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Time-dependent scattering on fractal measures.

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Abstract

We study the time-evolution for the Schrödinger equation and the wave equation on the line when the interaction term is a fractal measure. We relate the long-time localization properties of the wave-packets to a fractal dimension of the potential, namely the wavelet correlation dimension. For the wave equation, we also show how the wave-packets can be interpreted in terms of wavelet transform of the potential.

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1 Introduction

The aim of this paper is to study the time evolution generated by Schrödinger operators on the line when the potential term is a fractal measure. We wish to show how the time-space behaviour of the wave-packets mirrors the "fractality" of the scatterer. A first step has been made in the stationary representation ([GH96]), where the high-frequency behaviour of the scattering data has been related to some fractal dimension of the potential.

The problem we want to deal with can be put in the following abstract form. Let $\mathcal{H}$ be some Hilbert space and H a self-adjoint operator on $\mathcal{H}$, with domain $D(H)$. Consider the Cauchy problem of first and second order in time:

$$i\partial_t\Psi_t - H\Psi_t = 0, \quad \Psi_{t=0} = f \in D(H), \tag{1.1}$$

$$\partial_{tt}\Phi_t + H\Phi_t = 0, \quad \Phi_{t=0} = f \in D(H), \quad \partial_t\Phi_{t=0} = g \in D(H). \tag{1.2}$$

The first equation is the Schrödinger equation, which describes the evolution of quantum wave-packets under the action of the hamiltonian H. Its solution is uniquely determined by the initial state:

$$\Psi_t = e^{-itH}f, t \geq 0. \tag{1.3}$$

The conservation of energy is expressed by

$$\|\Psi_t\|^2 = \|f\|^2. \tag{1.4}$$

The second equation can be viewed as the wave equation governing e.g the evolution of electromagnetic waves. It can be solved by reduction to a first-order matricial equation. If H is non-negative, there exists a unique non-negative self-ajoint operator $H^{1/2}$ such that $(H^{1/2})^2 = H$ and $D(H)$ is dense in $D(H^{1/2})$. The unique solution of (1.2) is then given by (see e.g [Yaf80]):

$$\Phi_t = \cos(H^{1/2}t)f + H^{-1/2}\sin(H^{1/2}t)g, \; t \geq 0, \tag{1.5}$$

where the functions of H are defined by the usual functional calculus. The conservation of energy now reads

$$\|\partial_t\Phi_t\|^2 + \left\|H^{1/2}\Phi_t\right\|^2 = \|g\| + \left\|H^{1/2}f\right\|, \tag{1.6}$$

which can easily be seen by differentiating the left-hand side with respect to t.

In this paper, we will take $\mathcal{H} = L^2(\mathbb{R})$, the Hilbert space of square integrable functions on $\mathbb{R}$; H will be a generalized Schrödinger operator formally given by $H = -\Delta + V$, where V is a fractal measure on $\mathbb{R}$. At his point, we can rephrase more precisely the central question of this paper: can one retrieve some fractal characteristic (such as fractal dimension or self-similarity) of V via the long-time behaviour of the wave-packets Ψ_t as well as Φ_t ?

We will proceed as follows. We first set the mathematical frame of the problem and derive asymptotic formulae for the functions Ψ_t and Φ_t. Then we introduce our main tool to characterize the potential, namely the wavelet transform and the wavelet correlation dimension. Then we show how the probability of presence of the wave-packets in different regions can be related to the wavelet correlation dimension of the potential. For the wave equation, we interpret the wave-packets Φ_t in terms of wavelet transform of the potential, which can be used to analyse the local regularity of this latter.

2 Scattering formalism for measures

We equip $L^2(\mathbb{R})$ with the usual scalar product $< f,g >= \int \overline{f}g$ and adopt the notation $\|f\|$ for the induced norm. Let H_0 be the self-adjoint realization of the negative Laplacian $-\Delta$ on $L^2(\mathbb{R})$, with domain $D(H_0) = W^2(\mathbb{R})$, where $W^p(\mathbb{R})$ is the Sobolev space of functions such that

$$\int dk\ (1+k^{2p}) \left|\widehat{f}(k)\right|^2 < \infty.$$

Let V be a finite real Borel measure on $\mathbb{R}$. The operator sum $H = H_0 + V$ is in general not defined on $D(H_0)$, but there exists a unique self-adjoint operator H with domain $D(H) \subset W^1(\mathbb{R})$ corresponding to this formal expression in the sense of quadratic forms (see e.g [Bra85]). The action of this operator is given by

$$< f, Hg >= \int dx \overline{f'(x)} g'(x) + \int \overline{f(x)} g(x) V(dx)$$

for all $f \in W^1(\mathbb{R})$ and $g \in D(H)$. It has been shown in [Gue] that an explicit scattering theory can be constructed for such operators. We briefly summarize the results that are needed for the sequel. For $k \in \mathbb{R}$, the integral equations

$$\varphi_k^{\pm}(x) = e^{\pm ikx} + \frac{1}{2ik} \int e^{ik|x-y|} \varphi_k^{\pm}(y) V(dy) \tag{2.1}$$

admit unique continuous bounded solutions φ_k^+ and φ_k^-, respectively. Far from the scatterer, these functions behave like free waves; they have the following asymptotic expressions,

$$\varphi_k^{\pm}(x) \to \begin{cases} t_k e^{\pm ikx}, & x \to \pm\infty, \\ e^{\pm ikx} + r_k^{\pm} e^{\mp ikx}, & x \to \mp\infty, \end{cases} \tag{2.2}$$

which become exact if the measure V has compact support. The coefficients t_k, r_k^+, r_k^- are called transmission coefficients and reflection coefficients from the right and left, respectively. They can be expressed in terms of the functions $\varphi_k^{\pm}$:

$$t_k = 1 + \frac{1}{2ik} \int e^{\mp iky} \varphi_k^{\pm}(y) V(dy), \tag{2.3}$$

$$r_k^{\pm} = \frac{1}{2ik} \int e^{\mp iky} \varphi_k^{\pm}(y) V(dy). \tag{2.4}$$

The transmission and reflection coefficients satisfy a couple of relations which express the unitarity of the scattering matrix,

$$r_k^+ \overline{t_k} - t_k \overline{r_k^-} = 0,\ |t_k|^2 + \left|r_k^{\pm}\right|^2 = 1, \tag{2.5}$$

and, together with the scattering wave functions, obey symmetries which account for the time-reversal invariance property,

$$\overline{\varphi_k^{\pm}} = \varphi_{-k}^{\pm},\ \overline{t_k} = t_{-k},\ \overline{r_k^{\pm}} = r_{-k}^{\pm}. \tag{2.6}$$

At high-frequency, the transmission and reflection can be well estimated by the Born approximation. It has been shown in [GH96] that

$$r_k^+ = \frac{\widehat{V}(-2k)}{2ik} + O(k^{-2}), \tag{2.7}$$

where $\widehat{V}$ is the Fourier transform of the measure V,

$$\widehat{V}(\xi) = \int e^{-i\xi x} V(dx).$$

First order approximations can also be derived from (2.5) for r_k^- and t_k :

$$r_k^- = -\frac{\widehat{V}(2k)}{2ik} + O(k^{-2}), \ t_k = 1 + O(k^{-2}). \tag{2.8}$$

Spectral properties. Thanks to the properties of the two families of functions $\varphi_k^\pm$, the spectrum of H has been invastigated and the existence and completeness of the wave operators have been shown. Under the additional assumption the measure have two absolutely integrable moments, i.e

$$\int (1 + x^2) \, |V| \, (dx) < \infty, \tag{2.9}$$

the positive spectrum is purely absolutely continuous and the negative spectrum consists in a finite number of eigenvalues. The eigenvectors are those functions f of $W^1(\mathbb{R})$ for which

$$(-\Delta + V - E) f = 0 \tag{2.10}$$

for some $E < 0$, the above equation being understood in the sense of distributions. Let (E_λ) be the spectral family associated to H, i.e formaly $H = \int \lambda dE_\lambda$. For any states $f, g \in L^2(\mathbb{R})$, the associated spectral measure $\mu_{f,g}$, defined by

$$d\mu_{f,g}(\lambda) = (f, dE_\lambda g),$$

is absolutely continuous on the positive axis with density:

$$\frac{d\mu_{f,g}(\lambda)}{d\lambda} = m_{f,g}(\lambda) = \int dx dy \overline{f(y)} f(x) p(x, y, \lambda), \tag{2.11}$$

where the kernel $p(x, y, \lambda)$ is given by

$$p(x, y, \lambda) = \frac{1}{2\pi} \Re \left\{ \frac{\varphi_k^+(y) \varphi_k^-(x)}{k t_k} \right\} = \frac{1}{2\pi} \Re \left\{ \frac{\varphi_k^+(x) \varphi_k^-(y)}{k t_k} \right\}, \ k^2 = \lambda, \ k > 0. \tag{2.12}$$

Wave operators. If we introduce the unitary Fourier transform on $L^2(\mathbb{R})$,

$$\widehat{f}(k) = \frac{1}{\sqrt{2\pi}} \int dx \ e^{-ikx} f(x),$$

the wave operators

$$W^\pm = \text{s.} \lim_{t \to \pm\infty} e^{itH} e^{-itH_0} \tag{2.13}$$

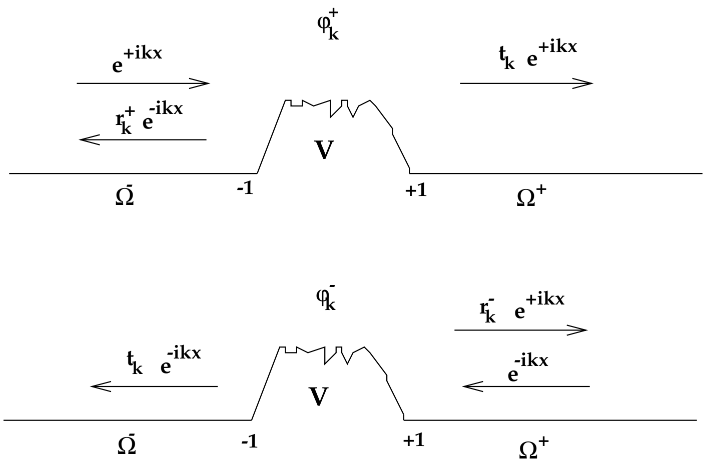

Figure 1: Scattering wave functions

are given by

$$W^+ f(x) = \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} dk\ \varphi_k^+(x) \widehat{f}(k) + \int_0^{+\infty} dk\ \overline{\varphi_k^-(x)} \widehat{f}(k) \right), \tag{2.14}$$

$$W^- f(x) = \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} dk\ \overline{\varphi_k^-(x)} \widehat{f}(k) + \int_0^{+\infty} dk\ \varphi_k^+(x) \widehat{f}(k) \right). \tag{2.15}$$

$$\tag{2.16}$$

$W_\pm$ are isometric operators whose range is $\mathcal{H}_{ac}$, the absolutely continuous subspace of $L^2(\mathbb{R})$ associated to H. Hence,

$$W_\pm^* W_\pm = 1,\ W_\pm W_\pm^* = P_{ac}, \tag{2.17}$$

where $*$ stands for the hermitian conjugation and P_{ac} is the orthogonal projector on $\mathcal{H}_{ac}$. We will assume for the moment $P_{ac} = 1$ (i.e the spectrum is purely positive and absolutely continuous), deferring to a later discussion the influence of bound states. The wave operators then provide a simple approximation of the evolution operator $e^{\pm itH}$ at large time:

$$\text{s.}\lim_{t\to+\infty} e^{\pm itH} - W_\pm e^{\pm itH_0} = 0. \tag{2.18}$$

In particular, for all initial state $f \in D(H)$ we have

$$\text{s.}\lim_{t\to+\infty} \Psi_t - \tilde{\Psi}_t = 0, \tag{2.19}$$

where $\Psi_t = e^{-itH} f$ and $\tilde{\Psi}_t = W_- e^{-itH_0} f$ is explicitable,

$$\tilde{\Psi}_t(x) = \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} dk\ e^{-itk^2} \overline{\varphi_k^-}(x) \widehat{f}(k) + \int_0^{+\infty} dk\ e^{-itk^2} \varphi_k^+(x) \widehat{f}(k) \right). \tag{2.20}$$

3 Time-evolution

We now wish to look at the time-space behaviour of the wave-packets Ψ_t and Φ_t defined in the introduction. For sake of simplicity, we will suppose V is a finite measure of compact support. In fact, the more general condition (2.9) also works, but brings into technical complications and carries no new essential idea. Without restriction, we can suppose that the support of V is included in the interval $[-1,+1]$. We call $\Omega^{\pm}$ the regions to the left and to the right of the support of V (see fig. 2),

$$\Omega^- =]-\infty,-1], \quad \Omega^+ = [1,+\infty[,$$

and $\chi_{\Omega^\pm}$ the corresponding characteristic functions, $\chi_{\Omega^\pm}(x) = 1$ if $x \in \Omega^{\pm}$, $\chi_{\Omega^\pm}(x) = 0$ else. We use the same notaion for the associated multiplication operator, i.e $(\chi_{\Omega^\pm} f)(x) = \chi_{\Omega^\pm}(x) f(x)$. Let us now define

$$p^{\pm}[f] = \lim_{t\to+\infty} \left\|\chi_{\Omega^\pm}\Psi_t\right\|^2, \ \Psi_t = e^{-iHt}f, \tag{3.1}$$

assuming for the moment that the limits exist. When f is normalized (i.e $\|f\| = 1$), these quantities can be interpreted as the final probability of finding a particle described by the wave function Ψ_t in the left, respectively right, region. Analogously, we define

$$p^{\pm}[f,g] = \lim_{t\to+\infty} \left\|\chi_{\Omega^\pm}\partial_t\Phi_t\right\|^2 + \left\|\chi_{\Omega^\pm}H^{1/2}\Phi_t\right\|^2, \tag{3.2}$$

with $\Phi_t = \cos(H^{1/2}t)f + H^{-1/2}\sin(H^{1/2}t)g$. The interpretation of $p^{\pm}[f,g]$ is similar. To avoid heavy notations, we have used the same letters $p^{\pm}$ for the quantities referring to Φ_t as well as Ψ_t. However, we implicitely refer to the one or the other according to whether the argument is $[f]$ or $[f,g]$.

3.1 Time evolution for the Schrödinger equation

The probability of presence $p^{\pm}[f]$ can be related to the transmission and reflection coefficients as follows.

Théorème 3.1 *For any initial state $f \in D(H)$ we have*

$$p^-[f] = \int_{-\infty}^{0} dk \ \left|\overline{r_k^+}\widehat{f}(-k) + \overline{t_k}\widehat{f}(k)\right|^2 \tag{3.1}$$

$$p^+[f] = \int_{0}^{+\infty} dk \ \left|r_k^-\widehat{f}(-k) + t_k\widehat{f}(k)\right|^2 \tag{3.2}$$

Proof. As before, define $\tilde{\Psi}_t = W_- e^{-itH_0}f$. In view of (2.18), we have

$$p^{\pm}[f] = \lim_{t\to+\infty} \left\|\chi_{\Omega^\pm}\tilde{\Psi}_t\right\|^2.$$

Now, since outside of the potential support the functions $\varphi_k^{\pm}$ are given by their asymptotic expressions (2.2), the function $\tilde{\Psi}_t$ takes a simple form there and we have

$$\chi_{\Omega^-}\tilde{\Psi}_t(x) = \frac{\chi_{\Omega^-}(x)}{\sqrt{2\pi}}\left(\int_{-\infty}^{0} dk e^{-itk^2}e^{ikx}\overline{t_k}\widehat{f}(k) + \int_{0}^{+\infty} dk[e^{-itk^2}e^{ikx}\widehat{f}(k) + e^{-itk^2}e^{-ikx}r_k^+\widehat{f}(k)]\right),$$

$$\chi_{\Omega^+}\tilde{\Psi}_t(x) = \frac{\chi_{\Omega^+}(x)}{\sqrt{2\pi}}\left(\int_{-\infty}^{0} dk e^{-itk^2} e^{ikx}\widehat{f}(k) + \int_{-\infty}^{0} dk[e^{-itk^2}e^{-ikx}\overline{r_k^-}\widehat{f}(k) + e^{-itk^2}e^{ikx}t_k\widehat{f}(k)]\right).$$

At this point, let us introduce the Fourier multipliers A_i defined by:

$$\begin{array}{ccc} \widehat{A_1 f}(k) = t_k \widehat{f}(k) & , & \widehat{A_2 f}(k) = \overline{t_k}\widehat{f}(k), \\ \widehat{A_3 f}(k) = \overline{r_k^+}\widehat{f}(-k) & , & \widehat{A_4 f}(k) = r_k^- \widehat{f}(-k), \end{array}$$

and the positive/negative frequencies projectors $P^{\pm}$:

$$\widehat{P^{\pm} f}(k) = \begin{cases} \widehat{f}(k) \text{ if } sign(k) = \pm 1 \\ 0 \text{ else} \end{cases},$$

then the previous equations may be rewritten more simply as

$$\begin{aligned} \chi_{\Omega^-}\tilde{\Psi}_t &= \chi_{\Omega^-} e^{-itH_0}[P^+ f + P^- A_3 f + P^- A_2 f], \\ \chi_{\Omega^+}\tilde{\Psi}_t &= \chi_{\Omega^+} e^{-itH_0}[P^- f + P^+ A_4 f + P^+ A_1 f]. \end{aligned}$$

Thus, the long-time behaviour of $\chi_{\Omega^{\pm}}\tilde{\Psi}_t$ depends on the localization properties of the free evolution operator e^{-itH_0}, that we summarize in

Lemme 3.2 *For all $f \in L^2(\mathbb{R})$ we have,*

$$\lim_{t\to+\infty} \left\|\chi_{\Omega^-} e^{-itH_0} f\right\|^2 = \int_{-\infty}^{0} dk \left|\widehat{f}(k)\right|^2, \tag{3.3}$$

$$\lim_{t\to+\infty} \left\|\chi_{\Omega^+} e^{-itH_0} f\right\|^2 = \int_{0}^{+\infty} dk \left|\widehat{f}(k)\right|^2. \tag{3.4}$$

Proof. It is well-known (see e.g [RS75], p 60) that for all $f \in L^2(\mathbb{R})$,

$$\text{s.}\lim_{t\to+\infty} e^{-itH_0} - V_t = 0, \tag{3.5}$$

where

$$V_t f(x) = \frac{1}{\sqrt{2it}} e^{ix^2/4t}\widehat{f}(x/2t).$$

Now

$$\left\|\chi_{\Omega^-} e^{-itH_0} f\right\|^2 = -\left\|\chi_{\Omega^-}(e^{-itH_0} - V_t)f\right\|^2 + 2\Re\left\{< \chi_{\Omega^-} e^{-itH_0} f, \chi_{\Omega^-}(e^{-itH_0} - V_t)f >\right\} + \left\|\chi_{\Omega^-} V_t f\right\|^2.$$

The first two terms on the right-hand side converge to zero. For the last term we have

$$\left\|\chi_{\Omega^-} V_t f\right\| = \frac{1}{2t}\int_{-\infty}^{-1} dx \left|\widehat{f}(x/2t)\right|^2 \longrightarrow \int_{-\infty}^{0} dk \left|\widehat{f}(k)\right|^2,\ t \to +\infty.$$

The proof for (3.4) is similar. □

A direct application of the last lemma yields the formula (3.1) and (3.2). Thanks to the relations (2.5), it can be easily verified that $p^+[f] + p^-[f] = 1$, that is no energy remains trapped in the region of interaction. □

Remark. We have assumed $P_{ac} = 1$, i.e the spectrum is purely absolutely continuous. However, if $P_{ac} \neq 1$, that is there are some bound states, then the last theorem still holds for $P_{ac}f$ in place of f.

3.2 Time evolution for the wave equation

We will use an eigenfunction expansion to compute the exact time-dependent solution of the wave equation. In the following, we will consider initial states with support in Ω^- only. All the results can be adapted in an obvious way to states with support in Ω^+. We begin with

Théorème 3.3 *For all bounded measurable real function η and all $f \in L^2(\mathbb{R})$ with support in Ω^- we have*

$$\eta(H)f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\ \eta(k^2)\varphi_k^+(x)\widehat{f}(k), \tag{3.1}$$

the equality holding in $L^2(\mathbb{R})$. If $f \in D(H)$, then both left and right-hand side are continuous function and thus the equality holds pointwise.

Proof. We will use the spectral density $m_{f,g}(\lambda)$ introduced in (2.11). For any $f, u \in C_0^\infty(\mathbb{R})$, we have

$$\begin{aligned} < u, \eta(H)f > &= \int_0^\infty d\lambda\ \eta(\lambda) m_{u,f}(\lambda) \\ &= \frac{1}{2\pi}\int_0^\infty d\lambda\ \eta(\lambda) \int dxdy\ \overline{u}(y) f(x) \Re \left\{ \frac{\varphi^+_{\sqrt{\lambda}}(y)\varphi^-_{\sqrt{\lambda}}(x)}{\sqrt{\lambda} t_{\sqrt{\lambda}}} \right\}, \end{aligned}$$

Interchanging the order of integration, which is justified since the inner integrals are absolutely convergent, we obtain

$$< u, \eta(H)f > = < u, Af >,$$

with

$$Af(y) = \frac{1}{2\pi}\int dx\ f(x) \int_0^\infty d\lambda\ \eta(\lambda) \Re \left\{ \frac{\varphi^+_{\sqrt{\lambda}}(y)\varphi^-_{\sqrt{\lambda}}(x)}{\sqrt{\lambda} t_{\sqrt{\lambda}}} \right\}.$$

Making the change of variables $\lambda = k^2$, using the symmetry properties of $\varphi_k^\pm$ and interchanging again the integrals yields finally

$$Af(y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\ \eta(k^2)\varphi_k^+(y) \int dx\ f(x) t_k^{-1} \varphi_k^-(x).$$

Since f is supported on Ω^-, we have $f(x)t_k^{-1}\varphi_k^-(x) = f(x)e^{-ikx}$ and

$$Af(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\ \eta(k^2)\varphi_k^+(y)\widehat{f}(k). \tag{3.2}$$

Now A can be extended to a bounded operator in $L^2(\mathbb{R})$. Hence, by density, $\eta(H)f = Af$ holds on $L^2(\mathbb{R})$. Note that Af is a continuous function (since φ_k^+ is continuous and the integral in (3.2) is uniformly convergent in y). If $f \in D(H)$, then $\eta(H)f \in D(H) \subset W^1(\mathbb{R})$ is also continous and thus the equality $\eta(H)f = Af$ holds pointwise. This concludes the proof. $\square$

By now we will assume that the initial wave-packet $\Phi_{t=0}$ is prepared by sending a incident wave train from the remote left on the scatterer. Thus, for $t<0$, the wave function is simply a (free) solution of the wave equation propagating to the right, that is of the form $\Phi_t(x)=s(x-t)$. Therefore, it is natural to impose the initial conditions

$$g=\partial_t\Phi_{t=0}=-\partial_x\Phi_{t=0}=-f'. \tag{3.3}$$

With these conditions we have

Proposition 3.4 *Let $f\in W^1(\mathbb{R})$ be a function with compact support in Ω^- and consider the solution of the wave equation with initial conditions (3.3), that is*

$$\Phi_t=\cos(H^{1/2}t)f-H^{-1/2}\sin(H^{1/2}t)f'.$$

Then

$$\chi_{\Omega^-}\Phi_t(x) = \chi_{\Omega^-}(x)[f(x-t)+A_3f(x+t)], \tag{3.4}$$

$$\chi_{\Omega^+}\Phi_t(x) = \chi_{\Omega^+}(x)A_1f(x-t), \tag{3.5}$$

where as before $\widehat{A_1f}(k)=t_k\widehat{f}(k)$ and $\widehat{A_3f}(k)=\overline{r_k^+}\widehat{f}(-k)$. For the associated probabilities of presence, we have

$$p^-[f,-f'] = 2\int dk\ k^2\left|r_k^+\right|^2\left|\widehat{f}(k)\right|^2 \tag{3.6}$$

$$p^+[f,-f'] = 2\int dk\ k^2\left|t_k\right|^2\left|\widehat{f}(k)\right|^2 \tag{3.7}$$

Proof. It follows from theorem (3.3) that

$$\Phi_t(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}dk\ e^{-ikt}\varphi_k^+(x)\widehat{f}(k).$$

Using the asymptotic expression (2.2) of φ_k^+ yields (3.4) and (3.5). Let us now compute $p^\pm[f,-f']$. By definition we have

$$\begin{aligned}\partial_t\Phi_t &= -H^{1/2}\sin(H^{1/2}t)f-\cos(H^{1/2}t)f'\\ H^{1/2}\Phi_t &= H^{1/2}\cos(H^{1/2}t)f-\sin(H^{1/2}t)f',\end{aligned}$$

as soon as $f\in D(H^{1/2})$ and $f'\in L^2(\mathbb{R})$. This is the case here since a function in $W^1(\mathbb{R})$ with compact support outside the potential is also in $D(H)\subset D(H^{1/2})$. Using theorem (3.3) again, we obtain

$$\begin{aligned}\partial_t\Phi_t &= -\frac{1}{\sqrt{2\pi}}\int dk\ e^{-ikt}\varphi_k^+\ ik\widehat{f}(k),\\ H^{1/2}\Phi_t &= \frac{1}{\sqrt{2\pi}}\int dk\ e^{-ikt}\varphi_k^+\ |k|\,\widehat{f}(k),\end{aligned}$$

and thus

$$\begin{aligned}(\chi_{\Omega^-}\partial_t\Phi_t)(x) &= -\chi_{\Omega^-}(x)[f'(x-t)+A_3f'(x+t)],\\ (\chi_{\Omega^+}\partial\Phi_t)(x) &= -\chi_{\Omega^+}(x)A_1f'(x-t),\\ (\chi_{\Omega^-}H^{1/2}\Phi_t)(x) &= -i[P^+f'(x-t)-P^-f'(x-t)+P^-A_3f'(x+t)-P^+A_3f(x+t)],\\ (\chi_{\Omega^+}H^{1/2}\Phi_t)(x) &= -i[P^+A_1f'(x-t)-P^-A_1f'(x-t)].\end{aligned}$$

Now, for any function $h \in L^2(\mathbb{R})$ we have

$$\begin{aligned}
\lim_{t\to+\infty} \left\|\chi_{\Omega^\pm} h(\cdot \pm t)\right\| &= 0, \\
\lim_{t\to+\infty} \left\|\chi_{\Omega^\mp} h(\cdot \pm t)\right\| &= \|h\| .
\end{aligned}$$

Hence,

$$\begin{aligned}
\lim_{t\to+\infty} \left\|\chi_{\Omega^-} \partial_t \Phi_t\right\|^2 &= \left\|A_3 f'\right\|^2 \\
\lim_{t\to+\infty} \left\|\chi_{\Omega^+} \partial_t \Phi_t\right\|^2 &= \left\|A_1 f'\right\|^2 \\
\lim_{t\to+\infty} \left\|\chi_{\Omega^-} H^{1/2} \Phi_t\right\|^2 &= \left\|P^- A_3 f' - P^+ A_3 f'\right\|^2 = \left\|A_3 f'\right\|^2, \\
\lim_{t\to+\infty} \left\|\chi_{\Omega^+} H^{1/2} \Phi_t\right\|^2 &= \left\|P^+ A_1 f' - P^- A_1 f'\right\|^2 = \left\|A_1 f'\right\|^2,
\end{aligned}$$

which leads to equations (3.6) and (3.7). Note that we have, because $supp(f) \in \Omega^-$,

$$\left\|H^{1/2} f\right\|^2 =< Hf, f >=< H_0 f, f >= \int dk\ k^2 \left|\widehat{f}(k)\right|^2,$$

and thus

$$p^+[f, -f'] + p^-[f, -f'] = 2 \int dk\ k^2 \left|\widehat{f}(k)\right|^2 = \left\|f'\right\|^2 + \left\|H^{1/2} f\right\|^2 .$$

This conveys the fact that the wave-packets have travelled away fronm the scatterer. □

4 The wavelet correlation dimension

The formalism which has been presented above allows to treat potentials given by finite measures, in particular fractal measures. By "fractal" we mean, loosely speaking, a measure which has a non-trivial fractal dimension. They are several definitions of the fractal dimension of a measure, such as the box dimension, the information, the correlation dimension, to cite only a few. Historically, these dimensions have been introduced in the frame of dynamical systems and ergodic theory and characterize essentially probabilty measures. Recenlty ([Hol94]), a new set of fractal dimensions has been constructed by means of wavelet transform, namely the *fractal wavelet dimensions*. These dimensions are well adapted to signed or complex measures. One of them, the so-called wavelet correlation dimension is particulary suited for scattering problems (see [GH96],[GSH97]). The wavelet correlation dimension is basically a generalization of the usual correlation dimension, so it is natural to introduce this latter first.

The correlation dimension. For a positive measure μ, one definition ([Pes93]) of the correlation dimension is the following. Consider the function :

$$\Omega_\mu(r) = \int \mu(dx)\ \mu\left(B(x, r)\right),$$

where $B(x, r)$ is the ball of radius r around the point x. The quantities:

$$D^+[\mu] = \limsup_{r\to 0} \frac{\log \Omega_\mu(r)}{\log r}, \qquad D^-[\mu] = \liminf_{r\to 0} \frac{\log \Omega_\mu(r)}{\log r},$$

are called respectively upper and lower correlation dimension of μ. When the upper and lower dimension coincide, their common value it simply called correlation dimension. For positive finite measures, the correlation dimension has been related to the mean quadratic Fourier asymptotics:

Théorème 4.1 ([GH97]) *For any finite positive measure μ on $\mathbb{R}$,*

$$\lim_{p\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \int_{|\xi|\leq p} d\xi\, |\widehat{\mu}(\xi)|^2}{\log \xi} = 1 - D^{\mp}[\mu]. \tag{4.1}$$

For signed or complex measures, the correlation dimensions are not sufficient to characterise the Fourier asymptotic. The relevant dimensions in this case are the upper and lower wavelet dimensions. Intuitively, the reason is the following. To compute the "classical" fractal dimensions, one sooner or later tests the measures with boxes, whose size becomes smaller and smaller. But it has no sense to test signed measures with boxes because cancellations may appear which corrupt the information. The idea of wavelet dimensions is to replace the boxes by functions which oscillate faster and faster. At this point, let us briefly recall some elementary facts about the wavelet analysis.

Wavelet analysis. The wavelet transform can be seen as a generalization of the Fourier transform; while the latter consists in decomposing a function in pure frequencies, the former allows a decomposition on a basis of wavelets, which are basically functions well localized both in position and frequency. The minimal requirement for a function g to be a wavelet is that it be in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and have zero mean, that is $\int g = 0$. Along these lines, we will adopt the following notational convention. For all function g, we write

$$g_a(x) = a^{-1}g(a^{-1}x),\ g_{b,a}(x) = a^{-1}g(a^{-1}(x-b)),\ \tilde{g}(x) = g(-x)$$

for the dilated, resp. translated and dilated, resp. symmetric version. The wavelet transform of a function or more generally a distribution s with respect to a wavelet g is given by

$$\mathcal{W}_g s(b,a) = \int dt\ a^{-1}\overline{g(a^{-1}(t-b))}s(t) = \int dt\overline{g_{b,a}}(t)s(t). \tag{4.2}$$

This expression can also be seen as a convolution,

$$\mathcal{W}_g s(b,a) = (\widetilde{g}_a * s)(b), \tag{4.3}$$

or in the Fourier space,

$$\mathcal{W}_g s(b,a) = \int dk\ \overline{\widehat{g}(ak)}e^{ibk}\widehat{s}(k) =< \widehat{g}_a, \widehat{s} > . \tag{4.4}$$

Intuitively, the wavelet transform acts as a filter selecting the details present in s at scale a and position b. It is therefore a tool to analyse the local regularity of functions. The accuracy of the wavelet analysis depends on the choice of the wavelet. Convenient classes of wavelets are the following. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of smooth functions which, together with their derivatives, decrease faster than any polynomial, that is

$$\max_{q\leq N} \sup_{x\in\mathbb{R}} |x^q g(x)| < \infty,\ N = 0, 12...$$

Let $\mathcal{S}_+(\mathbb{R})$ be the subspace of Schwartz functions with no negative frequencies (that is $\widehat{g}(\xi) = 0$ if $\xi \leq 0$), $\mathcal{S}_-(\mathbb{R})$ the subspace of Schwartz functions with no positive frequencies, and $\mathcal{S}_o(\mathbb{R}) = \mathcal{S}_+(\mathbb{R}) \oplus \mathcal{S}_-(\mathbb{R})$. Function in $\mathcal{S}_+(\mathbb{R})$, $\mathcal{S}_-(\mathbb{R})$ or $\mathcal{S}_o(\mathbb{R})$ have all moments vanishing, that is

$$\int dx\ x^n g(x) = 0,\ \ n = 0, 1, 2, ..,$$

or, what amounts to the same, have a flat Fourier transform near zero,

$$\widehat{g}(\xi) = O(\xi^\infty),\ \ \xi \to 0,$$

where the convenient notation "$O(\xi^\infty)$" means $O(\xi^n)$ for all n.

The wavelet correlation dimension. The wavelet correlation dimension is now constructed in the following way. Take some wavelet g in $\mathcal{S}_+(\mathbb{R})$ and set

$$G_g(a) = \|\mathcal{W}_g\mu(\cdot, a)\|^2 = \int d\omega\ \ |\widehat{g}(a\omega)|^2\, |\widehat{\mu}(\omega)|^2$$

and

$$\Gamma_g(a) = \min\left\{ \int_a^1 \frac{d\alpha}{\alpha} G_g(\alpha), \int_0^a \frac{d\alpha}{\alpha} G_g(\alpha) \right\}.$$

Then the (upper and lower) wavelet correlation dimensions $\kappa_2^\pm(\mu)$ are defined by

$$\kappa_2^+(\mu) = \limsup_{a\to 0} \frac{\log \Gamma_g(a)}{\log a}, \qquad \kappa_2^-(\mu) = \liminf_{a\to 0} \frac{\log \Gamma_g(a)}{\log a},$$

and do not depend on the choosen wavelet g in $\mathcal{S}_+(\mathbb{R})$ provided $g \neq 0$. Note that a wavelet in $\mathcal{S}_\pm(\mathbb{R})$ cannot have compact support (otherwise its Fourier transform would be analytic and hence could not vanish on a semi-axis without vanishing everywhere). However, admitting an arbitrarily small error on the dimensions $\kappa_2^\pm$, on always may suppose that the analysing wavelet is compactly supported. This is useful in the application.

The wavelet correlation dimension of a measure governs the L^2 asymptotic of its Fourier transform, as shows the following:

Théorème 4.2 ([Hol94]) *If μ is a finite complex measure, then either $\mu \notin L^2(\mathbb{R})$ and*

$$\lim_{p\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \int_0^p d\xi\, |\widehat{\mu}(\xi)|^2}{\log p} = -\kappa_2^\mp[\mu] \tag{4.5}$$

or $\mu \in L^2(\mathbb{R})$ and

$$\lim_{p\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \int_p^{+\infty} d\xi\, |\widehat{\mu}(\xi)|^2}{\log p} = -\kappa_2^\mp[\mu]. \tag{4.6}$$

If μ is real, one can take indiscriminately a wavelet g in $\mathcal{S}_o(\mathbb{R})$, $\mathcal{S}^+(\mathbb{R})$ or $\mathcal{S}^-(\mathbb{R})$.

Note that the classical correlation dimension and the wavelet correlation are essentially the same in case of positive measures. Precisely,

$$-\kappa_2^\pm[\mu] = 1 - D^\pm[\mu]$$

provided $\mu \notin L^2(\mathbb{R})$, which is always the case for non-trivial singular measures.

One could ask to which extend the (wavelet) correlation dimension is a relevant characterization of a fractal measure. Of course a single fractal dimension is not enough to describe a fractal measure. In general, an infinite set of so-called generalized fractal dimensions ([HP83]) is necessary to characterize a fractal measure (at least when this latter is strictly or statistically self-similar). However, having found a non-trivial correlation dimension (i.e $0 < D^{\pm} < 1$ or $-1 < \kappa_2^{\pm} < 0$) tells us that the analysed object has small-scales properties. In fact, the lower wavelet correlation dimension can be seen as an index of regularity. The higher is κ_2^-, the more regular is the measure. The two extreme cases are $\kappa_2^- = -1$ for point measures and $\kappa_2^- = +\infty$ for smooth functions. For arbitrary singular measures, $-1 \leq \kappa_2^- \leq 0$. The class $L^2(\mathbb{R})$ is characterized by $\kappa_2^- > 0$.

5 Recovering the correlation dimension

We shall now propose a method to retrieve the wavelet correlation dimension of the potential from the probabilities of presence $p^{\pm}[f]$ and $p^{\pm}[f, -f']$ by choosing an appropriate family of initial states. As usual, we begin with the quantum evolution.

5.1 Schrödinger equation

Suppose we can prepare some initial state f in $\mathcal{S}_+(\mathbb{R})$. Then

Proposition 5.1 *Let $f \in \mathcal{S}_+(\mathbb{R})$. For all $b \in \mathbb{R}$, we have*

$$p^-[f_{b,a}] = a^2 \left\|\mathcal{W}_g V(\cdot, a)\right\|^2 + O(a^2), \ a \to 0, \tag{5.1}$$

where the wavelet $g \in \mathcal{S}_+(\mathbb{R})$ is defined by $\widehat{g}(k) = (2k)^{-1}\widehat{f}(k/2)$.

Proof. Thanks to the Born approximation we have

$$\begin{aligned}
p^-[f_{b,a}] &= \int_{-\infty}^{0} dk\ \left|\overline{r_k^+}\right|^2 \left|\widehat{f_{b,a}}(-k)\right|^2 \\
&= \int_0^{+\infty} dk\ \left|r_k^+\right|^2 \left|\widehat{f_{b,a}}(k)\right|^2 \\
&= \int_0^{+\infty} dk\ \left|\widehat{V}(2k)\right|^2 (2k)^{-2} \left|\widehat{f_{b,a}}(k)\right|^2 + O\left(\int_0^{+\infty} dk\ k^{-3} \left|\widehat{f_{b,a}}(k)\right|^2\right) \\
&= a^2 \int_0^{+\infty} dk\ \left|\widehat{V}(k)\right|^2 \left|\widehat{g}(ak)\right|^2 + O(a^2) \\
&= a^2 \left\|\mathcal{W}_g V(\cdot, a)\right\|^2 + O(a^2),
\end{aligned}$$

which is the desired result. □

The influence of bound states. Up to now, we have supposed there were no bound states, that is $P_{ac} = 1$. In fact, the eventual presence of bound states does not affect the last result, as shows the next

Proposition 5.2 *Suppose $P_{ac} \neq 1$. Then, for all $f \in \mathcal{S}_+(\mathbb{R})$ and $b \notin supp(V)$,*

$$p^-[f_{b,a}] = p^-[P_{ac} f_{b,a}] + O(a^{\infty}) = a^2 \left\|\mathcal{W}_g V(\cdot, a)\right\|^2 + O(a^2). \tag{5.2}$$

Proof. The theorem (3.1) still applies to any initial state in $\mathcal{H}_{ac}$. Hence,

$$p^-[P_{ac}f_{b,a}] = \int_{-\infty}^{0} dk \left| \overline{r_k^+} (P_{ac}f_{b,a})^\wedge(-k) + \overline{t_k}(P_{ac}f_{b,a})^\wedge(k) \right|^2 \tag{5.3}$$

Now, take an orthonormal basis $(\eta_1, ..\eta_n)$ of the space of bound states and call P_{bs} the orthogonal projector on this basis. The states $f_{b,a}$ can be decomposed as

$$f_{b,a} = P_{ac}f_{b,a} + P_{bs}f_{b,a} = P_{ac}f_{b,a} + \sum_{j=1}^{n} < f_{b,a}, \eta_j > \eta_j.$$

The scalar products $< f_{b,a}, \eta_j >$ are nothing else than wavelet transforms of the eigenvectors η_j,

$$< f_{b,a}, \eta_j >= \mathcal{W}_f \eta_j(b, a).$$

Now, outside of the support of V, the η_j are smooth and rapidly decreasing because they are merely exponential functions (see equation (2.10)). Hence the wavelet coefficients in this region are rapidly decreasing at small scales (see e.g [Hol95]), that is, for fixed b,

$$|< f_{b,a}, \eta_j >| \leq O(a^\infty), \tag{5.4}$$

Using (5.3) and (5.4), we obtain after some elementary estimates

$$p^-[P_{ac}f_{b,a}] = \int_{0}^{+\infty} dk \left|r_k^+\right|^2 \left|\widehat{f_{b,a}}(k)\right|^2 + O(a^\infty),$$

and we end the proof as in proposition (5.1) □

A straightforward consequence of propositions (5.1) and (5.2) is

Corrolaire 5.3 *Define*

$$\rho(b, a) = \int_a^1 \frac{d\alpha}{\alpha^3} \; p^-[f_{b,\alpha}].$$

If $P_{ac} \neq 1$, then for all $b \notin supp(V)$,

$$\lim_{a \to 0} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \rho(b, a)}{\log a} = \min\{\kappa_2^\pm[V], 0\}.$$

If $P_{ac} = 1$, this holds true for all $b \in \mathbb{R}$.

Thus, the behaviour of $p^-[f]$ as one varies the characteristic scale of the initial state f is governed by the wavelet correlation dimension of the potential, provided this latter is sufficiently singular (at least non square integrable). Analogous results can be stated for $p^+[f]$ in place of $p^-[f]$. It suffices to take a wavelet in $\mathcal{S}_-(\mathbb{R})$ (no positive frequencies) instead of $\mathcal{S}_+(\mathbb{R})$.

Remark. The assumption that the initial states belong to $\mathcal{S}_\pm(\mathbb{R})$ is somewhat unrealistic. Indeed, such functions cannot have compact support, otherwise their Fourier transform would be analytic and hence could not vanish on a semi-axis without vanishing everywhere; so these states have to be prepared and controlled not only far from the target, but also in the region of interaction, which is practically impossible. However, the functions $f_{b,a}$ in $\mathcal{S}_\pm(\mathbb{R})$ are rapidly decreasing. So if they are localized remote from the potential region, only their tail interacts with the scatterer with a contribution of order $O(a^\infty)$. Hence, we can prepare initial states with compact support remote from the potential and apply propositions (5.1) and (5.2) as if they were in $\mathcal{S}_\pm(\mathbb{R})$.

5.2 Wave equation

We now adapt the previous method to the case of the wave equation.

Proposition 5.4 *Let $f \in \mathcal{S}_0(\mathbb{R})$ with support in Ω^-. Then for all $b \in \mathbb{R}$,*

$$p^-[f_{b,a}, -(f_{b.a})'] = \frac{1}{4}\,\|\mathcal{W}_f V(\cdot, a/2)\|^2 + O(1),\ \ a \to 0. \tag{5.1}$$

As a result we have

$$\lim_{a\to 0}\ \begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\log \rho(b,a)}{\log a} = \min\{\kappa_2^{\pm}[V], 0\}.$$

with

$$\rho(b,a) = \int_a^1 \frac{d\alpha}{\alpha} p^-[f_{\alpha,b}, -(f_{\alpha,b})'].$$

Proof. The proof is analogous to the proof of proposition (5.1). □

6 An experimental wavelet transform

We wish now to derive further results in the case of the wave equation relating the wave function Φ_t to the wavelet transform of the potential. Take $f \in \mathcal{W}^1(\mathbb{R})$ with compact support in Ω^- and such that $\left|\widehat{f}(k)\right| \leq O(k^{-2}),\ k \to 0$, and let us adopt the notation

$$\Phi_t(x,a) = \cos(H^{1/2}t) f_a - H^{-1/2}\sin(H^{1/2}t)(f_a)'. \tag{6.1}$$

By proposition (3.4) we have for every x

$$\chi_{\Omega^-}\Phi_t(x,a) = \chi_{\Omega^-}(x)[f_a(x-t) + A_3 f_a(x+t)]$$

If we introduce F the inverse Fourier transform of r^+_{-k}:

$$\widehat{F}(k) = r^+_{-k} = i\frac{\widehat{V}(2k)}{2k} + O(k^{-2}), \tag{6.2}$$

we obtain, for all a, b and for t large enough,

$$\begin{aligned}\Phi_t(b-t,a) &= F * \overline{\tilde{f}}_a(b)\\ &= \mathcal{W}_{\overline{\tilde{f}}}F(b,a)\\ &= \mathcal{W}_h V(b/2,a) + O(a),\end{aligned} \tag{6.3}$$

where $\widehat{h}(k) = -(2ik)^{-1}\widehat{f}(-k/2)$. Thus, the wave function is constant along the trajectories $x(t) = b-t$. Its value there can be interpreted, upon the Born error term, as a wavelet coefficient of the potential, where the time is the position parameter and the inverse characteristic frequency of the initial state is the scale parameter. The error term in the wavelet coefficient is of order $O(a)$ and thus becomes negligible at some small scales if V is sufficiently singular. Thus, analysing the long-time behaviour of the wave function for initial states with increasing central frequency makes possible to detect the local singularities of the potential.

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