ON THE MULTIPLICITY OF EIGENVALUES OF THE LAPLACIAN ON SURFACES*†

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Abstract

We show that the multiplicity of the eigenvalues of the Laplace Beltrami operator on compact Riemannian surfaces with genus zero is bounded by $m(\lambda_k) \leq 2k-3$ for $k \geq 3$. Here we label the eigenvalues in the following way: $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \dots$.

Let (M, g) be a compact connected Riemannian surface M with genus zero and smooth metric g and without boundary. We consider the corresponding eigenvalue problem on $L^2(M, g)$

$$-\Delta u_k = \lambda_k u_k, \qquad k = 1, 2 \dots \tag{1}$$
$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \dots .$$

In 1976 Cheng [4] showed in an important paper that the multiplicity of λ_2, $m(\lambda_2)$ satisfies

$$m(\lambda_2) \leq 3 \tag{2}$$

and this is sharp for the sphere S^2 with the canonical metric. The proof of (2) is based on the investigation of the zeros of eigenfunctions u_k, Courant's nodal theorem and a suitable version of Euler's formula for polyhedra. Using these ideas one of us, [11], generalized (2) to higher eigenvalues and showed

$$m(\lambda_k) \leq 2k - 1. \tag{3}$$

*Supported by Ministerium für Wissenschaft und Verkehr der Republik Österreich
†Work supported by the European Union TMR grant FMRX-CT 96-0001
1991 Mathematics Subject Classification 35B05

(3) holds in fact under more general conditions, e.g. for planar domains with Dirichlet or Neumann boundary conditions. We should also mention that there are interesting results concerning $m(\lambda_2)$ for surfaces with genus > 0, [6, 11, 2, 12]. For n-dimensional manifolds with $n \geq 3$ no such bounds to multiplicities of eigenvalues can be true [5, 7]. In this paper we improve upon (3).

Theorem 1 *Let (M, g) be as above, then*

$$m(\lambda_k) \leq 2k - 3 \tag{4}$$

for $k \geq 3$

Remarks.

(i) This will follow from a sharper result which will be given below.

(ii) (4) remains true for a Schrödinger equation on M, say, with $V \in C^\infty(M)$. (1) is then replaced by

$$(-\Delta + V)u_k = \lambda_k u.$$

(iii) For our case Weyl's eigenvalue asymptotics with two terms implies that for fixed M, [8]

$$\overline{\lim_{k\to\infty}} m(\lambda_k) k^{-1/2} < \infty$$

and indeed for the canonical S^2 this limit > 0. However, if we instead define

$$\mathcal{M}(k) = \sup_{(M,g)} m(\lambda_k)$$

for compact surfaces with genus zero, we only know that

$$1/2 \leq \overline{\lim_{k\to\infty}} \frac{\ln \mathcal{M}(k)}{\ln k} \leq 1.$$

(iv) For the fixed membrane case with smooth boundary related results have been obtained recently [9]. But the methods employed for the proof of the bounds to the multiplicities of eigenvalues are rather different.

Before proceeding we have to introduce some notation and recall some well known results on eigenfunctions of the Laplacian on M which will allow for a stronger result. Let u_k satisfy (1) and let

$$\mathcal{N}(u_k) = \{x \in M : u_k(x) = 0\}. \tag{5}$$

The nodal domains of u_k are the connected components of $M \setminus \mathcal{N}(u_k)$. We denote the number of nodal domains by $\mu(u_k)$. Courant's nodal theorem tells us that [3]

$$\mu(u_k) \leq k. \tag{6}$$

Suppose $m(\lambda_k) = l$ and denote the corresponding eigenspace by U_k^l, then each $u \in U_k^l$ satisfies $\mu(u) \leq k$. We can formulate now a somewhat stronger version of Theorem 1.

Theorem 2 *Suppose λ is an eigenvalue of* (1) *with eigenspace $U(\lambda)$. Suppose that for each $u \in U$, $\mu(u) \leq l$, $l \geq 2$, then* $\dim U \leq \max(2l-3, 3)$.

Remarks.

(i) Clearly Theorem 2 implies Theorem 1 via Courant's nodal theorem.

(ii) It is sharp for $l = 2$ for the sphere. For $l = 3$ we have no explicit example in the sense that we have only examples of multiplicity 3 with eigenfunctions with $\mu(u) = 2$, but for $l = 4$ it is again sharp: Consider the canonical S^2. We have $0 = \lambda_1 < \lambda_2 = \cdots = \lambda_4 < \lambda_5 = \cdots = \lambda_9 < \lambda_{10}$. But the eigenspace associated to λ_5 has the property that each function in it has at most 4 nodal domains, see [10]. We do not know whether there is actually a case where $m(\lambda_4) = 5$. For $l > 4$ we do not know whether Theorem 2 can be improved.

Next we recall some results on $\mathcal{N}(u_k)$. See [1, 4].

Proposition 1 *$\mathcal{N}(u_k)$ consists of a finite number of C^2-immersed circles. The number of points where these circles meet is finite. At a point where l nodal lines meet, u_k vanishes of order l and the zero set is in a neighborhood C^1-diffeomorphic to the zero set of a harmonic homogeneous polynomial of degree l.*

Harmonic homogeneous polynomials h_l in two variables of degree l have a simple representation in polar coordinates:

$$h_l = r^l(a \cos l\alpha + b \sin l\alpha), \qquad a, b \in \mathbf{R} \tag{7}$$

Next we give a suitable version of Euler's theorem for polyhedra, see e.g [10, 11].

Proposition 2 *Suppose u satisfies (1) for some λ and suppose that $y_i \in M$, is the set of points where u has zeros of order $\rho_i > 1, \quad i = 1, \ldots, N$. Define*

$$\nu(u) = \sum_{i=1}^{N} (\rho_i - 1).$$

Denote by $\xi(u)$ the number of connected components of $\mathcal{N}(u)$, then

$$\mu(u) = \xi(u) + \nu(u) + 1 \tag{8}$$

Remarks.

(i) Clearly here enters the assumption that M has genus zero.

(ii) Armed with proposition 1, proposition 2 and Courant's nodal theorem we already can prove (3) : Assume for contradiction that $m(\lambda_k) \geq 2k$. Fix $y \in M$. (7) implies that there is a function $u_y \in U_k^{2k}$, the eigenspace associated to λ_k, such that u_y has a zero of order $\geq k$ at y, (a simple exercise in linear algebra). Proposition 2 implies that $\mu(u_y) \geq 1 + (k-1) + 1 = k+1$ contradicting Courant's nodal theorem.

Proof of Theorem 2.

We assume first for contradiction that for some $k \geq 3$, in some eigenspace U^{2k-1} of dimension $2k-1$, $\sup_{u \in U^{2k-1}} \mu(u) \leq k$. Fix $y \in M$. Then there are as a consequence of (7) and Proposition 1 two linearly independent functions, $v_1,\ v_2 \in U_k^{2k-1}$ such that v_1 and v_2 vanish at $y \in M$ with order $k-1$. Proposition 2 together with (7) imply that $v_i, i = 1, 2$ cannot vanish of order $> k-1$ anywhere on M since this would lead to a contradicition to our assumption on $\sup_{u \in U} \mu(u)$.

Lemma 1 *There is a $v \in \operatorname{span}(v_1, v_2)$ such that $\mu(u) > k$.*

Proof of Lemma 1. First we note that proposition 2 implies that

$$\mu(v_i) = k, \quad i = 1, 2$$

Without loss we can assume that (again using local polar coordinates centered at y) that

$$v_1 = r^{k-1} \sin(k-1)\alpha + \mathcal{O}(r^k),$$

$$v_2 = r^{k-1}\cos(k-1)\alpha + \mathcal{O}(r^k),$$

since if v_1 and v_2 would both start with the same homogeneous harmonic polynomial of degree $k-1$ we would obtain by linear combination a zero of order k.

Consider v_1. We can associate the nodal lines (actually $2k-2$ rays) starting from the origin with $\alpha_1 = 0, \alpha_2 = \frac{\pi}{k-1}, \ldots, \alpha_{2k-2} = \frac{(2k-3)\pi}{k-1}$ with rays tangent to the nodal lines with argument α_i. We observe that each component $\gamma_{i,j}$ of $\mathcal{N}(v_1)\backslash\{0\}$ is a loop hitting the origin along the directions α_i, α_j. So again by proposition 2, $\mathcal{N}(v_1)\backslash\{0\}$ consist of $k-1$ nonintersecting loops.

Consider the linear combinations parametrized by θ

$$w_\theta = v_1\cos(k-1)\theta + v_2\sin(k-1)\theta\ . \tag{9}$$

Obviously $w_0 = v_1$ and

$$|w_0 + w_{\pi/(k-1)}| = \mathcal{O}(r^k).$$

Hence $w_{\pi/(k-1)} = -w_1$ since if this would not hold we would have $w_{\pi/(k-1)} + w_0 = \mathcal{O}(r^k)$ again leading to too many nodal domains.

The nodal sets of w_0 and $w_{\pi/(k-1)}$ are equal. There must be in $\mathcal{N}(w_0)\backslash\{0\}$ a loop $\gamma_{0,j}$ for some $j \leq 2k-3$ where j must be odd, since an even j would lead to intersecting loops. Now consider $\mathcal{N}(w_{\pi/(k-1)})\backslash\{0\}$. Its zero set must contain a loop $\gamma_{1,j+1}$ for $j < 2k-3$ or $\gamma_{2k-3,0}$. Hence $\gamma_{0,j}$ must intersect $\gamma_{1,j+1}$ if $j < 2k-3$ or γ_{2k-3}, a contradiction.

We proceed by assuming for contradiction that

$$\sup_{u\in U^{2k-2}} \mu(u) \leq k \tag{10}$$

in some eigenspace U^{2k-2} of dimension $2k-2$. Arguing as above we see that for all $y \in M$ there is a unique u_y (up to a multiplicative factor) with a zero of order $k-1$ at y.

This means that to each y we can associate $k-1$ lines in the tangent space T_yM, which meet transversally at y. Let U^{2k-2} be as above and suppose that $\phi_1, \ldots, \phi_{2k-2}$ are orthonormal eigenfunctions which span U^{2k-2}. Any normalized u_y can be uniquely represented by

$$u_y = \sum_{i=1}^{2k-2} c_i(y)\phi_i, \quad \sum_{i=1}^{2k-2} c_i^2(y) = 1. \tag{11}$$

Since u_y has by assumption a zero of order $k-1$, the $c_i(y)$ are determined by the requirement (up to the sign) that

$$\sum_{i=1}^{2k-2} c_i(y)(\partial^\alpha \phi_i)(y) = 0, \quad \sum_{i=1}^{2k-2} c_i^2 = 1, \quad |\alpha| \leq k-2 \tag{12}$$

where α is the usual multi-index. From (7) we infer that (12) is a system of $2k-1$ linear equations for the $c_i(y)$. Obviously the rank of the $(2k-1)\times(2k-2)$ matrix with the elements $(\partial^\alpha\phi_i)(y)$ must be $2k-2$ for all y. If it would be smaller for some y_0 we could again find two linear independent u_{y_0} and obtain a contradiction as above. So the $c_i(y)$ are unique up to the sign and depend smoothly on the $(\partial^\alpha\phi_i)(y)$. In particular $(\partial^\alpha u_y)(y) \in C^\infty(M)$ for $|\alpha| = k-1$ implying that the position of the $k-1$ lines for each T_yM depend smoothly on y. The Euler characteristic of M is not zero and this will lead finally to the desired contradiction.

To make this precise let $\pi : TM \to M$ be the tangent bundle. For each $x \in M$ we define $2k-2$ unit vectors $l^1, \ldots, l^{2k-2} \in T_xM$ such that they are tangent to the nodal lines (actually $2k-2$ rays) of u_x starting from the point x, hence $l^i \neq l^j$ for $i \neq j$. Define f by

$$f : x \to \{l_x^i\}_{i=1}^{2k-2}$$

then f is a $2k-2$ valued section of $\pi : TM \to M$. Let E denote the set of all values of f, a submanifold of TM. The restriction of π is a covering map to $E \to M$. Since M is simply connected, E is a disjoint union of manifolds E_i such that π induces a diffeomorphism $E_i \to M$ for each i. Each inverse $(\pi|E_i)^{-1} : M \to E_i \subset TM$ is a smooth vectorfield on M without zero which contradicts the hairy ball theorem.

Acknowledgement. It is a pleasure to thank Peter Michor for many helpful discussions.

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