}\}.$$

Now let **hub** $\notin U(h)$. Then G-induction makes at least one step, G contains more that one vertex and $\mathcal{A} * \mathcal{E}$ is non-empty, whence Note 6 can be applied. Then:

(26) follows from $(\mathbf{Q4}_i)$.

(F1) follows from $(\mathbf{Q1}_i)$.

(F2) follows from $(\mathbf{Q2}_i)$ and $(\mathbf{S3}_i)$.

(F3) follows from $(\mathbf{Q3}_i)$.

(F4) follows from (38).

(F5) follows from $(\mathbf{Q4}_i)$, Note 17, $(\mathbf{Q5}_i)$ and Lemma 20 □

Thus Theorem 2 is proved □

9. Proof of Theorem 3

9.1. Main Lemma

Proof of Theorem 3 is similar to and based on the proof of Theorem 2. In this case for every inner point v and every $k \in \mathbf{Z}_{++}$ we construct two finite families $\Phi'(v,k)$ and $\Phi''(v,k)$ of point-sets which have the following three properties (where $C = \text{const}$):

$$\forall\ h \in H\ :\ (x(v) \geq k) \implies \exists\ S \in \Phi'(v,k) \cup \Phi''(v,k)\ :\ S \subseteq W(h). \tag{39}$$

$$\sum\{\theta^{|S|}\ :\ S \in \Phi'(v,k)\} \leq \begin{cases} \theta + (C\theta)^2 & \text{if } k = 1\,, \\ (C\theta)^{k+1} & \text{if } k > 1 \end{cases} \tag{40}$$

$$\sum\{\theta^{|S|}\ :\ S \in \Phi''(v,k)\} \leq \text{const} \cdot t(v) \cdot (C \cdot \theta)^{k+1+\text{const}\cdot \mathsf{size}}. \tag{41}$$

Lemma 22. Given families $\Phi'(v,k)$ and $\Phi''(v,k)$ for all $v \in \mathbf{V}_{inner}$, $k \in \mathbf{Z}_{++}$ and $C = \text{const}$ which satisfy (39), (40) and (41). Then Theorem 2 holds.

Proof: Any constant T given, while $t(v) < T^{\mathsf{size}}$, the right part of (41) is still less than $\text{const} \cdot (\text{const} \cdot \theta)^{k+1+\text{const}\cdot\mathsf{size}}$. Then we can choose such size_0 that for all $\mathsf{size} \geq \mathsf{size}_0$ this expression is $O(\theta^{k+1})$. Thus, from (39), then (40) and (41):

$$Prob(x(v) \geq k) \leq \sum\{Prob(W(h) \supseteq S) \;:\; S \in \Phi'(v,k) \cup \Phi''(v,k)\} =$$
$$\sum\{\theta^{|S|} \;:\; S \in \Phi'(v,k) \cup \Phi''(v,k)\} \leq \begin{cases} \theta + (C\theta)^2 & \text{if } k = 1\,, \\ (C\theta)^{k+1} & \text{if } k > 1\,, \end{cases}$$

which proves (12). Then (13) follows from (12) and from the fact that $Prob(x(v) > 0) \geq \theta$ for any inner v $\square$

9.2. Span

Note 18. If a connected truss $\mathcal{T}$ consists of edgers on V', then $\mathsf{diam}(\mathrm{RANGE}(\mathcal{T})) \leq \mathrm{const} \cdot |\mathcal{T}|$.

Note 19. $\mathsf{diam}(S_1 \cup S_2) \;\leq\; \mathsf{diam}(S_1) + \mathsf{diam}(S_2) + \mathsf{dist}(S_1, S_2)$.

Denote $\mathsf{span}(S) = \mathsf{diam}(S) + \lambda$ and call two point-sets λ-CLOSE if the distance between them does not exceed λ. (Note that any non-empty set is λ-close to itself.) Call a family of point-sets λ-CONNECTED if it is possible to go from any its element to any other, every step made between λ-close sets.

Lemma 23. If a family F of point-sets is λ-connected, then $\mathsf{span}(\mathrm{UNION}(F)) \leq \sum\{\mathsf{span}(S),\ S \in F\}$.

Proof: By induction, based on Note 19 $\square$

Lemma 24. For any $v \in V$

$$\mathsf{span}(N_G^{tran}(v)) \leq \sum\{\mathsf{span}(\mathbf{N}_G^{tran}(w)),\ w \in \mathbf{N}_G(v)\} + \lambda.$$

Proof: This follows from Lemma 23 $\square$

9.3. Infinite Representation of The Finite Case

We are considering a process P_{size} on a finite volume $V_{\mathsf{size}} = \mathbf{Z}_{\mathsf{size}}^{\dim} \cdot Time$. Let us call elements of the finite volume size-points. Let $V_\infty = \mathbf{Z}_\infty^{\dim} \cdot Time$ stand for the corresponding infinite volume whose elements are called ∞-points now. The map $\mathsf{fin} \;:\; V_\infty \mapsto V_{\mathsf{size}}$ is defined as follows: $\mathsf{fin}(s,t)=(s',t)$ where components of s' are residues of components of s modulo size. The opposite map $\mathsf{inf}\ :\ V_{\mathsf{size}}\mapsto V_\infty$ is defined by

$$\forall\ (s,t)\in V_{\mathsf{size}}\ :\ \mathsf{inf}(s,t)=(s,t)\in V_\infty.$$

Call elements of the finite and infinite hidden spaces HIDDEN size-CONFIGURATIONS and ∞-CONFIGURATIONS. To every hidden size-configuration h there corresponds a periodical hidden ∞-configuration $h'=\mathsf{INF}(h)$ defined by the rule

$$\forall\ v\in V_\infty\ :\ h'(v)=h(\mathsf{fin}(v)).$$

We shall use the auxiliary infinite process P'_∞ on V_∞ induced by the finite hidden measure η_{size} with this map.

Note 20. $\forall\ v\in V_\infty\ :\ P'_\infty|_v\equiv P_{\mathsf{size}}|_{\mathsf{fin}(v)}$.

Say that a set $S\subset V_\infty$ OVERLAPS if there are different $v,v'\in S\ :\ \mathsf{fin}(v)=\mathsf{fin}(v')$.

Note 21. If $S\subset V_\infty$ does not overlap, then $|\mathsf{fin}(S)|=|S|$.

Note 22. If $S\subset V_\infty$ overlaps, then $\mathsf{diam}(S)\geq\mathsf{size}$.

9.4. Description of the families $\Phi'(\ \cdot\ ,\ \cdot\ )$ and $\Phi''(\ \cdot\ ,\ \cdot\ )$

To define $\Phi'(v,k)$**:** One element of it is $\{v\}$. Otherwise a point-set $S\subset\mathbf{V}_{\mathsf{size}}$ belongs to $\Phi'(v,k)$ if it is representable as $S=\mathsf{fin}(S_\infty)$ where S_∞ is such a subset of V_∞ for which there are such trusses $\mathcal{A}$ and $\mathcal{E}$ on V_∞, that $(\mathcal{A},\mathcal{E},S_\infty)\in\Psi(v,k)$ and S_∞ does not overlap.

Lemma 25. Family $\Phi'(v,k)$ satisfies (40).

Proof: is like that of (27) $\square$

To define $\Phi''(v,k)$**:** A point-set $S\subset V_{\mathsf{size}}$ belongs to $\Phi''(v,k)$ if it is representable as $S=\mathsf{fin}(S_\infty)$ where S_∞ is such a subset of V_∞ for which there are such trusses $\mathcal{A}$ and $\mathcal{E}$ on V_∞ and such a point $w\in\mathbf{N}^{tran}(v)$ that $(\mathcal{A},\mathcal{E},S_\infty)\in\Psi(w,k)$ and S_∞ does not overlap and

$$\mathsf{span}(\text{RANGE}(\mathcal{A}*\mathcal{E}))\geq(\mathsf{size}-\lambda)/n. \tag{42}$$

Lemma 26. Family $\Phi''(v,k)$ satisfies (41).

Proof: From (42) and definition of span

$$\mathsf{diam}(\text{RANGE}(\mathcal{A} * \mathcal{E})) \geq \frac{\mathsf{size} - \lambda}{n} - \lambda.$$

Then from **(F2)** and Note 18

$$|\mathcal{A} * \mathcal{E}| \geq \text{const} \cdot \mathsf{size} - \text{const}.$$

Due to **(F1)** and **(F2)** we can apply Lemma 12 which gives

$$|\mathcal{E}| \geq \text{const} \cdot \mathsf{size} - \text{const}.$$

Then from **(F5)**, the fact that S_∞ does not overlap and Note 21

$$|S| = |S_\infty| \geq C_1 \mathsf{size} - C_2,$$

where $C_1 > 0$ and C_2 are constants. Now let us prove (41) by evaluating the cardinalities of the sets

$$\Phi''_j(v,k) = \{S \in \Phi''(v,k) \ : \ |S| = j\}.$$

As we have proved, $|\Phi''_j(v,k)| = 0$ for all $j < C_1 \mathsf{size} - C_2$.

Now let $j \geq C_1 \cdot \mathsf{size} - C_2$. ¿From **(F5)** $|\mathcal{E}| \leq r \cdot (j-k)$, then from Lemma 12 $|\mathcal{A} * \mathcal{E}| \leq \text{const} \cdot j$, then from Lemma 13 $|\Phi''_j(v,k)| \leq \text{const}^j$. Thus,

$$\sum\{\theta^{|S|},\ S \in \Phi''(v,k)\} \ \leq$$
$$\text{const} \cdot \mathsf{size}^{\dim} \cdot t(v) \cdot \sum_{j=C_1 \mathsf{size} - C_2}^{\infty} (\text{const} \cdot \theta)^j \ \leq \ \text{const} \cdot \mathsf{size}^{\dim} \cdot t(v) \cdot (\text{const} \cdot \theta)^{\text{const} \cdot \mathsf{size}}.$$

For small enough θ, this is less than the right part of (41) $\square$

9.5. Final

In section 3 we have represented the finite process P_{size} as a special infinite process P'_∞, to which we can apply constructions of the infinite case. Given a ∞-point v, a number k and a hidden ∞-configuration h such that $x(v) \geq k > 0$, we obtain the triplet $(\mathcal{A},\ \mathcal{E},\ S)$ defined by (38). Let $\mathcal{T}_\infty(v,h,k) = \mathcal{A} * \mathcal{E}$ stand for the concatenation of the first two terms of this triplet, and $S_\infty(v,h,k)$ stand for the last term of this triplet.

Lemma 27. If $S_\infty(v,h,k)$ overlaps, then there exists a size-point $\text{ERSATZ}(v,h,k) \in \mathbf{N}^{tran}(v)$ such that $S_\infty(\text{ERSATZ}(v,h,k),h,k)$ does not overlap, and

$$\mathsf{span}(\text{RANGE}(\mathcal{T}(\text{ERSATZ}(v,h,k),\ h,\ k))) \geq (\mathsf{size} - \lambda)/n.$$

Proof: Take the set of those points $w \in \mathbf{N}_G^{tran}(v)$ for which $S_\infty(w,h,k)$ overlaps, take a point in this set whose time is the smallest, and call this point PIT. Let us prove by contradiction that among G-neighbors of PIT there is at least one which fits our need. Assume the contrary:

$$\forall\ w \in N_G(\text{PIT})\ :\ \mathsf{span}(\text{RANGE}(\mathcal{T}(w,h,k))) < (\mathsf{size} - \lambda)/n.$$

Then from **(F5)** and Lemma 24

$$\mathsf{span}(S_\infty(\text{PIT},h,k)) \le \mathsf{span}(N_G^{tran}(\text{PIT})) < \mathsf{size}$$

which contradicts Note 22 □

Now define a set $S_{\mathsf{size}} \in \Phi'(v,k) \cup \Phi''(v,k)$ for any size-point v, hidden size-configuration h and $k \in [1,x(v)]$:

(43) $$S_{\mathsf{size}}(v,\ h,\ k) = \mathsf{fin}(S_\infty(\mathsf{inf}(v), \mathsf{INF}(h), k))$$

if $S_\infty(\mathsf{inf}(v), \mathsf{INF}(h), k)$ does not overlap, and

(44) $$S_{\mathsf{size}}(v,\ h,\ k) = \mathsf{fin}(S_\infty(\text{ERSATZ}(\mathsf{inf}(v)),\ \mathsf{INF}(h),\ k))$$

if $S_\infty(\mathsf{inf}(v), \mathsf{INF}(h), k)$ overlaps.

Note 23. In both cases of the definition of S_{size}, (39) is assured.

Lemma 28. Every $S_{\mathsf{size}}(v,h,k)$ defined by (43) belongs to $\Phi'(v,k)$ and every $S_{\mathsf{size}}(v,h,k)$ defined by (44) belongs to $\Phi''(v,k)$.

Proof: The first assertion can be proved as in the infinite case. Let us prove the second. All assertions but the last one were proved when treating of the infinite case. The last assertion (42) follows from Lemma 27 □

Thus Theorem 3 is proved □

10. Proof of Theorem 4

Note 24. If a point-set Tower satisfies (24), then any $v \in V$ belongs to Tower shifted by some $s \in \mathbf{Space}$.

Now take any steady set Tower which satisfies (24) and separate it into parts by the rule:

$$\mathsf{Floor}_i = \{v \in \mathsf{Tower}\ :\ (i-1)\cdot\mathsf{mem} \le t(v) < i\cdot\mathsf{mem}\}.$$

Note 25. $\forall\, i \;:\; 1 \le |\mathsf{Floor}_i| \le \text{const}$.

Lemma 29. $\forall\, v \in V \;:\; \mathbf{E}_\theta(v) \ge \theta^{\text{const}} \cdot t(v)/\mathsf{mem}$, where const is that of Note 25.

Proof: First we prove that

$$\forall\, v \in \mathsf{Floor}_i \;:\; E_\theta(\min\{x(v) \;:\; v \in \mathsf{Floor}_i\}) \ge \theta^{\text{const}} \cdot i.$$

This is because

$$\begin{aligned}
&\min\{x(v),\ v \in \mathsf{Floor}_{i+1}\} = \\
&\min\{\mathbf{trans}(x(N(v))) + h(v),\ v \in \mathsf{Floor}_{i+1}\} \ge \\
&\min\{\mathbf{trans}(x(N(v))),\ v \in \mathsf{Floor}_{i+1}\} + \min\{h(v),\ v \in \mathsf{Floor}_{i+1}\} \ge \\
&\min\{x(v),\ v \in \mathsf{Floor}_{i}\} + \min\{h(v),\ v \in \mathsf{Floor}_{i+1}\}.
\end{aligned}$$

(Expectation of the last term is no less than θ^{const}.) Hence and from Note 24 Lemma 29 follows $\square$

Theorem 4 follows from Lemma 29 $\square$

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