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# Two-body Elliptic Model in proper variables: Lie-algebraic forms and their discretizations. [1]

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## Abstract

Two Lie algebraic forms of the 2-body Elliptic Calogero model are presented. Translation-invariant and dilatation-invariant discretizations of the model are obtained.

## 1. Introduction

The general $N$-body Lame operator is given by

$$L \equiv \Delta - n(n+1) \sum_{i>j}^{N} \mathcal{P}(x_i - x_j) \ . \tag{1.1}$$

where $n$ is positive integer and the Weierstrass function $\mathcal{P}(x)$ is a doubly-periodic meromorphic function, which obeys the equation

$$\mathcal{P}'^2 = 4(\mathcal{P} - e_1)(\mathcal{P} - e_2)(\mathcal{P} - e_3) \equiv 4\mathcal{P}^3 - g_2\mathcal{P} - g_3 \ .$$

Here $e_1, e_2, e_3$ such that $e_1 + e_2 + e_3 = 0$ are called the roots of the Weierstrass function. Olshanetsky and Perelomov have shown (see, for example, [1]) that the problem (1.1) is completely integrable.

## 2. Lie-algebraic analysis

For a two-body case we need to make a slight modification in (1.1) In particular, adding an extra constant term

$$\tilde{L}^{(2)} \equiv \Delta^{(2)} - \frac{n(n+1)}{2}\left[\mathcal{P}\left(\frac{x_1 - x_2}{2}\right) + A\right] \tag{2.1}$$

where $A(\omega, \omega')$ is a real constant with the condition that $A(\omega, \infty) = \frac{1}{3}(\frac{\pi}{2\omega})^2$. Here $\omega, \omega'$ are the half-periods of the $\mathcal{P}$–function. In particular, one can choose $A(\omega, \omega') = \frac{1}{3}(\frac{\pi}{2\omega})^2$. Hermiticity of $L$ requires that one of the periods be real. Let it be $\omega$. The extra term in (2.1) just changes the reference point for energies.

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Hereafter we assume that the center-of-mass coordinate $X = \sum x_i$ is separated and the remaining part of the Laplace operator is already written in Perelomov relative coordinates [4]

$$y_i = x_i - \frac{X}{2} \ , \qquad i = 1, 2 \ , \tag{2.2}$$

and, say, $y_1 = \frac{x_1 - x_2}{2} = -y_2 \equiv y$, and, thus the Laplace operator is equal to

$$\Delta^{(2)} = 2\partial^2_{XX} + \frac{1}{2}\partial^2_{yy} \ . \tag{2.3}$$

The operator $\tilde{L}^{(2)}$ after separation of the center-of-mass motion becomes

$$\tilde{L}^{(2)}_{rel} = \{2\tilde{L}^{(2)}\}|_{rel} = \partial^2_{yy} - n(n+1)[\mathcal{P}(y) + A] \ , \tag{2.4}$$

and has an important feature of coinciding either with the Hamiltonian of Calogero model (both periods tend to infinity, thus $A = 0$) or to the Sutherland model (if the second [complex] period tends to infinity while $\omega$ is kept fixed; in this case $A = \frac{3g_3}{2g_2}$). This operator (2.4) is the Weierstrass form of the Lame operator from textbooks (see, for example, [5, 6]).

In the rest of the analysis we assume that $n$ is an even number. We study pure polynomial solutions (non-vanishing at the points, where $\mathcal{P}(y) = e_i, \ i = 1, 2, 3$). These solutions are called the Lame polynomials of the first type (see, for example, [5]). Implicitly, it means that we are exploring periodic-periodic solutions of the Lame equation.

The well-known algebraic form of the Lame operator (2.4) occurs if a new variable, $\xi = \mathcal{P}(y)$, is introduced:

$$\tilde{L}^{(2)}_{rel} = (4\xi^3 - g_2\xi - g_3)\partial^2_{\xi\xi} + (6\xi^2 - \frac{g_2}{2})\partial_\xi - n(n+1)(\xi + A) \ . \tag{2.5}$$

The operator (2.5) can be immediately rewritten in terms of the generators of the $sl(2)$-algebra:

$$J^+_k = \rho^2 \frac{\partial}{\partial \rho} - k\rho \ , \ J^0_k = \rho \frac{\partial}{\partial \rho} - \frac{k}{2} \ , \ J^-_k = \frac{\partial}{\partial \rho} \ , \tag{2.6}$$

and then the following expression for (2.5) emerges[4] [2]:

$$\tilde{L}^{(2)}_{rel} = J^+_{\frac{n}{2}}[4J^0_{\frac{n}{2}} + (3n+2)] - g_2\left(J^0_{\frac{n}{2}} + \frac{n+2}{4}\right)J^-_{\frac{n}{2}} - g_3 J^-_{\frac{n}{2}} J^-_{\frac{n}{2}} \ , \tag{2.7}$$

in the representation (2.6) of spin $\frac{n}{2}$. It is evident that the operator (2.5) has $(\frac{n}{2}+1)$ polynomial eigenfunctions.

It is worth mentioning that for even values $n$ in (2.4) there also exist three different kinds of eigenfunctions more, those proportional to $\eta_{ij} = (\xi - e_i)^{1/2}(\xi - e_j)^{1/2}, i \neq j = 1, 2, 3$. One can show that making the gauge rotation of (2.5): $\eta^{-1}_{ij}\tilde{L}^{(2)}_{rel}\eta_{ij}$ we again get the operator (2.5) with different coefficients in the part of it which is linear function in derivative. In particular, the only new term which occurs is $\xi\partial_\xi$. This operator can be also rewritten in terms of the generators of the $sl(2)$-algebra (2.6) but with spin $(\frac{n}{2} - 1)$ [2, 3]. This proves that the operator (2.5) contains $\frac{n}{2}$ polynomial eigenfunctions. They are nothing but Lame polynomials of the third type.

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[4] $\rho$ is replaced by $\xi$

Now we pose the question of what are the proper coordinates in these the *algebraic* and also *Lie-algebraic* forms of the rational, trigonometric and elliptic models occur, if corresponding limits are taken? After some analysis we arrive at the following expression

$$\rho = -\frac{1}{(\mathcal{P}(\frac{x_1-x_2}{2})+A)} \equiv -\frac{1}{(\mathcal{P}(y)+A)}\ , \tag{2.8}$$

where, for the sake of convenience, we have put

$$A \ \equiv\ \frac{\alpha^2}{12}\ .$$

In trigonometric limit $\alpha = \frac{\pi}{4\omega}$. The variable $\rho$ obeys the following equation

$$(\rho')^2 = -4\rho - \alpha^2\rho^2 - (12A^2 - g_2)\rho^3 - (4A^3 - g_2 A + g_3)\rho^4\ . \tag{2.9}$$

The trigonometric limit turns out to correspond to the disappearance of the $\rho^3$ and $\rho^4$ terms, while in the rational limit the $\rho^2$ term also vanishes. Then, in these limits, (2.4) becomes

$$\tilde{L}^{(2)}_{rel} \ =\ \partial^2_{yy} - \frac{n(n+1)\alpha^2}{4\sin^2(\frac{\alpha y}{2})}\ ,$$

or

$$=\ \partial^2_{yy} - \frac{n(n+1)}{y^2}\ ,$$

correspondingly. It is worth mentioning the 'embedding chain' rule of degeneration:

$$\rho(y) \to \frac{2(\cos\alpha y - 1)}{\alpha^2} = -\frac{4}{\alpha^2}\sin^2\frac{\alpha y}{2} \to -y^2\ . \tag{2.10}$$

Now we calculate the gauge rotated $\tilde{L}^{(2)}_{rel}$ operator in the relative coordinate $\rho(y)$

$$\begin{aligned} h \ &=\ \rho^{-\frac{n}{2}}\{\partial^2_{yy} - n(n+1)[\mathcal{P}(y)+A]\}\rho^{\frac{n}{2}} = \\ &= -\{4\rho + 12A\rho^2 + (12A^2-g_2)\rho^3 + (4A^3 - g_2A + g_3)\rho^4\}\partial^2_\rho \\ &+\{2(2n-1) + 12(n-1)A\rho + (n-\frac{3}{2})(12A^2-g_2)\rho^2 + (n-2)(4A^3-g_2A+g_3)\rho^3\}\partial_\rho \\ &-3An^2 - \frac{n(n-1)}{4}(12A^2-g_2)\rho - \frac{n(n-2)}{4}(4A^3-g_2A+g_3)\rho^2 \end{aligned} \tag{2.11}$$

This expression also can be represented in terms of $sl_2$–generators (2.6) of spin $\frac{n}{2}$,

$$\begin{aligned} h \ =\ & -(4A^3 - g_2A + g_3)J^+_{\frac{n}{2}}J^+_{\frac{n}{2}} - (12A^2 - g_2)J^+_{\frac{n}{2}}\{J^0_{\frac{n}{2}} - \frac{n(n-2)}{4}\} \\ & -12AJ^0_{\frac{n}{2}}(J^0_{\frac{n}{2}} - \frac{n}{2}) - 4(J^0_{\frac{n}{2}} - \frac{3n-2}{4})J^-_{\frac{n}{2}} + \frac{3An^2}{4} \end{aligned} \tag{2.12}$$

It is worth noting that the gauge factor $\rho^{\frac{n}{2}}$ in the rational and trigonometric limits degenerates to the Vandermonde determinant factor or its trigonometric generalization respectively, which we also gauged away in the Calogero and Sutherland models [7].

The trigonometric limit corresponds to zeroing of two expressions: $12A^2 - g_2 = 0$ and $4A^3 - g_2A + g_3 = 0$, which leads to the vanishing of the terms containing the positive-root $J^+$-generator. The spectrum (2.11) is

$$\epsilon_k = -3A(2k-n)^2,\ k = 0, 1, \ldots, \frac{n}{2}\ .$$

Following a general criterion we conclude that the operator (2.11) preserves the infinite flag of spaces of polynomials and the corresponding problem is exactly solvable [3]. The rational limit occurs if $A = g_2 = g_3 = 0$. Since in the rational limit the harmonic oscillator interaction is omitted, there are no polynomial eigenfunctions. It is known that the eigenvalues corresponding to the polynomial eigenfunctions and the eigenfunctions themselves of (2.5), (2.11) are the branches of an $(\frac{n}{2}+1)$-sheeted Riemann surface in the variable $g_2(g_3)$ characterized by the square-root branch points only [2]. In the trigonometric limit the residues of these branch points vanish and this surface splits off into separate sheets.

There is an intrinsic hierarchy of the algebraic form (2.11) of the Hamiltonian (2.4): elliptic (grading +2), trigonometric (grading 0), rational (grading -1). The elliptic case of the grading (+1) [5] has no meaning in our study, since there is no solution of (2.9) satisfying the 'embedding chain' rule (2.10). In principle, the variable (2.8) can be modified by allowing higher order terms in (10) and insisting on fulfillment of (2.10). It would then become a hyper-elliptic variable.

The above analysis is performed for periodic-periodic solutions of the Lame equation, for the case of an even number of zones, $n = 2k, k = 1, 2, 3\ldots.$ A similar study can be carried out for other types of solutions (see discussion above), gauging away the factor $\rho^{\frac{n}{2}-1}$ in (2.4) in addition to the factor $\eta_{ij}$. Finally, we get algebraic and Lie-algebraic forms of the Lame equation similar to (2.11)–(2.12). Similar results can be obtained for the case of an odd number of zones.

As a certain line of possible development one can substitute the generators of the $gl(2|k)$-superalgebra instead of those of $sl_2$ (see [8]) and explore the matrix QM models those will occur. We will discuss this exercise elsewhere.

## 3. Translation invariant discretization

The next problem we study is how to discretize the Lame operator in a translation invariant way, while simultaneously preserving the property of polynomiality of the eigenfunctions and also isospectrality. Again we restrict our analysis to the case of the periodic-periodic solutions and an even number of zones, $n = 2k, k = 1, 2, 3\ldots$, respectively.

Let us introduce a translationally-covariant finite-difference operator

$$\mathcal{D}_+ f(x) = \frac{(e^{\delta\partial_x} - 1)}{\delta} f(x) \ = \ \frac{f(x+\delta) - f(x)}{\delta} \ , \qquad (3.1)$$

where $\delta$ is the parameter. It is not difficult to show that the canonically conjugate operator to $\mathcal{D}_+ f(x)$ is of the form [9]

$$x(1 - \delta\mathcal{D}_-) f(x) = x e^{-\delta\partial_x} f(x) \ = \ x f(x - \delta) \ , \qquad (3.2)$$

where $\mathcal{D}_+ \to \mathcal{D}_-$ if $\delta \to -\delta$. Thus the operators (3.1) and (3.2) span the Heisenberg algebra, $[a, b] = 1$. One can easily show that using the operators (3.1)–(3.2) we can construct a representation of the $sl_2$ algebra in terms of finite-difference operators,

$$J_k^+ = x(\frac{x}{\delta} - 1) e^{-\delta\partial_x}(1 - k - e^{-\delta\partial_x}) \ ,$$

$$J_k^0 = \frac{x}{\delta}(1 - e^{-\delta\partial_x}) - \frac{n}{2} \ , \ J^- = \frac{1}{\delta}(e^{\delta\partial_x} - 1) \ , \qquad (3.3)$$

[5] $4A^3 - g_2 A + g_3 = 0$ and hence the rhs in (2.9) is a cubic polynomial in $\rho$

or, equivalently,

$$J_n^+ = x(1-\frac{x}{\delta})(\delta^2\mathcal{D}_-\mathcal{D}_- - (n+1)\delta\mathcal{D}_- + k) \ ,$$

$$J_k^0 = x\mathcal{D}_- - \frac{k}{2} \ , \ J^- = \ \mathcal{D}_+ . \tag{3.4}$$

In the limit $\delta \to 0$, the representation (3.3)–(3.4) coincides with (2.6). The finite-dimensional representation space for (2.6) and for (3.3)–(3.4) occurring for the integer values of $k$ is the same. Again that is a linear space of polynomials of degree not higher than $k$.

**1.** It is quite obvious that any operator written in terms of the generators of the algebra $sl_2$ has the same eigenvalues for both representations (2.6) and (3.3)–(3.4) (for the proof and a discussion see [9]). Let us take the operator (2.5) and construct the isospectral finite-difference operator. In order to do it we take the representation (2.7) of this operator and substitute the $sl_2$ generators in the form (3.3) or (3.4). Finally, we arrive at the following finite-difference operator,

$$\frac{4\xi^{(3)}}{\delta^2}e^{-3\delta\partial_\xi} \ - 2\frac{\xi^{(2)}}{\delta}\left(4\frac{\xi}{\delta}-5\right)e^{-2\delta\partial_\xi} \ + \left\{4\frac{\xi^{(3)}}{\delta^2}+6\frac{\xi^{(2)}}{\delta} - \left[\frac{g_2}{\delta^2}n(n+1)\right]\xi\right\}e^{-\delta\partial_\xi}$$
$$+ \ 2\frac{g_2}{\delta^2}\xi + \frac{g_2}{2\delta} - \frac{g_3}{\delta^2} - n(n+1)A$$
$$- \ \left(\frac{g_2}{\delta^2}\xi + \frac{g_2}{2\delta} - 2\frac{g_3}{\delta^2}\right)e^{\delta\partial_\xi} \ - \ \frac{g_3}{\delta^2}e^{2\delta\partial_\xi} \ . \tag{3.5}$$

The spectral problem corresponding to the operator (3.5) can be written as the 6-point finite-difference equation

$$\frac{4\xi^{(3)}}{\delta^2}f(\xi-3\delta) \ - 2\frac{\xi^{(2)}}{\delta}\left(4\frac{\xi}{\delta}-5\right)f(\xi-2\delta) \ +$$
$$\left\{4\frac{\xi^{(3)}}{\delta^2}+6\frac{\xi^{(2)}}{\delta} - [\frac{g_2}{\delta^2}n(n+1)]\xi\right\}f(\xi-\delta) \ + \ \left[2\frac{g_2}{\delta^2}\xi + \frac{g_2}{2\delta} - \frac{g_3}{\delta^2} - n(n+1)A\right]f(\xi)$$
$$- \ \left(\frac{g_2}{\delta^2}\xi + \frac{g_2}{2\delta} - 2\frac{g_3}{\delta^2}\right)f(\xi+\delta) \ - \ \frac{g_3}{\delta^2}f(\xi+2\delta) \ = \ \lambda f(\xi) \ , \tag{3.6}$$

where $\lambda$ is the spectral parameter and $\xi^{(n+1)} = \xi(\xi-\delta)(\xi-2\delta)\ldots(\xi-n\delta)$ is the so called 'quasi-monomial'.

Polynomial solutions of the equation (3.6) are associated with the Lame polynomials of the first type. They can be constructed as

$$\tilde{P}_{\frac{n}{2}}(\xi) = \sum_{i=0}^{\frac{n}{2}} \ell_i \xi^{(i)} \ , \tag{3.7}$$

where $\ell_i$ are the coefficients the Lame polynomial of the first type. We call these polynomials 'the 1-associated finite-difference Lame polynomials of the first type'. One show that there exist slightly modified 6-point finite-difference equations of the same functional form as (3.6) having polynomial solutions of the form (3.7) associated with the Lame polynomials of the second, third and fourth types. All their eigenvalues corresponding to the polynomial eigenfunctions have no dependence on the parameter $\delta$.

**2.** Now we proceed to study the translation invariant discretization of the operator (2.11). We substitute the $sl_2$-generators of the form (3.3)–(3.4) into (2.12) and arrive at

$$
\begin{aligned}
& -(4A^3-g_2A+g_3)\frac{\rho^{(4)}}{\delta^2}e^{-4\delta\partial_\rho} \\
& + \left[2(4A^3-g_2A+g_3)\left(\frac{\rho}{\delta}-\frac{n}{2}-2\right)-(12A^2-g_2)\right]\frac{\rho^{(3)}}{\delta}e^{-3\delta\partial_\rho} \\
& - \left\{(4A^3-g_2A+g_3)\delta^2\left(\frac{\rho}{\delta}-\frac{n}{2}-1\right)\left(\frac{\rho}{\delta}-\frac{n}{2}-2\right)\right. \\
& \left. -(12A^2-g_2)\delta\left(2\frac{\rho}{\delta}-n-\frac{5}{2}\right)-12A\right\}\frac{\rho^{(2)}}{\delta^2}e^{-2\delta\partial_\rho} \\
& - \left\{(12A^2-g_2)\delta\left(\frac{\rho}{\delta}-\frac{n}{2}-\frac{1}{2}\right)\left(\frac{\rho}{\delta}-\frac{n}{2}-1\right)-24A\left(\frac{\rho}{\delta}-\frac{n}{2}-\frac{1}{2}\right)+\frac{4}{\delta}\right\}\frac{\rho}{\delta}e^{-\delta\partial_\rho} \\
& - \left[12A\left(\frac{\rho^2}{\delta^2}-n\frac{\rho}{\delta}+\frac{n^2}{4}\right)-8\frac{\rho}{\delta^2}+\frac{2(2n-1)}{\delta}\right] \\
& -\frac{4}{\delta}\left(\frac{\rho}{\delta}-n+\frac{1}{2}\right)e^{\delta\partial_\rho}\ . \qquad (3.8)
\end{aligned}
$$

The spectral problem corresponding to the operator (3.8) can be written as the 6-point finite-difference equation

$$
\begin{aligned}
& -(4A^3-g_2A+g_3)\frac{\rho^{(4)}}{\delta^2}f(\rho-4\delta) \\
& + \left[2(4A^3-g_2A+g_3)\left(\frac{\rho}{\delta}-\frac{n}{2}-2\right)-(12A^2-g_2)\right]\frac{\rho^{(3)}}{\delta}f(\rho-3\delta) \\
& - \left\{(4A^3-g_2A+g_3)\delta^2\left(\frac{\rho}{\delta}-\frac{n}{2}-1\right)\left(\frac{\rho}{\delta}-\frac{n}{2}-2\right)\right. \\
& \left. -(12A^2-g_2)\delta\left(2\frac{\rho}{\delta}-n-\frac{5}{2}\right)-12A\right\}\frac{\rho^{(2)}}{\delta^2}f(\rho-2\delta) \\
& - \left\{(12A^2-g_2)\delta\left(\frac{\rho}{\delta}-\frac{n}{2}-\frac{1}{2}\right)\left(\frac{\rho}{\delta}-\frac{n}{2}-1\right)-24A\left(\frac{\rho}{\delta}-\frac{n}{2}-\frac{1}{2}\right)+\frac{4}{\delta}\right\}\frac{\rho}{\delta}f(\rho-\delta) \\
& - \left[12A\left(\frac{\rho^2}{\delta^2}-n\frac{\rho}{\delta}+\frac{n^2}{4}\right)-8\frac{\rho}{\delta^2}+\frac{2(2n-1)}{\delta}\right]f(\rho) \\
& -\frac{4}{\delta}\left(\frac{\rho}{\delta}-n+\frac{1}{2}\right)f(x+\delta) \ = \ \lambda f(\rho)\ . \qquad (3.9)
\end{aligned}
$$

Polynomial solutions of the equation (3.9) are associated with the Lame polynomials of the first type. They can be constructed as

$$
\tilde{P}_{\frac{n}{2}}(\rho) \ = \ \sum_{i=0}^{\frac{n}{2}} \ell_{\frac{n}{2}-i}\rho^{(i)}\ , \qquad (3.10)
$$

(cf.(3.7)), where $\ell_i$ are the coefficients of the Lame polynomial of the first type. We call these polynomials 'the 2-associated finite-difference Lame polynomials of the first type'. One can show that there exist slightly modified 6-point finite-difference equations of the same functional form as (3.9) having polynomial solutions of the form (3.10) associated with the Lame polynomials of the second, third and fourth types. All their eigenvalues corresponding to the polynomial eigenfunctions have no dependence on the parameter $\delta$. In the trigonometric limit the coefficients in front of the $f(\rho-4\delta), f(\rho-3\delta)$ terms disappear and (3.9) becomes the 4-point finite-difference equation.

## 4. Dilatation invariant discretization

Now we proceed to study a dilatationally-invariant discretization of the Lame operator (2.4). The prescription of the discretization we are going to use is a preservation of the polynomiality of the eigenfunctions under discretization.

Let us introduce a dilatationally-invariant, finite-difference operator. One of the simplest operator possessing such a property is known in the literature as the Jackson symbol (see e.g. [10, 11])

$$Df(x) = \frac{f(x)-f(qx)}{(1-q)x} \ , \tag{4.1}$$

where $q \in C$. The Leibnitz rule for the operator $D$ is

$$Df(x)g(x) = (Df(x))g(x) + f(qx)Dg(x) \ .$$

One can construct the algebra of the first-order finite-difference operators $D$ possessing finite-dimensional irreps for generic $q$ [12],

$$\begin{aligned} \tilde{J}_n^+ &= x^2 D - \{n\}x \\ \tilde{J}_n^0 &= xD - \hat{n} \\ \tilde{J}_n^- &= D, \end{aligned} \tag{4.2}$$

where $\{n\} = \frac{1-q^n}{1-q}$ is the so called $q$-number and $\hat{n} \equiv \frac{\{n\}\{n+1\}}{\{2n+2\}}$. If $n$ is non-negative integer, the operators (4.2) possess a common finite-dimensional invariant subspace realized by the polynomials of degree not higher than $n$, similar to what happens to the algebra (2.6). The operators (4.2) span the algebra $sl_{2q}$ (see the discussion in [3]). In the limit $q \to 1$ the generators (4.2) become (2.6) and the $sl_{2q}$ algebra reduces to the standard $sl_2$ algebra.

**1.** In [13] (see also [3]) a theorem was proven, which showed that a linear differential (finite-difference) operator has a certain number of polynomial eigenfunctions if and only if it admits a representation in terms of the generators (2.6) of the $sl_2$-algebra (the generators (4.2) $sl_{2q}$-algebra). In particular, the theorem implies that any deformation of the positive-grading part of the operator (2.5):

$$\tilde{L}_+^{(2)} = 4\xi^3\partial^2_{\xi\xi} + 6\xi^2\partial_\xi - n(n+1)\xi \ , \tag{4.3}$$

must map a polynomial of degree $n/2$ onto itself [6]. A minimal and perhaps natural deformation of the operator (4.3), which occur after replacing $\partial_\xi \to D_\xi$ is given by

$$\tilde{L}_{+,def}^{(2)} = 4\xi^3 D^2_{\xi\xi} + 6\xi^2 D_\xi - 4\left\{\frac{n}{2}\right\}\left[\left\{\frac{n}{2}-1\right\}+\frac{3}{2}\right]\xi \ , \tag{4.4}$$

[6]There are no restrictions on the deformation of the numerical coefficients in the remaining part of the operator (2.6) – they can be deformed in any way we like

which can be immediately rewritten as a linear combination of $\tilde{J}^+_{\frac{n}{2}}\tilde{J}^0_{\frac{n}{2}}$ and $\tilde{J}^+_{\frac{n}{2}}$ terms. In general this result can be treated in such a way that the $q$-deformation of the Lame equation (2.4) is reduced to the $q$-deformation of its coupling constant

$$n(n+1) \to 4\left\{\frac{n}{2}\right\}\left[\left\{\frac{n}{2}-1\right\}+\frac{3}{2}\right] . \tag{4.5}$$

Similar deformations can be carried out for other algebraic forms of the Lame operator associated with the Lame polynomials of the second, third and fourth types.

**2.** In a similar way one can carry out a deformation of another algebraic form of the Lame operator (2.11). Again a deformation of only the positive-grading part of (2.11)

$$\begin{aligned} h_+ \;=\; & -[(12A^2-g_2)\rho^3+(4A^3-g_2A+g_3)\rho^4]\partial^2_\rho \\ & +[(n-\frac{3}{2})(12A^2-g_2)\rho^2+(n-2)(4A^3-g_2A+g_3)\rho^3]\partial_\rho \\ & -\frac{n(n-1)}{4}(12A^2-g_2)\rho-\frac{n(n-2)}{4}(4A^3-g_2A+g_3)\rho^2 \ , \end{aligned} \tag{4.6}$$

is essential for a preservation of the polynomiality of the eigenfunctions. Under these requirement the simplest and perhaps the most natural deformation is given by the following expression,

$$\begin{aligned} h_{+,def} \;=\; & -[(12A^2-g_2)\rho^3+(4A^3-g_2A+g_3)\rho^4]D^2_\rho \\ & +[\left[\left\{\frac{n}{2}-1\right\}+\left\{\frac{n}{2}-\frac{1}{2}\right\}gg](12A^2-g_2)\rho^2+2\left\{\frac{n}{2}-1\right\}(4A^3-g_2A+g_3)\rho^3]D_\rho \\ & -\left\{\frac{n}{2}\right\}\left\{\frac{n}{2}-\frac{1}{2}\right\}(12A^2-g_2)\mathring{h}o-\left\{\frac{n}{2}\right\}\left\{\frac{n}{2}-1\right\}(4A^3-g_2A+g_3)\rho^2 \ . \end{aligned} \tag{4.7}$$

This expression can be immediately rewritten as a linear combination of $\tilde{J}^+_{\frac{n}{2}}\tilde{J}^+_{\frac{n}{2}}$, $\tilde{J}^+_{\frac{n}{2}}\tilde{J}^0_{\frac{n}{2}}$ and $\tilde{J}^+_{\frac{n}{2}}$ terms. Similar deformations can be carried out for other algebraic forms of the Lame operator associated with the Lame polynomials of the second, third and fourth types.

In conclusion one should mention that another dilatation-invariant deformation of the Lame equation based on a different form of the dilatation-invariant finite-difference operator was studied in [14]. It is worth emphasizing that this study was done in relation to the problem of Bloch electrons in a magnetic field (Azbel-Hofstadter problem).

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