Two-particle scattering theory for anyons

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Abstract

We consider potential scattering theory of a nonrelativistic quantum mechanical 2-particle system in $\mathbb{R}^2$ with anyon statistics. Sufficient conditions are given which guarantee the existence of Møller operators and the unitarity of the S–matrix. As examples the rotationally invariant potential well and the δ–function potential are discussed in detail. In case of a general rotationally invariant potential the angular momentum decomposition leads to a theory of Jost functions. The anyon statistics parameter gives rise to an interpolation for angular momenta analogous to the Regge trajectories for complex angular momenta. Levinson's theorem is adapted to the present context. In particular we find that in case of a zero energy resonance the statistics parameter can be determined from the scattering phase.

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I. INTRODUCTION

In recent years the theory of identical quantum mechanical particles with braid group statistics has received increasing attention (for a recent review and references on anyon physics see e.g. 35, 39). The original observation that in two dimensional configuration space $\mathbb{R}^2$ identical particles may obey statistics different from Bose or Fermi statistics is due to J.M. Leinaas and J. Myrheim[34]. They based the notion of quantum statistics on the topological structure of the classical configuration space for identical particles. The relevant symmetry group is then shown to be the braid group which replaces the permutation group. Models for particles with a one–dimensional representation of the braid group were first discussed by F. Wilczek, who coined the name anyons for particles with these new statistics[57] (see also 20, 22, 21). Wilczek suggested the following physical picture of anyons: Magnetic flux tubes (vortices) are attached to either charged bosons or fermions. The latter then give rise to arbitrary Aharonov–Bohm phases when transported along paths exchanging the particle positions. The magnetic flux tubes are described by long range gauge potentials whose curvature vanishes and can be related to Chern–Simons theory 56, 57. This point of view was taken up by several authors (e.g. 7, 23) leading to a second quantized version of anyons obtained by coupling a Chern–Simons U(1)–gauge potential to a matter field[27].
Also the general case, where the finite dimensional representation of the braid group is not one–dimensional, has been considered. The corresponding particles are called "plektons"[14,15]. The relevance of braid group statistics in conformal quantum field theory was realized by Tsuchiya and Kanie[53] and in algebraic quantum field theory by J. Fröhlich[16] and K. Fredenhagen, K.–H. Rehren and B. Schroer[14,15].
However, in the discussion of particles with braid group statistics the main focus is on anyons (abelian statistics), since in many particle theory they might provide an explanation for the fractional quantum Hall effect and for high T_c superconductivity (for a review see e.g. 45, 50 and 58, 36). In this article we want do discuss non–relativistic two–particle potential scattering theory for anyons. This is done in the framework of ordinary quantum mechanics, i.e. we insert a gauge potential of the Aharonov–Bohm type in the center of mass Hamiltonian. Emphasis is then put on showing how well known techniques of scattering theory extend to the case of anyon statistics. The motivation is twofold. On the one hand scattering theory is a powerful tool in spectral analysis and thus might be helpful for a better understanding of fractional statistics. On the other hand scattering data can be used to compute the virial coefficients of an interacting anyon gas[9]. Hence, it is possible to infer bulk properties from scattering theory. The latter are of main interest in the investigation of the above mentioned phenomena.
The first calculation of the second virial coefficient for the two–particle anyon system was done by Arovas et al.[7]. They considered the case when only the statistics is present corresponding to the case of Aharonov–Bohm scattering[1,48]. See 30 and references therein for recent articles on the second virial coefficient of *interacting* anyon systems. In a forthcoming publication we intend to apply the results of this article to the calculation of the second virial coefficient.

The article is organized as follows. In section II we review the quantum mechanics of two "free" anyons in order to introduce our notation and to keep the paper self–contained.

In particular, we give the energy eigenfunctions, the resolvent (Green's function), and the propagator. We also recapitulate the relation of the free anyon system and Aharonov–Bohm scattering which will be used in our discussion of the differential cross–section in section IV. In appendix A we recall the equivalent differential geometric formulation in terms of vector bundles, which in the case of anyons are line bundles. In particular, we recall that there exists a canonical hermitian connection encoding the statistics such that the "free" Hamiltonian is the canonically associated Bochner Laplacean.
In section III we consider scattering theory for the interacting anyon system obtained by adding a potential to the center of mass Hamiltonian. We give sufficient conditions for the existence of the Møller operators, which also cover the non spherical symmetric case. Applying the Kuroda–Birman theorem[32] we derive the unitarity of the resulting S–matrix.
In section IV we discuss the differential cross–section with the modifications necessary to accommodate anyon statistics. We show that the scattering amplitude splits into two parts, one describing the effects of the statistics the other the interaction represented by the potential.
In section V Jost functions are introduced which depend on the statistics parameter for anyons. The latter enters in the form of continuous angular momentum. This establishes a connection with Regge trajectories in the theory of complex angular momenta. We conclude section V by showing how Levinson's theorem, which relates the scattering phase shift to the number of bound states, carries over to the present situation. In case that the Jost function vanishes at zero energy we derive an explicit formula giving the statistics parameter in terms of the scattering phase shift.
Section VI and VII are devoted to explicitly solvable examples. In section VI we examine the δ-potential. This case also figures under the name of anyons without hard–core condition. The corresponding resolvent is calculated in closed form in appendix D and the bound state problem is then considered. We also remark on the modification of Levinson's theorem and find an additional formula relating the statistics parameter to the scattering phase. In section VII we discuss the square well potential. The Jost function is calculated and Regge trajectories are plotted which show the dependence of the point spectrum on angular momentum and the statistics parameter. In section VI and VII we provide numerical examples for the differential cross–section which display the interpolation between Bose and Fermi statistics when the statistics parameter for anyons is varied.

The results presented here are based in part on the diploma thesis of two of the authors (C.K.[31] and G.L.[33]).

Throughout the article we will work in atomic units, $\hbar = e = m = 1$, where m denotes the mass of the particles. In particular this sets the reduced mass of a two–particle system equal to 1/2. In estimates we make the convention that $C, C(\epsilon)$ etc. denote generic constants depending on ϵ etc.

II. QUANTUM MECHANICS OF TWO ANYONS

The theory of identical particles with statistics differing from Bose or Fermi statistics may show up when the configuration (or momentum) space of one particle is the two dimensional

Euclidean space. Henceforth we will often use the complex plane $\mathbb{C}$ to describe such a space. For the configuration space $\mathbb{C} \times \mathbb{C}$ of two non identical particles with points labeled by (z_1, z_2) the relative coordinate $z = z_1 - z_2$ changes into $-z$ if the coordinates z_1 and z_2 of the two particles are interchanged. The basic observation of Leinaas and Myrheim was, that by leaving out the case where the two particles are at the same point, i.e. where $z = 0$, the configuration space in the center of mass frame of two identical particles should be the space obtained from $\mathbb{C}^\star = \mathbb{C} \setminus \{0\}$ by identifying the points z and $-z$. This space is therefore the orbit space $\mathbb{C}^\star/\mathbb{Z}_2$, where $\mathbb{Z}_2 = \{+1, -1\}$ acts in the obvious way as a transformation group on $\mathbb{C}^\star$. This space is also obtained from the closed upper half–plane $\mathbb{H}$ in $\mathbb{C}$ minus the origin, i.e the set $\mathbb{H} \setminus \{0\}$, by identifying the points x and $-x$ on the real axis. Geometrically this leads to a cone with removed apex as configuration space. The obvious choice of polar coordinates $(r, \theta) \in \mathbb{R}^+ \times [0, \pi)$ on $\mathbb{H}$ carries over to the cone $\mathbb{C}^\star/\mathbb{Z}_2$. We take the state vectors of the system to be the square integrable functions on the cone $\mathcal{L}^2(\mathbb{C}^\star/\mathbb{Z}_2)$ or equivalently the π–periodic functions on the punctured plane $\mathbb{C}^\star$. The fact that the cone's apex or respectively the plane's origin is removed, allows for the particles to carry flux–tubes. This corresponds to the physical picture introduced by F. Wilczek (see e.g. 57, 58), who named such particles *anyons*. The flux–tubes are taken into account by inserting a gauge potential of Aharonov–Bohm type

$$A_\alpha = \frac{\alpha}{r}\, e_\theta, \qquad \alpha \in [0, 1] \tag{II.1}$$

into the center of mass Hamiltonian. Here α is the so called statistics parameter and e_θ denotes the unit vector corresponding to the polar angle. Choosing units in such a way that $\hbar = 1$ and setting the mass of the particles to one, the resulting Hamiltonian has the form

$$H_0(\alpha) = -(\nabla + iA_\alpha)^2 = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\left(\frac{\partial}{\partial \theta} + i\alpha\right)^2$$

in polar coordinates. This operator will be considered to be the free Hamiltonian in the center of mass frame for a two–particle anyon system with statistics parameter $\alpha \in [0, 1]$. If $\alpha = 0$ the particles actually behave like bosons, while for $\alpha = 1$ they behave like fermions. In appendix A we give a short review of a mathematically precise formulation of this model in terms of vector bundles, following 40, 41 (see also 25, 26. There we also argue why $\alpha = 0, 1$ corresponds to bosons and fermions respectively and why we may restrict the parameter α to the interval $[0, 1]$.

We now determine the spectrum and the eigenfunctions of $H_0(\alpha)$. We start with the decomposition

$$\mathcal{L}^2(\mathbb{C}^\star/\mathbb{Z}_2) \cong \mathcal{L}^2(\mathbb{R}^+, r\, dr) \otimes \mathcal{L}^2(S^1, d\theta)\ ,$$

where the points in S^1, the unit circle in $\mathbb{C}$, are parameterized as $\exp(2i\theta)$ with $0 \leq \theta < \pi$. This leads to the decomposition

$$\mathcal{L}^2(\mathbb{C}^\star/\mathbb{Z}_2) \cong \bigoplus_{m \in \mathbb{Z}} \mathfrak{h}_{2m}\ , \qquad \mathfrak{h}_{2m} = \mathcal{L}^2(\mathbb{R}^+, r dr) \otimes \left\{ \frac{e^{2im\theta}}{\sqrt{\pi}} \right\}. \tag{II.2}$$

This decomposition is of course related to the following fact. The rotation group $SO(2) \cong U(1)$ maps $\mathbb{C}^\star$ into itself and commutes with the action of $\mathbb{Z}_2$, thus acts on $\mathbb{C}^\star/\mathbb{Z}_2$ and defines a unitary action on $L^2(\mathbb{C}^\star/\mathbb{Z}_2)$. Also $H_0(\alpha)$ commutes with this action of $\mathrm{U}(1)$, it is diagonal w.r.t. the decomposition, i.e.

$$H_0(\alpha) = \underset{m\in\mathbb{Z}}{\oplus} H_{0,2m}(\alpha)$$

with

$$H_{0,2m}(\alpha) \;=\; -\left\{\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(2m+\alpha)^2}{r^2}\right\}$$

on $\mathcal{L}^2(\mathbb{R}^+, r\,dr) \;\cong\; \mathfrak{h}_{2m}$. To find all solutions of the stationary Schrödinger equation for $H_0(\alpha)$ it therefore suffices to find the solutions of the Bessel equation for each $m \in \mathbb{Z}$

$$\left\{\frac{d^2}{dr} + \frac{1}{r}\frac{d}{dr} - \frac{(2m+\alpha)^2}{r^2} + E\right\} R(r) = 0 \;. \tag{II.3}$$

For definiteness we choose those solutions of equations (II.3) which are regular near $r = 0$. This gives the following improper eigenfunctions

$$\phi_{\alpha;m,E}(r,\theta) = \frac{e^{2im\theta}}{\sqrt{2\pi}} \cdot J_{|2m+\alpha|}(\sqrt{E}r) \;.$$

In what follows we will use the notations

$$k := \sqrt{E}, \qquad \mu := \mid 2m+\alpha \mid \;.$$

The identity

$$\int_0^\infty J_\mu(at) J_\mu(bt)\, t dt = \frac{1}{\sqrt{ab}}\,\delta(a-b)\;, \qquad \text{for } a,b>0$$

gives the following orthogonality and completeness relations

$$\langle \phi_{\alpha;m,E} \mid \phi_{\alpha;m',E'}\rangle \;=\; \delta_{m,m'}\,\delta(E-E') \tag{II.4}$$

$$\sum_{m=-\infty}^{+\infty} \int_0^\infty dE\, \phi_{\alpha;m,E}(r,\theta)\,\overline{\phi_{\alpha;m,E}(r',E')} = \frac{1}{r}\,\delta(r-r')\,\delta(\theta-\theta') \;. \tag{II.5}$$

We will also need the integral kernel for the resolvent $R_{0,\alpha}(z) := (H_0(\alpha)-z)^{-1}$ also called Green's function, given by

$$\langle r,\theta \mid R_{0,\alpha}(k \pm i\varepsilon) \mid r',\theta'\rangle = \pm\frac{i}{2}\sum_{m=-\infty}^{+\infty} e^{2im(\theta-\theta')} J_\mu(kr_<)\, H_\mu^\pm(kr_>) \;. \tag{II.6}$$

Here we have used the standard convention $r_> := \max(r,r')$ and $r_< := \min(r,r')$. To abbreviate notation we will sometimes make the convention that $H_\mu^\pm$ denotes the first and second Hankel function $H_\mu^{(1)}$ and $H_\mu^{(2)}$ respectively. I_μ is the modified Bessel function and

K_μ is the MacDonald function (see e.g. 54). To establish (II.6) one uses the well–known formula (see e.g. 37).

$$\int_0^\infty k\,dk\,\frac{J_\mu(kr)\,J_\mu(kr')}{k^2+c^2} = I_\mu(cr_<)\,K_\mu(cr_>) \quad [\mathrm{Re}\,\mu > -1, \mathrm{Re}\,c > 0].$$

Finally we give the kernel of of the unitary time evolution $\exp\left(-itH_0(\alpha)\right)$:

$$\mathfrak{K}_\alpha(r,\theta;r',\theta';t) := \left\langle r,\theta \;\middle|\; e^{-itH_0(\alpha)} \;\middle|\; r',\theta' \right\rangle \tag{II.7}$$

We can make this explicit by adapting a calculation, which is quite standard in the context of the Aharonov–Bohm effect[1]. Along the lines of 4, 19 we obtain the following result:

$$\mathfrak{K}_\alpha(r,\theta;r',\theta';t) = \mathfrak{K}_{\alpha,0}(r,\theta;r',\theta';t) + \hat{\mathfrak{K}}_\alpha(r,\theta;r',\theta';t)\ . \tag{II.8}$$

Here the two subexpressions $\mathfrak{K}_{\alpha,0}$ and $\hat{\mathfrak{K}}_\alpha$ are given as

$$\begin{aligned}
\mathfrak{K}_{\alpha,0}(r,\theta;r',\theta';t) =& \frac{1}{2\pi it}\, e^{-\frac{1}{4it}(r^2+r'^2)}\, e^{i\alpha(\theta-\theta'-\frac{\pi}{2}\,\mathrm{sgn}(\theta-\theta'))} \times \qquad \text{(II.9)}\\
&\times \cos\left(-\frac{\alpha\pi}{2}\,\mathrm{sgn}(\theta-\theta') + \frac{rr'}{2t}\cos(\theta-\theta')\right)\\
\hat{\mathfrak{K}}_\alpha(r,\theta;r',\theta';t) =& \frac{i}{2\pi t}\cdot\frac{\sin(\pi\alpha)}{\pi}\, e^{-\frac{1}{4it}(r^2+r'^2)} \times I_\alpha(\tfrac{rr'}{2t},\theta-\theta')\ , \qquad \text{(II.10)}
\end{aligned}$$

where we have introduced

$$I_\alpha(\rho,\chi) = \int_{-\infty}^{+\infty} dy\, e^{i\rho\cosh y}\,\frac{e^{-y\alpha}}{1-e^{-2y-2i\chi}}\ . \tag{II.11}$$

The formal relation to the Aharonov–Bohm effect used in the derivation of (II.9) and (II.10) does not come by accident. In fact, the Hamiltonian of the two–anyon system coincides with the Hamiltonian of the Aharonov–Bohm effect when restricted to the subspace of symmetric wave functions. As has been realized before (see e.g. 58) one can exploit this by describing the anyonic dynamics with the help of Aharonov–Bohm scattering and thus demonstrating the non–trivial character of the statistical interaction. The description of the Aharonov–Bohm effect in terms of scattering theory was first given by Aharonov and Bohm themselves and later taken up by several authors, among others 24, 48, 49. We will mostly follow the discussion presented in 48 because there the time–dependent as well as the time-independent scattering formalism are considered. The wave operators and the scattering operator of the Aharonov–Bohm effect are formally defined by

$$\Omega^\pm_{AB} := \operatorname*{s–lim}_{t\to\pm\infty} e^{itH_0(\alpha)}e^{-itH_0} \qquad \text{and} \qquad S_{AB} := (\Omega^+_{AB})^*\Omega^-_{AB} \tag{II.12}$$

respectively. Here and henceforth H_0 denotes the bosonic Hamiltonian $H_0(\alpha=0)$ and the symbol s–lim stands for the strong operator limit. Furthermore, by writing (II.12) we have implied the restriction of the Aharonov–Bohm scattering to the subspace of symmetric functions or equivalently to the space $\mathcal{L}^2(\mathbb{C}^\star/\mathbb{Z}_2)$. It has been shown[48] that the wave operators exist and are complete, whence the scattering operator is unitary. For later use we give the explicit form of the integral kernels of $\Omega^{\pm}_{AB}$, i.e. the stationary scattering states, which can be obtained by symmetrizing the results given in 48, 49,

$$\langle r,\theta \mid \Omega^{\pm}_{AB} \mid k,\theta'\rangle = \frac{1}{\pi}\sum_{m\in\mathbb{Z}}(\pm i)^{|2m+\alpha|}J_{|2m+\alpha|}(kr)e^{i\,2m(\theta-\theta')}, \tag{II.13}$$

where $|k,\theta'\rangle$ denotes the symmetric plane wave, i.e.

$$\langle r,\theta \mid k,\theta'\rangle = \tfrac{1}{\pi}\cos(kr\cos(\theta-\theta'))\ .$$

The scattering phase shift corresponding to S_{AB} was first derived by Henneberger[24]. Using his result we can rewrite the scattering operator in the compact form

$$S_{AB} = e^{i\pi\alpha}P_{\geq} + e^{-i\pi\alpha}P_{<} \tag{II.14}$$

with $P_{\geq}$ and $P_{<}$ denoting the spectral projections onto the subspaces of positive respectively negative angular momentum. Equation (II.14) serves physical intuition by clarifying the effect of the gauge potential A_α defined in (II.1).
However, of practical importance is the integral kernel of the S–matrix in momentum space which was first given in 48. After symmetrization we obtain for $\alpha\in[0,1]$

$$\langle k,\theta \mid S_{AB}\mid k',\theta'\rangle = 4\delta(k^2-k'^2)\left[\cos\pi\alpha\,\delta(\Theta)+\frac{\sin\pi\alpha}{2\pi i}\,PV\,(1+i\cot\Theta)\right]. \tag{II.15}$$

Here, $\Theta=\theta-\theta'$ and PV stands for the principal value prescription. From (II.15) one immediately derives the scattering amplitude

$$\begin{aligned} f_{AB}(k,\Theta) &= \left(\tfrac{\pi}{ik}\right)^{\frac{1}{2}}\langle\theta\mid T_{AB}(E=k^2)\mid\theta'\rangle \\ &= \left(\tfrac{\pi}{ik}\right)^{\frac{1}{2}}2\left[(\cos\pi\alpha-1)\,\delta(\Theta)+\frac{\sin\pi\alpha}{2\pi i}\,PV\,(1+i\cot\Theta)\right], \end{aligned} \tag{II.16}$$

where $\langle\theta\mid T_{AB}(E)\mid\theta'\rangle$ denotes the on-shell matrix element of $T_{AB}:=S_{AB}-\mathbf{1}$ defined by the equation

$$\langle k,\theta\mid T_{AB}\mid k',\theta'\rangle = 2\delta(E-E')\,\langle\theta\mid T_{AB}(E)\mid\theta'\rangle$$

with $E=k^2$ and $E'=k'^2$. Note, that equation (II.16) differs from what one would get when symmetrizing the amplitude in the way calculated by Aharonov and Bohm[1]. In their result the contribution of the δ–function was left out. This violates the unitarity of the S–matrix as was pointed out by Ruijsenaars[48]. Thus, away from the forward direction we end up with the following expression for the differential cross–section

$$\begin{aligned} \frac{d\sigma_{AB}}{d\theta} &= |f_{AB}(k,\theta)|^2 \\ &= \frac{\sin^2\pi\alpha}{\pi k}\left(1+\cot^2\theta\right), \quad \theta\neq 0, \end{aligned} \tag{II.17}$$

which has a nontrivial angular dependence. Note that the total cross–section is infinite if $\alpha\notin\mathbb{Z}$ because of the singular contribution in the forward direction as displayed in (II.16). The latter has been interpreted to be characteristic of anyon statistics reflecting the long range nature of the statistical interaction[58,p23]. For $\alpha=0,1$, i.e. for bosons and fermions, the cross section vanishes. This is obvious from equation (II.14) which shows that bosons are not scattered at all while fermions pick up a factor minus one.

III. POTENTIAL SCATTERING

In this section we discuss two–particle potential scattering for anyons. Time dependent scattering theory involves the comparison between two dynamics, one of which is considered to be "free" in an appropriate sense. Here, we shall consider the "free" dynamics to be the one defined by $H_0(\alpha)$. This appears to be a natural choice, if one views statistics to be an inherent property of the particles. This is a generalization of the fermionic picture, where usually the free time evolution acts on the space of *antisymmetric* functions. Note, however, that this "free" dynamics contains an interaction given by the long range gauge forces encoding the statistics. In particular, non–integer values of α lead to a highly non–trivial "free" time evolution similar to the Aharonov–Bohm effect (see our discussion at the end of section II). As a consequence not all methods used in potential scattering theory may be applied. For example, we do not know how two adopt Enss'[13] method, which makes use of Fourier transformation and is therefore not suitable in the present context. Thus, in order to prove the existence of the wave operators and the unitarity of the S–matrix, we will rely on Cook's method (see e.g. 46) and the Kuroda–Birman theorem[32] respectively.

Following our discussion in section II we will formulate the two particle scattering theory in the center of mass system using the Hilbert space $\mathcal{L}^2(\mathbb{C}^\star/\mathbb{Z}_2)$ with $H_0(\alpha) = -\Delta_\alpha$ being the free Hamiltonian. Let V, the potential, be a measurable function on $\mathbb{C}^\star/\mathbb{Z}_2$. For example V may result from a function, also denoted by $V(z)$, on $\mathbb{C}$ with $V(-z) = V(z)$ and which acts as a multiplication operator. We set

$$H(\alpha) := H_0(\alpha) + V = -\Delta_\alpha + V$$

as an operator on $\mathcal{L}^2(\mathbb{C}^\star/\mathbb{Z}_2)$. At this point we are not concerned with giving a criterium for self–adjointness of $H_0(\alpha)$; below we specify a certain class of *spherically symmetric* potentials for which this operator is self–adjoint. The wave operators $\Omega^\pm_\alpha$ for the pair $(H(\alpha), H_0(\alpha))$ are defined as follows

$$\Omega^\pm_\alpha := \operatorname*{s\text{–}lim}_{t\to\pm\infty} e^{itH(\alpha)}\, e^{-itH_0(\alpha)}$$

provided the strong operator limit, denoted by s–lim, exists. Note that in the present context the absolute continuous spectrum of $H_0(\alpha)$ is the positive real axis including the origin and the associated space is all of $\mathcal{L}^2(\mathbb{C}^\star/\mathbb{Z}_2)$. The S–matrix is then defined as

$$S_\alpha := (\Omega^+_\alpha)^* \Omega^-_\alpha$$

Now we have the

Theorem III.1. *Let $(1{+}r)V$ be in $\mathcal{L}^2(\mathbb{C}^\star/\mathbb{Z}_2)$. Then the wave operators $\Omega^\pm_\alpha$ for the pair $(H(\alpha), H_0(\alpha))$ exist for all α.*

According to Cook's theorem (see e.g. 46) it suffices to prove the following lemma.

Lemma III.1. *Under the conditions on V stated in the Theorem one has*

$$\int_t^\infty \left\| V\, e^{-isH_0(\alpha)}\, \phi \right\| ds < \infty$$

for a dense set of ϕ's and for some $t = t(\phi) < \infty$.

We will provide two proofs of this lemma. The first one will use the explicit form of the integral kernel of $e^{-itH_0(\alpha)}$ given in expressions (II.9), (II.10) and (II.11). The second proof will make use of the asymptotic form of Bessel functions near the origin and at infinity.

1. Proof of lemma III.1: It suffices to consider the case $\alpha \in (0,1)$ since the cases $\alpha = 0$ and $\alpha = 1$ are the already well known bosonic and fermionic situation respectively. We will choose ϕ to be of the form $r\phi(r,\theta) = \frac{\partial}{\partial r}\Phi(r,\theta)$ with $\Phi \in C_0^\infty(\mathbb{C}^\star/\mathbb{Z}_2)$. It is easy to see that such ϕ's form a dense set.
Using (II.8), we write

$$\left|(e^{-itH(\alpha)}\phi)(r,\theta)\right| \;\leq\; F_{\alpha,0}(r,\theta;t) + \hat{F}_\alpha(r,\theta;t)$$

with

$$F_{\alpha,0}(r,\theta;t) \;=\; \left|\int_0^\infty \int_0^\pi \mathfrak{K}_{\alpha,0}(r,\theta;r',\theta';t)\,\phi(r',\theta')\,r'dr'd\theta'\right|$$

$$\hat{F}_\alpha(r,\theta;t) \;=\; \left|\int_0^\infty \int_0^\pi \hat{\mathfrak{K}}_\alpha(r,\theta;r',\theta';t)\,\phi(r',\theta')\,r'dr'd\theta'\right|.$$

We start with an estimate of $F_{\alpha,0}$. Partial integration w.r.t r' gives

$$F_{\alpha,0}(r,\theta;t) \;\leq\; \frac{1}{t^2}\int_0^\infty \int_0^\pi G_0(r,\theta;r',\theta';t)\,|\Phi(r',\theta')|\;dr'd\theta'$$

where G_0 satisfies an estimate of the form

$$0 \leq\; G_0(r,\theta;r',\theta';t) \;\leq\; (1+r)\,C$$

uniformly in $r,\theta,t,\alpha \in (0,1)$ and (r',θ') in the support of Φ. This gives

$$F_{\alpha,0}(r,\theta;t) \;\leq\; C\,\frac{(1+r)}{t^2}$$

and hence

$$\int_0^\infty \int_0^\pi \left|V(r,\theta)\,F_{\alpha,0}(r,\theta;t)\right|^2 r\,dr\,d\theta \leq \frac{C}{t^4}\int_0^\infty \int_0^\pi \left|(1+r)\,V(r,\theta)\right|^2 r\,dr\,d\theta \tag{III.1}$$

uniformly in $\alpha \in (0,1)$. We turn to an estimate of $\hat{F}_\alpha$. By (II.10)

$$0 \leq \hat{F}_\alpha(r,\theta;t) \leq \frac{C}{t}\cdot\left|\int_0^\infty \int_0^\pi e^{-\frac{1}{4it}(r^2+r'^2)}\,I_\alpha(\rho,\chi)\,\phi(r',\theta')\,r\,dr'\,d\theta'\right|$$

with the notation

$$\rho = \frac{rr'}{2t}, \quad \chi = \theta - \theta'.$$

Adding and subtracting $I_\alpha(0,\chi)$ gives

$$
\begin{aligned}
0 &\le \hat{F}_\alpha(r,\theta;t) && \text{(III.2)}\\
&\le \frac{C}{t}\int_0^\infty\int_0^\pi |I_\alpha(\rho,\chi)-I_\alpha(0,\chi)|\;|\phi(r',\theta')|\;r'\,dr'\,d\theta' \\
&+\frac{C}{t}\left|\int_0^\infty\int_0^\pi e^{-\frac{1}{4it}(r^2+r'^2)}\,I_\alpha(0,\chi)\,\frac{\partial}{\partial r'}\Phi(r',\theta')\,dr'd\theta'\right| \;. && \text{(III.3)}
\end{aligned}
$$

We need the following lemma which will be proved in Appendix B.

Lemma III.2. *The quantity $I_\alpha(\rho,\chi)$ satisfies the estimates*

$$
\begin{aligned}
|I_\alpha(\rho,\chi)| &\le C\\
|I_\alpha(\rho,\chi)-I_\alpha(0,\chi)| &\le C(\epsilon)\,\rho^\epsilon
\end{aligned}
$$

uniformly in ρ and χ for all $0\le\epsilon<\min(\alpha,1-\alpha)$.

We use this lemma combined with a partial integration w.r.t. r' in the second term of the r.h.s of (III.2) to obtain

$$
\left|\hat{F}_\alpha(r,\theta;t)\right| \;\le\; C(\epsilon)\left(\frac{r^\epsilon}{t^{1+\epsilon}}+\frac{1}{t^2}\right)\;.
$$

Therefore, since $r^\epsilon\;\le 1+r$ we finally arrive at

$$
\begin{aligned}
&\int_0^\infty\int_0^\pi |V(r,\theta)|^2\left|\hat{F}_\alpha(r,\theta;t)\right|^2 r\,dr\,d\theta \le \\
&\qquad C(\epsilon)\left(\frac{1}{t^2}+\frac{1}{t^{1+\epsilon}}\right)^2\int_0^\infty\int_0^\pi(1+r)^2\,|V(r,\theta)|^2 r\,dr\,d\theta. \qquad \text{(III.4)}
\end{aligned}
$$

Combining (III.1) and (III.4) shows that $\left\|Ve^{-itH_0(\alpha)}\right\|$ is integrable in t on the interval $[1,\infty)$ say, concluding the 1. Proof of lemma III.1. □

2. Proof of lemma III.1: For fixed and given $\alpha\in(0,1)$ we use the spectral decomposition of $H_0(\alpha)$ to obtain an α-dependent unitary equivalence

$$
\hat{V}_\alpha:\;\mathcal{L}^2(\mathbb{C}^\star/\mathbb{Z}_2)\;\to\;\hat{\mathcal{L}}=\bigoplus_{m\in\mathbb{Z}}\mathcal{L}^2(\mathbb{R}^+,dE) \qquad \text{(III.5)}
$$

defined by

$$
\psi\longmapsto\hat{\psi}\equiv\{\hat{\psi}_m\}_{m\in\mathbb{Z}} \qquad , \qquad \hat{\psi}_m(E)\;=\langle\phi_{\alpha;m,E}\mid\psi\rangle\;\;,
$$

such that $H_0(\alpha)$ just turns into a multiplication operator $\widehat{H_0(\alpha)}$:

$$
(\widehat{H_0(\alpha)\psi})_m(E)=(\widehat{H_0(\alpha)}\hat{\psi})_m(E)=E\,\hat{\psi}_m(E).
$$

Via this isomorphism we have

$$
\begin{aligned}
&\left\|Ve^{-itH_0(\alpha)}\right\|^2 = \\
&\qquad \sum_{m,m'\in\mathbb{Z}}\int_0^\infty dE\int_0^\infty dE'\,e^{-it(E'-E)}\,\overline{\hat{\psi}_m(E)}\,v_\alpha^{m,m'}(E,E')\,\hat{\psi}_{m'}(E')\;, \qquad \text{(III.6)}
\end{aligned}
$$

where

$$v_\alpha^{m,m'}(E,E') = \left\langle \phi_{\alpha;m,E} \mid V^2 \phi_{\alpha;m',E'} \right\rangle \ .$$

We now choose the following dense subspace $\hat{\mathcal{D}} \subset \mathcal{D}(\widehat{H_0(\alpha)})$ in $\hat{\mathcal{L}}$

$$\hat{\mathcal{D}} = \left\{ \hat{\psi} \in \hat{\mathcal{L}} \mid \hat{\psi}_m \in C_0^\infty(\mathbb{R}^+) \text{ and } \hat{\psi}_m \equiv 0 \text{ for almost every } m \right\} \ .$$

Correspondingly we set $\mathcal{D} = (\hat{V}_\alpha)^{-1}\hat{\mathcal{D}}$. We need the following lemma, which will be proved in Appendix B.

Lemma III.3. *Let V satisfy the conditions of the Theorem. For arbitrary m and m' the functions $v_\alpha^{m,m'}(E,E')$ have measurable partial derivatives up to order 3 in E and E' on $(0,\infty)$, which are essentially bounded on compact sets.*

With this lemma at hand we now proceed as follows. We use the identity

$$e^{itE} \ = \ \left(\frac{1}{it}\frac{\partial}{\partial E}\right)^n \ e^{itE}$$

to perform 3 partial integrations w.r.t. E in (III.6). This gives for $\psi \in \mathcal{D}$

$$\begin{aligned} \left\| V e^{-itH_0(\alpha)}\, \psi \right\|^2 &\le \frac{1}{t^3} \sum_{m,m'} \int dE \int dE' \left| \partial_E^3 \left\{ \overline{\hat{\psi}_m(E)}\, v_\alpha^{m,m'}(E,E') \right\} \right| \, \left| \hat{\psi}_{m'}(E') \right| \\ &\le C\, \frac{1}{t^3} \end{aligned}$$

which again shows that $\left\| V e^{-itH_0(\alpha)}\, \psi \right\|$ is integrable in t in the interval $[1,\infty)$ say. This concludes the second proof of lemma III.1. $\square$

Now we turn to the situation, where the potential is centrally symmetric, i.e. where $V = V(r)$. According to 11 it is possible to define $H(\alpha) = H_0(\alpha) + V$ as a self–adjoint operator, if the potential is of the form $\frac{\gamma}{r} + \frac{\beta}{r^b} + W(r)$ say, where γ and β are arbitrary reals, $b \in [0,2)$ and W is a bounded function on $\mathbb{R}^+$. In general $H(\alpha)$ is obviously diagonal w.r.t. the decomposition (II.2) such that

$$H(\alpha) = \bigoplus_{m\in\mathbb{Z}} H_{2m}(\alpha) \quad \text{with} \quad H_{2m}(\alpha) \ = \ H_{0,2m}(\alpha) + V$$

acting on $\mathcal{L}^2(\mathbb{R}^+, rdr)$. This leads to a corresponding decomposition for the wave operators and the S–matrix

$$\Omega_\alpha^\pm \ = \bigoplus_{m\in\mathbb{Z}} \Omega_{\alpha,2m}^\pm, \quad \text{where} \quad \Omega_{\alpha,2m}^\pm \ = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH_{2m}(\alpha)}\, e^{-itH_{0,2m}(\alpha)} \tag{III.7}$$

and

$$S_\alpha = \bigoplus_{m\in\mathbb{Z}} S_{\alpha,2m} \quad \text{with} \quad S_{\alpha,2m} \ = \ (\Omega_{\alpha,2m}^+)^*\, \Omega_{\alpha,2m}^-.$$

The Kuroda–Birman theorem[32] now leads to the following result.

Theorem III.2. *Let the centrally symmetric potential V satisfy*

$$\int_0^1 |V(r)|\, r\, dr \;+\; \int_1^\infty |V(r)|\, dr \;<\; \infty.$$

Then the wave operators $\Omega^{\pm}_{\alpha,2m}$ exist and are complete. In particular the S–matrices $S_{\alpha,2m}$ are unitary and so are the S_α.

We recall that the wave operators for an arbitrary pair (H, H_0) are called complete if they are unitary when considered as operators from the absolutely continuous subspace of H_0 to the absolutely continuous subspace of H. Note that these conditions on V in the central symmetric case are weaker than the conditions used in theorem III.1.

Proof. We have to show that the operator

$$|V|^{\frac{1}{2}}\,(\,H_{0,2m}(\alpha) \;+\; k^2)^{-1}\, V^{\frac{1}{2}}$$

in $\mathcal{L}^2(\mathbb{R}^+, r dr)$ has finite trace class norm $\|\cdot\|_1$, where $V^{\frac{1}{2}}(r) \;= \mathrm{sgn}(V(r)) \cdot |V(r)|^{\frac{1}{2}}$. Here it suffices to choose any $k^2 \;>\; 0$. By (II.6) we have

$$\Big\langle r \;\Big|\; \big(H_{0,2m}(\alpha) \;+\; k^2\big)^{-1} \;\Big|\; r' \Big\rangle = I_\mu(kr_<)\, K_\mu(kr_>)$$

with $\mu = \;|2m + \alpha|$. Therefore we obtain the following a priori estimate

$$\Big\|\, |V|^{\frac{1}{2}} \big(H_{0,2m}(\alpha) \;+\; k^2\big)^{-1}\; V^{\frac{1}{2}} \Big\|_1 \le \int_0^\infty |V(r)|\; |I_\mu(kr)|\; |K_\mu(kr)|\, r\, dr\;. \tag{III.8}$$

The following estimates for I_μ, K_μ can be derived from the asymptotic behavior near the origin and at infinity (see e.g. 54, 37)

$$\begin{array}{lll} \text{for } kr \le 1: & |I_\mu(kr)| \;\le C(\mu)\,(kr)^{\mu} & |K_\mu(kr)| \;\le\; C(\mu)\,(kr)^{-\mu} \\ \text{for } kr \ge 1: & |I_\mu(kr)| \;\le C(\mu)\,\dfrac{e^{kr}}{\sqrt{kr}} & |K_\mu(kr)| \;\le C(\mu)\,\dfrac{e^{-kr}}{\sqrt{kr}} \end{array}$$

Now we choose $k = 1$. For this choice of k the right hand side of (III.8) is bounded by

$$C(\mu)\left\{\int_0^1 |V(r)|\, r\, dr + \int_1^\infty |V(r)|\, dr\right\}$$

concluding the proof of the theorem. □

IV. THE DIFFERENTIAL CROSS–SECTION

We now turn to the discussion of the two–particle scattering cross–section. The conventional approach when dealing with identical particles is to compute first the cross–section for *distinguishable* particles and in a second step to symmetrize or anti–symmetrize it in order to describe boson or fermion scattering. This is allowed because all observables commute with the projection operators onto the subspaces of symmetric and antisymmetric wave functions respectively (see e.g. 51 for a detailed account on this issue). In the context of anyon statistics this procedure is not applicable since no corresponding projection operators on the anyonic Hilbert spaces are available. Hence, we will investigate the two–particle cross–section in the center of mass system in a way similar to the single particle case. Doing this one encounters an additional difficulty, namely the presence of the gauge potential A_α encoding the statistics. The latter gives rise to a non–trivial differential cross–section even if no *dynamical* potential is present, i.e. $V \equiv 0$. This should not come as a surprise since the "free" dynamics generated by $H_0(\alpha)$ is described by Aharonov–Bohm scattering in the infinite time limit (see our discussion in section II). Thus, the cross–section will display both the statistical as well as the dynamical interaction represented by A_α and V respectively.

Let us first consider the simplest case where $\alpha = 0$, i.e. the particles are bosons. Then the natural choice of a basis for describing the scattering is given by the *symmetric* plane waves $|k,\theta\rangle$, because the incoming asymptote is usually taken to have a sharply peaked momentum distribution. Moreover, we recall that the localization of some state ψ at large times can be determined by means of its Fourier transform,

$$\lim_{t\to\pm\infty}\int_{\mathcal{C}}|e^{-itH_0}\psi|^2=\int_{\mathcal{C}}d\theta\,k\,dk\,|\,\langle k,\theta\mid\psi\rangle\,|^2, \tag{IV.1}$$

where $\mathcal{C} := \{(k,\theta) : k \in \mathbb{R}_+,\ \theta \in (a,b) \subset [0,\pi]\}$ denotes a cone in real space on the left hand side and in momentum space on the right hand side. The above identity can be found in several text books on scattering theory, see e.g. 44.

If $\alpha \notin \mathbb{Z}$ the flux–tube destroys translation invariance, whence the plane waves are not eigenstates of $H_0(\alpha)$. Thus, we have to look for an appropriate replacement such that the new basis diagonalizes $H_0(\alpha)$ and we still can form an incident wave packet with sharply peaked momentum distribution. It turns out that this can be achieved by making use of Aharonov–Bohm scattering theory. We assign to each symmetric plane wave denoted by $|k,\theta\rangle$ the corresponding stationary Aharonov–Bohm scattering state, that is

$$|k,\theta\rangle^{in} := \Omega^-_{AB}\,|k,\theta\rangle \quad \text{and} \quad |k,\theta\rangle^{out} := \Omega^+_{AB}\,|k,\theta\rangle \tag{IV.2}$$

The symbols $\Omega^\pm_{AB}$ stand for the Aharonov–Bohm wave operators as defined in (II.12). Note that each of the sets $\{\,|k,\theta\rangle^{in}\}$ and $\{\,|k,\theta\rangle^{out}\}$ forms a complete orthonormal system since the Aharonov–Bohm wave operators are unitary[48]. Moreover, the basis elements are (improper) eigenstates of $H_0(\alpha)$ due to the intertwining relation

$$H_0(\alpha)\Omega^\pm_{AB} = \Omega^\pm_{AB}H_0.$$

Their explicit form was given in section II equation (II.13). Now, by construction every wave packet in the $|k,\theta\rangle^{in}_{out}$ basis under the "free" anyonic time evolution $\exp(-itH_0(\alpha))$ approaches the corresponding wave packet in the plane wave basis $|k,\theta\rangle$ as $t \to \mp\infty$. Therefore, we shall refer to $|k,\theta\rangle^{in}_{out}$ as the anyonic state with incoming respectively outgoing (relative) momentum (k,θ). Note that according to their definition the "in" states are transformed into the "out" states by the Aharonov–Bohm S–matrix

$${}^{out}\langle k,\theta\mid k',\theta'\rangle^{in} = \langle k,\theta\mid S_{AB}\mid k',\theta'\rangle\,.$$

An additional argument for the usefulness of the two bases $|k,\theta\rangle^{in}_{out}$ is the following generalization of Dollard's theorem to anyonic dynamics,

$$\lim_{t\to\pm\infty}\int_{\mathcal{C}}|e^{-itH_0(\alpha)}\psi|^2=\int_{\mathcal{C}}d\theta\, k\, dk\, |{}^{in}_{out}\langle k,\theta\mid\psi\rangle\,|^2. \tag{IV.3}$$

The proof is immediate. By construction the state $e^{-itH_0}(\Omega^{\pm}_{AB})^*\psi$ approaches the anyonic state $e^{-itH_0(\alpha)}\psi$ in norm as $t\to\pm\infty$. Applying (IV.1) yields the desired relation.
We are now prepared to consider a single scattering event of two anyons when a potential V is present. Denote now by ψ the incoming wave function. According to (IV.3) the probability that the particles are scattered into $\mathcal{C}$ is given by

$$\lim_{t\to\infty}\int_{\mathcal{C}}|e^{-itH_0(\alpha)}S_\alpha\psi|^2=\int_{\mathcal{C}}d\theta k\, dk\, |{}^{out}\langle k,\theta\mid S_\alpha\psi\rangle\,|^2, \tag{IV.4}$$

where S_α is the scattering operator introduced in section III. On physical grounds ψ is assumed to have a sharply peaked momentum distribution at $t=-\infty$, whence it will be given in the basis $|k,\theta\rangle^{in}$. Therefore, we perform the following transformation

$$\begin{aligned}{}^{out}\langle k,\theta\mid S_\alpha\psi\rangle &= \int d\theta' k' dk'\, {}^{out}\langle k,\theta\mid S_\alpha\mid k',\theta'\rangle^{in}\ {}^{in}\langle k',\theta'\mid\psi\rangle\\ &= \int d\theta' k' dk'\ \left\langle k,\theta\,\middle|\,(\Omega^+_{AB})^*S_\alpha\Omega^-_{AB}\,\middle|\,k',\theta'\right\rangle\ {}^{in}\langle k',\theta'\mid\psi\rangle\,, \end{aligned}\tag{IV.5}$$

where in the second step we have used the defining relation (IV.2). Thus, we are lead to consider the operator

$$S^{tot}_\alpha := (\Omega^+_{AB})^* S_\alpha \Omega^-_{AB},$$

which can be identified with the S–matrix resulting from the wave operators

$$\Omega^{\pm}(H(\alpha),H_0)=\operatorname*{s-lim}_{t\to\pm\infty}e^{itH(\alpha)}e^{-itH_0}.$$

This can be easily verified by using the well known chain rule for wave operators (see e.g. 46)

$$\begin{aligned}\Omega^{\pm}(H(\alpha),H_0) &= \Omega^{\pm}(H(\alpha),H_0(\alpha))\Omega^{\pm}(H_0(\alpha),H_0)\\ &= \Omega^{\pm}_\alpha\,\Omega^{\pm}_{AB}.\end{aligned}\tag{IV.6}$$

Notice that the scattering operator S^{tot}_α incorporates the statistical as well as the dynamical interaction while S_α only takes care of the latter. This is best displayed by setting the potential equal to zero,

$$V\equiv 0 \quad\Rightarrow\quad S_\alpha=\mathbf{1},\ S^{tot}_\alpha=S_{AB}.$$

Therefore, in order to accommodate the anyon statistics of the particles we shall define the scattering amplitude f_α refering to the dynamical interaction V as follows. Denote by T^{tot}_α the operator

$$T^{tot}_\alpha := (\Omega^+_{AB})^*(S_\alpha-\mathbf{1})\Omega^-_{AB}=S^{tot}_\alpha-S_{AB}.$$

Then the scattering amplitude is given by

$$f_\alpha(k,\theta-\theta') := \left(\tfrac{\pi}{ik}\right)^{\frac{1}{2}} \left\langle \theta \;\middle|\, T^{tot}_\alpha(E=k^2) \,\middle|\; \theta' \right\rangle , \tag{IV.7}$$

where $\langle\theta \mid T^{tot}_\alpha(E) \mid \theta'\rangle$ denotes the on–shell matrix element defined by the relation

$$\left\langle k,\theta \;\middle|\, T^{tot}_\alpha \,\middle|\; k',\theta' \right\rangle = 2\delta(E-E') \left\langle \theta \;\middle|\, T^{tot}_\alpha(E) \,\middle|\; \theta' \right\rangle$$

with $E = k^2$ and $E' = k'^2$. From (IV.4) and (IV.5) one can now derive the differential cross–section analogously to the single-particle case, e.g. 51, 44. Away from the forward direction one ends up with the expression

$$\begin{aligned}\frac{d\sigma}{d\theta}(E_0,\theta_0) &= \pi E_0^{-\frac{1}{2}} \left| \left\langle \theta \;\middle|\, T^{tot}_\alpha(E_0) + T_{AB}(E_0) \,\middle|\; \theta_0 \right\rangle \right|^2 \\ &= |f_\alpha(k_0,\theta-\theta_0) + f_{AB}(k_0,\theta-\theta_0)|^2 \end{aligned} \tag{IV.8}$$

with $\theta \neq \theta_0$. Here, $E_0 = k_0^2, \theta_0$ determine energy and direction of the incident particle beam and $\langle\theta \mid T_{AB}(E) \mid \theta'\rangle$ denotes the on-shell matrix element of the operator $T_{AB} = S_{AB} - \mathbf{1}$ (see section II equation (II.17) for the explicit expression of the Aharonov–Bohm scattering amplitude f_{AB}). Note that we have a normalization factor π instead of 2π in (IV.8) because the polar angle is restricted to the interval $[0,\pi]$. As mentioned in section II the total Aharonov–Bohm cross–section is infinite for $\alpha \notin \mathbb{Z}$, whence the total cross–section corresponding to (IV.8) is infinite as well. However, the above differential cross–section coincides with the usual one for bosons or fermions if we set $\alpha = 0$ and $\alpha = 1$ respectively. This can be most easily seen by use of the defining relation (IV.2) for the "in" and "out" states which for the values $\alpha = 0, 1$ become symmetric and antisymmetric plane waves respectively (compare (II.13)). In case $\alpha = 1$, however, this is only true up to an angular dependent phase factor which comes in by representing fermions as bosons with attached flux–tubes. This does not influence the outcome since the differential cross–section is given by the square modulus of the scattering amplitude. Thus, as special cases we obtain the familiar result that the differential cross–section for bosons and fermions is given by the square modulus of the symmetrized and anti–symmetrized scattering amplitude respectively. In particular, f_{AB} vanishes and the total cross–section becomes finite for suitable short range interactions V.

V. JOST FUNCTIONS AND LEVINSON'S THEOREM

In this section we will introduce Jost functions[28] indexed by a continuous angular momentum and discuss their properties. This continuous parameter leads to an alternative formulation of the generalized Levinson theorem. We start by adapting the standard theory and results of Jost functions to the present situation (see e.g. 42, 43, 3, 51, 46). Conditions on the spherically symmetric potential will be presented at the appropriate places.
Consider the unitary map $U : \mathcal{L}^2(\mathbb{R}^+, rdr) \to \mathcal{L}^2(\mathbb{R}^+, dr)$ given by

$$\psi(r) \longmapsto \sqrt{r}\,\psi(r) \ . \tag{V.1}$$

Under this map the Hamiltonian $H_{2m}(\alpha) = H_{0,2m}(\alpha) + V$ turns into the operator

$$h_\mu \;=\; h_{0,\mu} \;+\; V \tag{V.2}$$

with $\mu = \mid 2m + \alpha \mid > 0$, where

$$h_{0,\mu} \;=\; -\left(\frac{d^2}{dr^2} \;-\; \frac{\mu^2 - \frac{1}{4}}{r^2}\right) \;. \tag{V.3}$$

In this section μ will be allowed to be any positive number. Also in this section the notation $\phi_0(r;k,\mu)$ will be reserved for the regular solutions (so called because of their behavior near $r=0$) of the free Schrödinger equation $(h_{0,\mu} - k^2)\,\psi = 0$ given as

$$\phi_0(r;k,\mu) \;=\; \sqrt{\tfrac{\pi k r}{2}}\, J_\mu(kr) \;. \tag{V.4}$$

With the help of these solutions the orthogonality and completeness relations (II.4) and (II.5) are now reformulated as

$$\int_0^\infty \phi_0(r;k,\mu)\;\phi_0(r';k,\mu)\;dk \;=\; \frac{\pi}{2}\,\delta(r-r') \tag{V.5a}$$

$$\int_0^\infty \phi_0(r;k,\mu)\;\phi_0(r;k',\mu)\;dr \;=\; \frac{\pi}{2}\,\delta(k-k') \;. \tag{V.5b}$$

We will also need the irregular, free solutions $\chi_0^\pm$ given as

$$\chi_0^\pm(r;k,\mu) \;=\; \pm i\,\sqrt{\tfrac{\pi k r}{2}}\, H_\mu^\pm(kr) \;,$$

With help of these solutions the free Green's functions read (compare (II.6))

$$G_0^\pm(r,r';k,\mu) = \left\langle r \;\middle|\; (h_{0,\mu} - k^2 \mp i\epsilon)^{-1} \;\middle|\; r' \right\rangle = \tfrac{1}{k}\,\phi_0(r_<;k,\mu)\;\chi_0^\pm(r_>;k,\mu) \;. \tag{V.6}$$

Definition V.1. *For given V the function $\phi = \phi(r;k,\mu)$ is the regular solution of the equation $(h_\mu - k^2)\,\psi = 0$ which for $r \to 0$ approximates the free, regular solution ϕ_0:*

$$\lim_{r\to 0}\left(\frac{2}{kr}\right)^{\mu+\frac{1}{2}} \frac{\Gamma(\mu+1)}{\sqrt{\pi}}\,\phi(r;k,\mu) \;=\; 1 \;.$$

As in the standard theory (see e.g. 42, 43) one establishes the following facts.

(A0) ϕ is the solution to the integral equation

$$\psi \;=\; \phi_0 \;-\; G_0^< \,V\,\psi \quad \text{with} \quad G_0^<(r,r';k,\mu) = \Theta(r-r')\,g_0(r,r';k,\mu) \;. \tag{V.7}$$

Here Θ denotes the Heaviside step function and g_0 is given as

$$g_0(r,r';k,\mu) = \tfrac{1}{k}[\phi_0(r;k,\mu)\,\chi_0^-(r';k,\mu) - \;\chi_0^-(r;k,\mu)\,\phi_0(r';k,\mu)\,] \;. \tag{V.8}$$

(B0) If the potential V satisfies the condition

$$\int_0^\infty r dr\,|V(r)|; < \infty$$

then the regular solution ϕ to the eigenvalue problem $(h_{0,\mu} + \lambda V - k^2)\,\psi = 0$ has a convergent power series expansion in λ of the form

$$\phi = \sum_{n=0}^{\infty} \lambda^n \phi_n \quad \text{with} \quad \phi_{n=0} = \phi_0 \quad \text{and} \quad \phi_n = -G_0^< V \phi_{n-1},\, n \geq 1 \,. \tag{V.9}$$

(C0) For fixed $\mu > 0$, $r > 0$ and real V the solution ϕ has an analytic continuation in k into the complex plane with a cut along the negative imaginary axis. There one has the relation ($k > 0$)

$$\phi(r; -ik - 0, \mu) = e^{i\pi(\mu+\frac{1}{2})}\,\phi(r; ik, \mu) = e^{2i\pi(\mu+\frac{1}{2})}\,\phi(r; -ik + 0, \mu) \,. \tag{V.10}$$

The proofs of (A0), (B0) and (C0) are as in ordinary 3-dimensional Schrödinger theory (see e.g. 43). As a byproduct of the proof one also has the two estimates:

$$|\phi_n(r; k, \mu)| < e^{|\mathrm{Im}\, k|\, r} \left(\frac{|k|\, r}{1 + |k|\, r}\right)^{\mu+\frac{1}{2}} \frac{C(\mu)^{n+1}}{n!} \left[\int_0^r dr'\, \frac{r'\,|V(r')|}{1 + |k|\, r'}\right]^n \tag{V.11a}$$

$$|\phi(r; k, \mu)| < C(\mu) e^{|\mathrm{Im}\, k|\, r} \left(\frac{|k|\, r}{1 + |k|\, r}\right)^{\mu+\frac{1}{2}} \tag{V.11b}$$

We will prove these estimates in the Appendix C. Relation (V.10) is a consequence of the relation

$$J_\mu(e^{in\pi}) = e^{in\pi\mu}\, J_\mu(z) \,.$$

We now introduce the irregular solutions of the eigenvalue equation with potential as those solutions $\chi^\pm$ which approximate the free irregular solutions $\chi_0^\pm$ at $r = \infty$.

Definition V.2. *The functions* $\chi^\pm = \chi^\pm(r; k, \mu)$ *are the solutions of the eigenvalue problem* $(h_\mu - k^2)\,\psi = 0$ *satisfying the boundary conditions at infinity of the form*

$$\lim_{r\to\infty} e^{\mp i(kr - \frac{\pi}{2}\mu + \frac{\pi}{4})}\, \chi^\pm(r; k, \mu) = 1 \,.$$

These solutions will be called Jost solutions. With the help of these solutions ϕ and $\chi^\pm$ the Green's function for the full Hamiltonian h_μ now takes a form analogous to the free case (see (V.6))

$$G^\pm(r, r'; k, \mu) = \left\langle r \,\middle|\, (h_\mu - k^2 \mp i\epsilon)^{-1} \,\middle|\, r' \right\rangle = \frac{\phi(r_<; k, \mu)\, \chi^\pm(r_>; k, \mu)}{W(\chi^\pm, \phi)} \,, \tag{V.12}$$

where W is the Wronskian, i.e.

$$W(\psi_1, \psi_2)(r) = \psi_1(r)\, \partial_r \psi_2(r) - \psi_2(r)\, \partial_r \psi_1(r) \,.$$

We recall that since ϕ and $\chi^\pm$ are solutions of the same eigenvalue equation, the Wronskian in (V.12) is actually independent of r, whence the notation.
Again one may establish the following facts (see e.g. 43).

(A$\pm$) The functions $\chi^\pm$ are the solutions to the integral equations

$$\psi \;=\; \chi_0^\pm \;-\; G_0^> \, V \, \psi \;, \quad G_0^>(r,r';k,\mu) := \Theta(r'-r)\, g_0(r,r';k,\mu) \;. \tag{V.13}$$

Here, g_0 denotes the function defined in (V.8) which can be rewritten as

$$g_0(r,r';k,\mu) = \tfrac{i}{2k}\left[\chi_0^+(r;k,\mu)\chi_0^-(r';k,\mu) - \chi_0^-(r;k,\mu)\chi_0^+(r';k,\mu)\right] \;.$$

(B$\pm$) If the potential V satisfies the condition

$$\int_0^\infty r dr\,(1+r)|V(r)| \;<\; \infty \;, \tag{V.14}$$

the irregular solutions $\chi^\pm$ to the eigenvalue problem $(h_{0,\mu} \,+\, \lambda V \,-\, k^2)\,\psi \;=\; 0$ with $\pm\,\mathrm{Im}\, k \,\geq\, 0$, $k \,\neq\, 0$ have a convergent power series expansion in λ of the form

$$\chi^\pm = \sum_{n=0}^\infty \lambda^n \chi_n^\pm \quad \text{with} \quad \chi_{n=0}^\pm = \chi_0^\pm \quad \text{and} \quad \chi_n^\pm = -G_0^> V \chi_{n-1}^\pm,\, n \geq 1 \;.$$

(C$\pm$) For V real and $\mu \,>\, 0$, $r \,>\, 0$ the functions $\chi^\pm$ are analytic in k for $\pm\,\mathrm{Im}\, k \,>\, 0$ with continuous extensions to the real axis except possibly for a singularity of order $\mu - \frac{1}{2}$ at $k=0$. In case V satisfies

$$\int_0^\infty r dr\,|V(r)|\,e^{\eta r} \;<\; \infty, \qquad \eta > \;0 \;, \tag{V.15}$$

then the functions $\chi^\pm$ have an analytic continuation into the domains given by $\pm\,\mathrm{Im}\, k \;>\; -\frac{\eta}{2}$ for all $\mu \,>\, 0\,, r \,>\, 0$ with the exception of a branch cut from $k=0$ to $k = \mp i \frac{\eta}{2}$. There one has the relation ($k \,>\, 0$)

$$\chi^\pm(r;\mp(ik+0),\mu) \;=\; \chi^\pm(r;\mp(ik-0),\mu) \;+\; 2i\cos\pi\mu\;\chi^\pm(r;\pm ik,\mu) \;.$$

The proof of these statements is similar to the one in the regular case and involves a proof of the bounds

$$\begin{aligned}\left|\chi_n^\pm(r;k,\mu)\right| \;< e^{\mp\,\mathrm{Im}(k) r}\left(\frac{|k|\,r}{1+\,|k|\,r}\right)^{-\mu+\frac{1}{2}} &\times \\ \times\; \frac{C(\mu)^{n+1}}{n!}\; &\left[\int_r^\infty dr'\, e^{\,|\mathrm{Im}\,k|\,r' \mp \mathrm{Im}(k) r} \frac{r'\,|V(r')|}{1+\,|k|\,r'}\right]^n\end{aligned} \tag{V.16}$$

and

$$\left|\chi^\pm(r;k,\mu)\right| \,<\, C(\mu) e^{\mp\,\mathrm{Im}(k) r}\left(\frac{|k|\,r}{1+\,|k|\,r}\right)^{-\mu+\frac{1}{2}} \tag{V.17}$$

for all $\mu > 0$ and $r > 0$. We will establish these bounds in appendix C. Due to the uniqueness of the solutions $\chi^\pm$ and the multivaluedness of the Hankel functions (see e.g. 37) one has the relation

$$\overline{\chi^\pm(r;k,\mu)} \;=\; \chi^\mp(r;\overline{k},\mu) = \;\mp i\, e^{\pm i\pi\mu}\, \chi^\pm(r,-\overline{k},\mu) \;.$$

We are now in the position to introduce the Jost functions in the present context and establish their properties. Since χ^+ and χ^- form a basis of solutions, ϕ is a linear combination of these two solutions. As in the standard theory (see e.g. 43) it turns out that this linear combination is of the form

$$\phi(r;k,\mu) = \tfrac{i}{2}\left[F(k,\mu)\chi^-(r;k,\mu) - F(-k,\mu)\chi^+(r;k,\mu)\right] \ ,$$

where the coefficient function $F = F(k,\mu)$ is called the *Jost function* given by the Wronskian

$$F = \tfrac{1}{k} W(\chi^+,\phi) \ . \tag{V.18}$$

The Jost function was first introduced by and named after R. Jost. Note, however, that our definition of F follows the convention introduced by R.G. Newton[43], which differs from the original one given in 29. We have the first main result concerning these Jost functions.

Theorem V.1. *Let the potential V satisfy the condition* (V.14). *Then for fixed $\mu > 0$ $F(k,\mu)$ extends to an analytic function in k in the open upper half plane, which is continuous up to the real axis with the exception of the origin.*
In case the potential satisfies the stronger condition (V.15), *$F(k,\mu)$ is analytic in k in $\{\, k \mid \operatorname{Im} k > -\frac{\eta}{2}\}$ except for a cut on the negative imaginary axis. Then F also satisfies the relation* ($k > 0$)

$$F(-ik-0,\mu) \ = \ -e^{2i\pi\mu}F(-ik+0,\mu) \ + \ 2\cos\pi\mu\, F(ik,\mu) \ .$$

The proof follows from the representation for F

$$F(k,\mu) \ = \ 1 + \frac{1}{k}\ \left\langle \chi_0^-(\,\cdot\,;k,\mu) \mid V\phi(\,\cdot\,;k,\mu)\right\rangle \ , \tag{V.19}$$

the estimate (V.11b) for ϕ and the estimate (V.17) which is also valid for χ_0^+. To establish this relation we use the integral equation (V.7) for ϕ which for large r gives

$$\begin{aligned}\phi(r;k,\mu) \xrightarrow{r\to\infty} \frac{i}{2}\Bigg[&\chi_0^-(r;k,\mu)\left\{1+\frac{1}{k}\ \left\langle\chi_0^-(\,\cdot\,;k,\mu)\mid V\phi(\,\cdot\,;k,\mu)\right\rangle\right\} \\ &- \chi_0^+(r;k,\mu)\left\{1+\frac{1}{k}\ \left\langle\chi_0^+(\,\cdot\,;k,\mu)\mid V\phi(\,\cdot\,;k,\mu)\right\rangle\right\}\Bigg] \ .\end{aligned} \tag{V.20}$$

Since $\chi^\pm(r;k,\mu)$ approaches $\chi_0^\pm(r;k,\mu)$ for $r\to\infty$, relation (V.19) follows from this behavior.

Note that making use of the analytic properties just proven there is an equivalent definition of the Jost function in terms of Fredholm theory, see e.g. 43, 47. Consider the operator $G_0^+(k,\mu)V$ with G_0^+ the free Green's operator and V satisfying relation (V.14). Then for k positive imaginary ($k \in i\mathbb{R}^+$) the operator $G_0^+(k,\mu)V$ is trace class, whence its Fredholm determinant $\det(\mathbf{1}+G_0^+(k,\mu)V)$ is well defined. It turns out that the Jost function is just the analytic continuation of this determinant to the upper half complex plane.

We note that the series expansion (V.9) for the regular solution ϕ gives the series expansion

$$F \;=\; 1 + \frac{1}{k}\sum_0^\infty F^n \quad \text{with} \quad F^n(k,\mu) \;=\; \left\langle \chi_0^-(\cdot\,;k,\mu) \mid V\phi^n(\cdot\,;k,\mu)\right\rangle \;.$$

The bound (V.11a) for ϕ_n combined with the bound (V.17) for χ_0^+ gives the bound

$$|F^n(k,\mu)| \;<\; C(\mu)\frac{C^n}{n!}\int_0^\infty dr\, e^{(\,|\mathrm{Im}\,k|\,-\mathrm{Im}\,k)r}\frac{|k|\,r}{1+\,|k|\,r}\,|V(r)|\;\;,$$

while the bound (V.11b) gives

$$|\;F(k,\mu)-1\;| \leq\; C(\mu)\int_0^\infty dr\, e^{(|\mathrm{Im}\,k|-\,\mathrm{Im}\,k)r}\,\frac{r|V(r)|}{1+|k|r}\;.$$

The importance of the Jost function stems from its role in the discussion of the S–matrix. Let $\Omega_\mu^\pm$ be the wave operators for the pair $(h_\mu, h_{0,\mu})$ such that one has (see (III.7) and (V.1))

$$\Omega_\mu^\pm \;=\; U\,\Omega_{\alpha,2m}^\pm\,U^{-1} \quad \text{and} \quad S_\mu \;=\; \left(\Omega_\mu^+\right)^*\,\Omega_\mu^- \;=\; U\,S_{\alpha,2m}\,U^{-1}\;.$$

Since S_μ commutes with $h_{0,\mu}$ there is a decomposition of S_μ in the form

$$S_\mu \;=\; \int_\oplus dk\,S(k,\mu)$$

with respect to the spectral decomposition of $h_{0,\mu}$. Since the spectrum of $h_{0,\mu}$ is not degenerate, $S(k,\mu)$ acts in a one–dimensional Hilbert space and therefore is a complex number of absolute value one, i.e.

$$S(k,\mu) \;=\; e^{2i\delta(k,\mu)}\;, k \geq 0\;, \mu \geq 0$$

with $\delta(k,\mu)$ being the phase shift. We define by

$$f(k,\mu) \;:=\; \frac{e^{2i\delta(k,\mu)}-1}{\sqrt{\pi i k}}$$

the partial wave amplitude for the angular momentum $\pm\mu$, such that we obtain the following partial wave decomposition of the scattering amplitude given in (IV.7)

$$\begin{aligned} f_\alpha(k,\theta-\theta') &= \left(\tfrac{\pi}{ik}\right)^{\frac12}\left\langle\theta\;\middle|\;T_\alpha^{tot}(k^2)\;\middle|\;\theta'\right\rangle \\ &= \left(\tfrac{\pi}{ik}\right)^{\frac12}\tfrac{1}{\pi}\sum_{m\in\mathbb{Z}} e^{-i\pi|2m+\alpha|}\left(e^{2i\delta(k,|2m+\alpha|)}-1\right)e^{i2m(\theta-\theta')} \\ &= \sum_{m\in\mathbb{Z}} e^{-i\pi|2m+\alpha|}f(k,|2m+\alpha|)\,e^{i2m(\theta-\theta')}. \end{aligned} \tag{V.21}$$

Here we have used the relation $\langle k,\theta\mid T_\alpha^{tot}\mid k',\theta'\rangle = {}^{out}\langle k,\theta\mid S_\alpha - \mathbf{1}\mid k',\theta'\rangle^{in}$ as well as the identities

$$\begin{aligned} \langle r,\theta \mid k,\theta'\rangle^{in}_{out} &= \sqrt{2\pi}\,\tfrac{1}{\pi}\sum_{m\in\mathbb{Z}}(\mp i)^{|2m+\alpha|}\phi_{\alpha;m,E}(r,\theta)\,e^{-i2m\theta'} \\ \langle \phi_{\alpha;m,E} \mid S_\alpha \phi_{\alpha;m',E'}\rangle &= \delta_{mm'}\,\delta(E-E')\,e^{2i\delta(k,|2m+\alpha|)}. \end{aligned}$$

Now set

$$\phi^{\pm}(r;k,\mu) \;=\; (\Omega^{\pm}_{\mu}\,\phi_0)(r;k,\mu)\ .$$

By construction one has a solution of the eigenvalue equation $(h_\mu - k^2)\psi = 0$ and $\phi^\pm$ satisfies the Lippmann-Schwinger equation

$$\phi^{\pm}(\cdot\,;k,\mu) \;=\; \phi_0(\cdot\,;k,\mu) - (h_\mu - k^2 \pm i0)^{-1}\,V\,\phi^{\pm}(\cdot\,;k,\mu)\ .$$

With the help of $\phi^\pm$ one may express the partial wave amplitude as

$$f(k,\mu) \;=\; -\tfrac{2}{k^2}\,(\tfrac{ik}{\pi})^{\frac{1}{2}}\,\big\langle \phi_0(\cdot\,;k,\mu) \mid V\,\phi^{+}(\cdot\,;k,\mu)\big\rangle\ .$$

The solution ϕ^+ of the Lippmann-Schwinger equation has the following asymptotic behavior for $r\to\infty$.

$$\phi^{+}(r;k,\mu) \xrightarrow[r\to\infty]{} \phi_0(r;k,\mu) + \tfrac{1}{2}\sqrt{\pi k}\,f(k,\mu)\,e^{i(kr-\frac{\pi}{2}\mu)}\ ,$$

where we have used $\chi_0^{+}(r;k,\mu) \xrightarrow[r\to\infty]{} e^{i(kr-\frac{\pi}{2}\mu+\frac{\pi}{4})}$. Alternatively, using the relation $\phi_0 = \frac{i}{2}\left(\chi_0^{-} - \chi_0^{+}\right)$ one gets

$$\phi^{+}(r;k,\mu) \xrightarrow[r\to\infty]{} \frac{i}{2}\left(\chi_0^{-}(r;k,\mu) - e^{2i\delta(k,\mu)}\,\chi_0^{+}(r;k,\mu)\right)\ .$$

With the same arguments as in the standard theory (see e.g. 42, 43, 46) one proves

Lemma V.1. *Let $\mu > 0$ be fixed and assume that V satisfies the condition* (V.14). *Then one has*

(i) *For k on the real axis ($k\in\mathbb{R}^+$) the following identities are valid:*

$$\phi(r;k,\mu) \;=\; F(k,\mu)\,\phi^{+}(r;k,\mu) \tag{V.22}$$

$$S(k,\mu) \;=\; e^{2i\delta(k,\mu)} \;=\; \frac{\overline{F}(k,\mu)}{F(k,\mu)} \;=\; \frac{F(-k,\mu)}{F(k,\mu)} \tag{V.23}$$

Furthermore, $F(k,\mu)$ can only vanish at the origin.

(ii) *The zeros of $F(k,\mu)$ in the upper half plane $\operatorname{Im} k > 0$ all lie on the positive imaginary axis and are simple. In particular $k\in i\mathbb{R}^+$ is a zero of $F(k,\mu)$ iff k^2 is the energy of a bound state of h_μ with angular momentum $\pm\mu$.*

(iii) *The following limit relation is valid in the closed upper half plane*

$$\lim_{k\to\infty} F(k,\mu) \;=\; 1\ .$$

Since the proof is analogous to the one given in three dimensional Schrödinger theory[42,43,46] we omit it here. Now, the famous Levinson's theorem follows as a corollary by use of the residue theorem. Only the so called resonance case, i.e. $F(k=0,\mu)=0$, requires some more work.

Theorem V.2. *Let* $\mu>0$ *and* V *satisfy* (V.14). *If* $F(k=0,\mu)\neq 0$ *one has the following relation between the phase shift* $\delta(k,\mu)$ *and the number* n_μ *of bound states with angular momentum* $\pm\mu$,

$$\delta(k=0,\mu)-\delta(k=\infty,\mu)=\pi n_\mu \ . \tag{V.24}$$

Proof. The proof of equation (V.24) is standard and can be found in e.g. 42, 43, 46, 51. For sake of completeness we recall the main arguments. Integrate the logarithmic derivative of F with respect to k over a closed semi–circle in the complex upper half plane centered at the origin. According to the preceding lemma the integrand has simple poles on the positive imaginary axis each of which corresponds to a bound state of h_μ. While the contribution of the semi–circle vanishes in the limit of infinite radius (see (III.4)) the integration over the real axis yields the phase difference by equation (V.23). Applying the residue theorem completes the proof. □

We comment on the results of the above theorem. Equation (III.5) generalizes Levinson's theorem to continuous angular momentum μ. One might wonder what happens when μ varies. On physical grounds one expects that for higher angular momentum μ there are fewer bound states since then the centrifugal barrier is stronger. Thus, when μ is steadily increased the zeros of the Jost function F move along the positive imaginary axis towards the origin. Every time a zero moves out of the upper half plane we get a discontinuous jump in the phase shift indicating the disappearance of a bound state.

Theorem V.3. *Let* $\mu\notin\mathbb{Z}$, $\mu>0$ *and assume* V *satisfies* (V.15). *In case* $F(k=0,\mu)=0$ *the above form of Levinson's theorem remains valid when* $\mu>1$ *and is modified as*

$$\delta(k=0,\mu)-\delta(k=\infty,\mu)=\pi(n_\mu+\mu) \tag{V.25}$$

when $\mu<1$.

So far we have not been able to settle the case $\mu\in\mathbb{Z}$ corresponding to boson and fermion statistics. For real anyons, however, note that in case of positive angular momentum we have $\mu=\alpha$ when $\mu<1$. Thus, equation (V.25) then specializes to

$$\delta(k=0,\alpha)-\delta(k=\infty,\alpha)=\pi(n_\mu+\alpha), \tag{V.26}$$

showing that one can determine the statistics parameter from the scattering phase independently of the detailed form of the short range potential V. Below we will find a similar formula for the delta potential. Setting $\alpha=1/2$ (i.e. for semions) in (V.25) we obtain the well known resonance case of three dimensional Schrödinger theory[42], since then the radial Schrödinger equation is formally identical to the one in three dimensions (compare (V.2) and (V.3)).

Proof. The strategy for proving (V.25) is similar to the one used for (V.24). The techniques used are analogous to those in three dimensional Schrödinger theory (see 42, 51) but the line of argument is slightly changed.
Because $F(k=0,\mu)=0$ we change the integration contour as follows. We cut the semi-circle at the origin and insert a small semi-circle C_ρ of radius $\rho \ll 1$ centered at the origin. Then there is an additional contribution to equation (V.24) namely,

$$\pi n'_\mu = \delta(k=0,\mu) - \delta(k=\infty,\mu) + \lim_{\rho\to 0} \frac{1}{2i} \oint_{C_\rho} d\ln F, \tag{V.27}$$

where n'_μ is the number of bound states with the exception of a possible zero energy eigen-state. In order to compute the second term of the right hand side in the above equation we only need to know the small energy behavior of F. Since we have assumed that the potential is decaying exponentially we can continue F in the lower half plane and expand it into a convergent power series around the origin for suitably small k. We start by rewriting expression (V.19),

$$\begin{aligned} F(k,\mu) &= 1 + \tfrac{1}{k} \left\langle \chi_0^-(\cdot\,;k,\mu) \mid V\phi(\cdot\,;k,\mu)\right\rangle \\ &= 1 + \tfrac{1}{k\sin\pi\mu}\big[\langle \phi_0(\cdot\,;k,-\mu) \mid V\phi(\cdot\,;k,\mu)\rangle \\ &\qquad - e^{-i\pi\mu}\langle \phi_0(\cdot\,;k,\mu) \mid V\phi(\cdot\,;k,\mu)\rangle\big]. \end{aligned} \tag{V.28}$$

Here we have used the identity $H^{(1)}_\mu(z) = \frac{1}{i\sin\pi\mu}[J_{-\mu}(z) - e^{-i\pi\mu}J_\mu(z)]$ which can be found in e.g. 54, 37. Note that in case of integer μ the Hankel function is defined by the limit of r.h.s. of the last equation. Then the above decomposition (V.28) is no longer valid, whence we restrict ourselves to the non-integer case. Both integrals in (V.28) converge uniformly as can be shown by use of the estimates

$$|\phi_0(r;k,\mu)\phi(r;k,\mu)| \le C(\mu) e^{2|\operatorname{Im} k|\, r} \left(\frac{|k|r}{1+|k|r}\right)^{2\mu+1}$$

$$|\phi_0(r;k,-\mu)\phi(r;k,\mu)| \le C(\mu) e^{2|\operatorname{Im} k|\, r} \left(\frac{|k|r}{1+|k|r}\right),$$

which follow from (V.11a) and (V.11b). The functions $\phi_0(r;k,-\mu)$ and $\phi_0(r;k,\mu)$ can be written as a convergent power series in k^2 times a factor $k^{-\mu+\frac{1}{2}}$ and $k^{\mu+\frac{1}{2}}$ respectively. This follows from their definition (V.4) and the series expansions of the Bessel functions $J_{-\mu}(kr)$, $J_\mu(kr)$, see e.g. 54, 37. From (V.9) we see that also $\phi(r;k,\mu)$ can be written as a convergent power series in k^2 times $k^{\mu+\frac{1}{2}}$, since the Green's function is an even function in k. Therefore, in a suitably small chosen neighborhood of $k=0$ the Jost function can be expanded in a convergent series of the form

$$F(k,\mu) = \sum_{n=0}^{\infty} a_n(\mu) k^{2n} + k^{2\mu} \sum_{n=0}^{\infty} b_n(\mu) k^{2n}, \quad |k| \ll 1. \tag{V.29}$$

Recall our convention to place te branch cut of F along the negative imaginary axis. The assumption $F(0,\mu)=0$ gives $a_0=0$. In order to deduce from this expansion the small energy behavior of F we need to know the first non-vanishing coefficient in (V.29). We claim $a_1(\mu) \neq 0$ for $\mu > 1$ and $b_0(\mu) \neq 0$ for $\mu < 1$. But then we have

$$F(k,\mu) = \begin{cases} \mathcal{O}(k^{2\mu}) & \text{if } \mu < 1 \\ \mathcal{O}(k^2) & \text{if } \mu > 1 \end{cases} .$$

Thus, evaluating the integral over the semi-circle C_ρ yields

$$\lim_{\rho \to 0} \frac{1}{2i} \oint_{C_\rho} d \ln F = \begin{cases} -\pi\mu & \text{if } \mu < 1 \\ -\pi & \text{if } \mu > 1 \end{cases} . \tag{V.30}$$

In order to prove the claim we consider the limit $\lim_{k\to 0} k^{-1} \frac{\partial}{\partial k} F$. If we can show that

$$\lim_{k \to 0} k^{-1} \frac{\partial}{\partial k} F(k,\mu) = \begin{cases} \infty & \text{if } \mu < 1 \\ C(\mu) \neq 0 & \text{if } \mu > 1 \end{cases}$$

then the claim follows as can be infered directly from (V.29). We calculate the expression $\frac{\partial}{\partial k} F$ by making use of the identity (V.18). Since the solutions χ^+, ϕ are ill defined for $k \to 0$ we consider instead the modified solutions

$$\Psi(r;k,\mu) := k^{\mu - \frac{1}{2}} \chi^+(r;k,\mu) \quad \text{and} \quad \Phi(r;k,\mu) := k^{-\mu - \frac{1}{2}} \phi(r;k,\mu)$$

which are both finite and non-vanishing at $k = 0$. This can be seen by taking the limit $k \to 0$ in the defining integral equations (V.7) and (V.13) respectively. In particular one then derives that $\Phi(r;0,\mu)$ behaves like $r^{\mu + \frac{1}{2}}$ for $r \to 0$ and $\Psi(r;0,\mu)$ like $r^{-\mu + \frac{1}{2}}$ for $r \to \infty$. In terms of Φ, Ψ the Jost function and its derivative w.r.t. k (denoted by a dot) then read

$$F = W(\Psi, \Phi) \quad \text{and} \quad \dot{F} = W(\dot{\Psi}, \Phi) + W(\Psi, \dot{\Phi}).$$

Since Ψ is a solution of the radial equation one easily verifies the identity

$$\frac{\partial}{\partial r} W[\Psi(r;k,\mu), \Psi(r;k',\mu)] = (k^2 - k'^2) \Psi(r;k,\mu) \Psi(r;k',\mu).$$

Differentiating w.r.t. k and setting subsequently $k = k'$ one obtains from this the relation

$$W[\dot{\Psi}(r;k,\mu), \Psi(r;k,\mu)] = -2k \int_r^\infty dr' \Psi(r';k,\mu)^2, \tag{V.31}$$

where the r.h.s. is finite for $\operatorname{Im} k > 0$. The assumption $F(0,\mu) = 0$ implies that the regular and the irregular solution are proportional when $k = 0$, i.e. $\Psi(r;0,\mu) = \kappa(\mu) \Phi(r;0,\mu)$ holds. Note that the proportionality factor $\kappa(\mu)$ is nonzero and finite because both solutions are non-vanishing. Thus, in the limit $k \to 0$ we have

$$\dot{F}(0,\mu) = \kappa(\mu)^{-1} W(\dot{\Psi}(r;0,\mu), \Psi(r;0,\mu)) + \kappa(\mu) W(\Phi(r;0,\mu), \dot{\Phi}(r;0,\mu)).$$

However, $\dot{F}(0,\mu)$ does not depend on r whence we can set $r = 0$ in the last equation. Since the regular solution Φ vanishes at $r = 0$ we obtain from (V.31),

$$\lim_{k\to 0} k^{-1}\frac{\partial}{\partial k}F(k,\mu) = \lim_{k\to 0} k^{-1}\kappa(\mu)^{-1}W[\dot\Psi(0;k,\mu),\Psi(0;k,\mu)]$$
$$= -2\kappa(\mu)^{-1}\int_0^\infty dr\,\Psi(r;0,\mu)^2$$
$$= -2\kappa(\mu)\int_0^\infty dr\,\Phi(r;0,\mu)^2 \tag{V.32}$$

The irregular solution Ψ falls off like $r^{-\mu+\frac{1}{2}}$ for large r and is proportional to the regular solution Φ at $k=0$, therefore the r.h.s. of (V.32) is finite for $\mu > 1$. This means that the regular solution is square integrable and hence a proper eigensolution of the full Hamiltonian h_μ. The number of bound states is thus $n_\mu = n'_\mu + 1$, whence we read of from (V.27) and (V.30) that Levinson's theorem also holds in case $F(0,\mu) = 0$ when $\mu > 1$.
For $\mu < 1$, however, the regular solution is no longer square integrable and the expression (V.32) is divergent. Hence the number of bound states is given by $n_\mu = n'_\mu$ and from (V.27) and (V.30) we infer the modified equation (V.25). This completes the proof. □

VI. THE δ–POTENTIAL

In this section we will introduce and discuss the δ–potential, often also called the contact potential or zero–range interaction potential because it describes an interaction which is "non–vanishing" only if the two particles are at the same place, i.e. if $z_{rel} = 0$. In the literature this figures also under the notion of anyons without hard–core condition[30]. See e.g. 38, 10, 5 for previous articles on this subject also in connection with field theoretic considerations.

To introduce the δ–potential we follow a standard strategy (see e.g. 2), i.e. we consider the free Hamiltonian (in the center of mass system) restricted to a definition domain of wave functions which vanish at $z_{rel} = 0$. The resulting operator has deficiency indices (1,1) and so there is a one–parameter family of self–adjoint extensions, one of which is of course the ordinary free Hamiltonian given as the Friedrich's extension. We discuss the bound state problem and the resulting scattering theory. We also check the validity of Levinson's theorem.

For a start, consider the operator

$$h_{0,\mu} = -\partial_r^2 + \frac{\mu^2 - 4^{-1}}{r^2}$$

on $\mathcal{L}^2(\mathbb{R}^+, dr)$ (recall (V.1)) with domain of definition $\mathcal{D}(h_{0,\mu})$ consisting of wave functions $\psi(r)$ having compact support away from the origin such that ψ and $\partial_r\psi$ are locally absolutely continuous and such that $h_{0,\mu}\psi$ (defined in the sense of distributions) is an element of $\mathcal{L}^2(\mathbb{R}^+, dr)$. Recall that locally absolutely continuous functions are such that their derivatives in the sense of distributions are locally integrable functions (see e.g. 52).

Let us now investigate the most general solution of the equation $h_{0,\mu}\psi = 0$, given by

$$c_1 \cdot r^{\frac{1}{2}+\mu} + c_2 \cdot r^{\frac{1}{2}-\mu}\ .$$

Using Weyl's terminology (consult e.g. 55), one has the limit point case at $r = \infty$. At $r = 0$ one also has the limit point case if and only if $\mu \geq 1$, otherwise one has the limit circle case. If now $0 \leq \mu < 1$, by Weyl's alternative there is exactly one solution of $h_{0,\mu}\psi = 0$, which is $\mathcal{L}^2$ near infinity. By construction this solution must be $\mathcal{L}^2$ at both infinity and at zero, and thus is follows that $\dim(\ker(h_{0,\mu} \pm i)) = 1$ by Weyl's alternative. Therefore the deficiency indices are (1,1) and there is a one–parameter family of s.a. extensions (see e.g. 47).

In terms of the statistics parameter α we have the following situation: If one chooses $\alpha = 1$, then $\mu = |2m + \alpha| \geq 1$ for all $m \in \mathbf{Z}$ and all the $\{h_{0,\mu}\}$ are essentially self–adjoint. Now, if $\alpha \in [0, 1)$ we have $\mu < 1$ if and only if $m = 0$. In this case, just as in the three dimensional case, the δ–potential affects only one of the angular momentum channels, namely the one associated with the smallest eigenvalue in the sense of the modulus (in three dimensions called the s–channel).

Let α be in $[0, 1)$. To construct the s.a. extensions of the operators $h_{0,\alpha}$ with domain $\mathcal{D}(h_{0,\alpha})$ given above, following 11, 2 we define the regular and irregular solutions of $h_{0,\alpha}\psi = 0$:

$$
\begin{aligned}
F_\alpha(r) &:= r^{\frac{1}{2}+\alpha} && \text{(regular)} \\
G_\alpha(r) &:= F_\alpha(r) \int_r^{r_0} dr' \, (F_\alpha(r'))^{-2} && \text{(irregular)}
\end{aligned}
$$

where $r_0 \in \mathbb{R}^+$ may be chosen arbitrary. The choice $r_0 = 1$ gives

$$
G_\alpha(r) = \begin{cases} \frac{1}{2\alpha}(r^{\frac{1}{2}-\alpha} - r^{\frac{1}{2}+\alpha}) & \alpha > 0 \\ -\sqrt{r}\ln r & \alpha = 0 \end{cases} .
$$

In order to explicitly construct the s.a. extensions of the $\{h_{0,\alpha}\}$ we take recourse to the following theorem, to be found in e.g. 55.

Theorem VI.1. *For $\alpha < 1$ the operator $h_{0,\alpha}$ with domain $\mathcal{D}(h_{0,\alpha})$ has a one–parameter family of s.a. extensions $\{h_{0,\alpha}(s)\}_{s\in\mathbf{R}}$ which are essentially s.a. on the domains*

$$
\mathcal{D}(h_{0,\alpha}(s)) = \{\psi \in \mathcal{D}(h^*_{0,\alpha}) \mid \lim_{r\downarrow 0} W(\psi_{\alpha,s}, \psi)(r) = 0\}.
$$

Here again W is the Wronskian and

$$
\psi_{\alpha,s} := G_\alpha + sF_\alpha.
$$

The choice $s = \infty$ gives the Friedrich's extension.

In appendix D we will prove the following

Theorem VI.2. *The integral kernel of the resolvent of $h_{0,\alpha}(s)$ for $\alpha < 1$ is given as*

$$
\begin{aligned}
&\frac{1}{h_{0,\alpha}(s) - k^2}(r, r') = \\
&\qquad = \frac{i\pi}{2}\sqrt{rr'}\, J_\alpha(kr_<)\, H^{(1)}_\alpha(kr_>) - A(k, \alpha; s) \cdot \sqrt{rr'}\, H^{(1)}_\alpha(kr)\, H^{(1)}_\alpha(kr') \ , \qquad \text{(VI.2)}
\end{aligned}
$$

where k satisfies $0 < \arg(k) < \pi$. $A(k, \alpha; s)$ *is given as:*

$$A(k, \alpha; s) = \frac{\pi}{2}\left[\left(\frac{k}{2}\right)^{-2\alpha}(2s\alpha - 1)\frac{1}{\sin\pi\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)} + \cot\pi\alpha - i\right]^{-1} \tag{VI.3}$$

Note that $\lim_{s\to\infty} A(k, \alpha; s) = 0$ such that with (VI.2) in the limit $s \to \infty$ we indeed recover the kernel of the resolvent of the Friedrich's extension (V.6). Taking the "bosonic limit" $\alpha \to 0$ one arrives at the expression:

$$A(k, 0; s) = \frac{\pi^2}{4}\left[\ln\frac{k}{2i} + \gamma + s\right]^{-1} \qquad (\gamma\text{: Euler's constant})$$

Combined with (VI.2) we have arrived at an expression for the resolvent of the δ–potential in the bosonic case, which agrees with 2.

We turn to the discussion of the bound states. It can be shown that the essential spectrum as a set is equal to $\mathbb{R}_0^+$ and that $\mathbb{R}^+$ is purely absolutely continuous. We therefore have to look for poles of the resolvent at energies $E_b = k_b^2 < 0$. By (VI.2) we have to search for the poles of $A(k, \alpha; s)$ as a function of k and by (VI.3) this gives the condition

$$k_b^{2\alpha} = 2^{2\alpha} e^{i\pi\alpha \bmod 2\pi i}\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)}(1 - 2\alpha s). \tag{VI.4}$$

The restriction $0 < \arg(k) < \pi$ now implies $0 < \arg(k^{2\alpha}) < 2\pi\alpha$. Applying this condition to (VI.4) leads to three different cases:

(i) $1 - 2\alpha s > 0$: Then (VI.4) may be solved to give

$$k_b = k_b(\alpha, s) = 2i\left(\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)}(1 - 2\alpha s)\right)^{\frac{1}{2\alpha}},$$

i.e. k_b is purely imaginary. The case $k_b = 0$ occurs when $\alpha \uparrow 1$. Note that this requires $0 \le s < \frac{1}{2}$. Hence there is exactly one bound state.

(ii) $1 - 2\alpha s < 0$: Since by definition $0 \le \alpha < 1$, it is easy to see that in this case there is no solution to equation (VI.4), only in the limit $\alpha \uparrow 1$ one obtains $k_b = 0$. This means that there is no bound state.

(iii) $1 - 2\alpha s = 0$: This immediately gives $k_b = 0$, which implies that there is no bound state.

For comparison we recall that in the three dimensional case the attractive δ–potential supports at most one bound state. A bound state exists if the particles obey bosonic statistics. For fermions there is none. In the present context a bound state always exists for $\alpha = 0$ (bosons) and disappears at the latest when $\alpha = 1$ (fermions). The fact that for $\alpha = 0$ there is always a bound state suggests that in two dimensions there is no such thing as a repulsive δ–potential.

Below we argue, that case (iii) corresponds to a zero energy resonance in three dimensional Schrödinger theory. Figure 1 gives a plot of $|k_b(\alpha, s)|$ as a function of s for various values of α in case (i), i.e. when $1 - 2\alpha s > 0$. To calculate the bound–state wave function ψ_b when $1 - 2\alpha s < 0$, we note that

$$\frac{1}{h_{0,\alpha}(s) - k^2} \approx \frac{1}{k_b^2 - k^2} P_b$$

for $k \approx k_b$ where P_b is the orthogonal projector onto the eigenspace of energy $E_b = k_b^2 < 0$. Since this pole of the resolvent is isolated, P_b may be calculated as a residue. The calculation is easy and shows that P_b is indeed one–dimensional and that the associated (normalized) eigenfunction ψ_b is given as

$$\psi_b(r; \alpha) = \left(\frac{2 \sin \pi\alpha}{\pi\alpha}\right)^{\frac{1}{2}} \sqrt{r\,|E_b|}\, K_\alpha(\,|k_b|\,r)$$

for $\alpha < 1$ and hence $|k_b(\alpha, s)| > 0$. For any fixed $\alpha < 1$ the wave function decays exponentially for large r, which guarantees square–integrability. In the limit $\alpha \uparrow 1$ the bound state wave function approaches $r^{-\frac{1}{2}}$ times a normalization constant, which is not square integrable. The case $\alpha = 1$ should therefore be considered as a resonance at $E = 0$.

We turn to a discussion of the resulting scattering theory. First we note that due to relation (VI.2) the difference of the resolvents for $h_{0,\alpha}(s)$ and the free Hamiltonian $h_{0,\alpha}(s = \infty)$ is a rank–one operator and therefore in particular trace class. Therefore the S–matrix exists and is unitary by the Kuroda–Birman theorem (see e.g. 32).

Theorem VI.3. *For given α and $E = k^2 > 0$ the outgoing states for the Hamiltonian $h_{0,\alpha}(s)$ are given as*

$$\phi^+(r; k, \alpha) = \phi_0(r; k, \alpha) + \frac{2i}{\pi} A(k, \alpha; s) \cdot \chi_0^+(r; k, \alpha)\ .$$

In particular the resulting partial wave amplitude and the phase shift take the form

$$f(k, \alpha; s) = \frac{4i}{\pi} \frac{A(k, \alpha; s)}{\sqrt{\pi i k}}\ , \tag{VI.5a}$$

$$e^{2i\delta(k,\alpha;s)} = 1 + \frac{4i}{\pi} A(k, \alpha; s)\ . \tag{VI.5b}$$

Proof. We start by rewriting the resolvent (VI.2) as

$$\frac{1}{h_{0,\alpha}(s) - k^2}(r, r') = \left(\phi_0(r_<; k, \alpha) + \frac{2i}{\pi} A(k, \alpha; s) \chi_0^+(r_<; k, \alpha)\right) \frac{i}{k} \chi_0^+(r_>; k, \alpha)\ . \tag{VI.6}$$

Now we observe that (see e.g. 2, p.37)

$$\lim_{\varepsilon \downarrow 0} \lim_{r' \to \infty} e^{-i(k+i\varepsilon)r'} \frac{1}{h_{0,\alpha}(s) - (k + i\varepsilon)^2}(r, r') \tag{VI.7}$$

must be proportional to the outgoing solution $\phi^+(r; k, \alpha)$. The claim now follows by applying (VI.7) to (VI.6). $\square$

In view of Levinson's theorem we would like to make the following comments on this last result: The limiting values of $\delta(k,\alpha;s)$ at $k \to 0^+$ and $k \to \infty$ are *a priori* determined up to an additive constant in $\pi \cdot \mathbb{Z}$, only. We shall choose these constants in such a way, that $\delta(k,\alpha;s)$ as a function of $k \in \mathbb{R}^+$ can be defined as a continuous function of k. Now, $\exp(i2\delta(k,\alpha;s))$ is of the form $(a(k,\alpha;s)+i)(a(k,\alpha;s)-i)^{-1}$. Here $a(k,\alpha;s)$ is a real–valued, continuous function of k. Explicitly

$$a(k,\alpha;s) = \frac{\pi}{2} A(k,\alpha;s)^{-1} + i$$

(This observation also shows that the right hand side of (VI.5b) is indeed unimodular.) Hence

$$\delta(k,\alpha;s) = \arg(a(k,\alpha;s) + i) \ .$$

Here the branch of $\arg(\cdot)$ is chosen such that $\delta(k,\alpha;s)$ is a continuous function of all its arguments. In particular we can achieve $\delta(k,\alpha;s) \in (0,\pi)$, for any k and fixed $\alpha < 1$ and s. For the cases, where there is a bound state, this is consistent with the attractive nature of the interaction. By the explicit form of $a(k,\alpha;s)$ it is easily seen, that

$$\begin{aligned} \delta(k,\alpha;s) &\xrightarrow{k\to 0^+} \pi \frac{1-\mathrm{sgn}(2\alpha s - 1)}{2} \\ \delta(k,\alpha;s) &\xrightarrow{k\to\infty} \pi\alpha \end{aligned}$$

The fraction $n := \frac{1-\mathrm{sgn}(2\alpha s-1)}{2}$ takes values in $\{0,1\}$ and is in fact nothing but the number of bound states given by (VI.4). We therefore have the following relation

$$\delta(0^+,\alpha;s) - \delta(\infty,\alpha;s) = \pi(n-\alpha) \ , \tag{VI.8}$$

which is *not* of the form one would naively expect in view Levinson's theorem V.2. In this context we note, that the statement of Levinson's theorem also fails in the non–anyonic three dimensional case, if a δ–interaction is considered. However, we obtain a formula similar to (V.26) relating the phase shift to the statistics parameter α.

By relations (IV.8) and (V.21), we obtain the following expression for the differential cross–section:

$$\frac{d\sigma}{d\theta} = \left| f(k,\alpha;s) \cdot e^{-i\pi\alpha} + f_{AB}(k,\theta) \right|^2 ,$$

where f_{AB} and f are given by expression (II.16) and (VI.5a), respectively. Inserting these explicit expressions, one obtains —away from the forward direction— the following relation

$$\frac{d\sigma}{d\theta} = \frac{1}{\pi k} \left| \sin\pi\alpha \ (\cot\theta - i) + e^{-i\pi\alpha} \frac{4i}{\pi} A(k,\alpha;s) \right|^2 .$$

So far, we have been unable to analyze this in a non–numerical way. Figures 2 and 3 show some numerical results of the differential cross–section as a function of the scattering angle θ. Displayed is the deviation from pure Aharonov–Bohm scattering which is asymptotically reached when $s \to \pm\infty$. Note the simultaneous cross over at $\theta = \pi\alpha$ for varying s in each of the plots with $s \neq \pm\infty$. In particular, the point corresponding to the cross over always lies on the graph ($s = \pm\infty$) associated with the Aharonov–Bohm cross–section. We like to point out that in both figures 2 and 3 only for $s = 0, 2$ and $\alpha = 0.5$ (semions) the differential cross–section actually vanishes for some scattering angle θ. In the other cases the cross–section becomes very small but is non–vanishing for all θ.

VII. THE SQUARE WELL POTENTIAL

Another explicitly solvable problem in scattering theory is given by the square well,

$$V(r) = \begin{cases} -V_0, & r \leq d \\ 0, & r > d \end{cases} \quad \text{with} \quad V_0 > 0.$$

The regular solution of the eigenvalue problem $(h_\mu - k^2)\psi = 0$ in the interval $0 < r \leq d$ is given by

$$\phi(r; k, \mu) = \left(\tfrac{k}{q}\right)^{\mu + \frac{1}{2}} \phi_0(r; q, \mu), \quad \text{where} \quad q = \sqrt{k^2 + V_0}.$$

For $r > d$ the eigensolutions of h_μ obey the free radial equation and hence the Jost solutions in this interval are just given by the free solutions,

$$\chi^{\pm}(r; k, \mu) = \chi_0^{\pm}(r; k, \mu), \quad r > d.$$

Making the usual requirement that all solutions are continuously differentiable we can evaluate the Wronskian $W(\chi^{\pm}, \phi)$ at $r = d$ and obtain the following expression for the Jost function,

$$F(k, \mu) = \left(\tfrac{k}{q}\right)^{\mu - \frac{1}{2}} \left[\chi_0^{+}(d; k, \mu)\phi_0(d; q, \mu - 1) - \tfrac{k}{q}\chi_0^{+}(d; k, \mu - 1)\phi_0(d; q, \mu)\right].$$

Here, we have used the common differentiation rules for Bessel and Hankel functions (see e.g. 54, 37). With the help of the Jost function we can now calculate the phase shift and the partial wave amplitude. We find

$$\tan\delta(k, \mu) = i\,\frac{F - \bar{F}}{F + \bar{F}} = \frac{qJ_\mu(kd)J_{\mu-1}(qd) - kJ_{\mu-1}(kd)J_\mu(qd)}{qY_\mu(kd)J_{\mu-1}(qd) - kY_{\mu-1}(kd)J_\mu(qd)}$$

$$f(k, \mu) = \frac{\bar{F} - F}{\sqrt{\pi i k}F} = \frac{-2}{\sqrt{\pi i k}}\,\frac{qJ_\mu(kd)J_{\mu-1}(qd) - kJ_{\mu-1}(kd)J_\mu(qd)}{qH_\mu^{(1)}(kd)J_{\mu-1}(pd) - kH_{\mu-1}^{(1)}(kd)J_\mu(qd)}.$$

Inserting the last expression into the formula (V.21) for the partial wave decomposition of the scattering amplitude we can calculate the differential cross sections for various parameters α. Note that for small momenta we can make use of the small energy behavior of the partial wave amplitude which is given by

$$k \to 0: \qquad f(k, \mu) = \mathcal{O}(k^{2\mu - 1}).$$

Thus, when calculating the scattering amplitude in the partial wave decomposition the partial wave amplitudes for higher angular momentum hardly contribute to the infinite sum in (V.21). According to our discussion in section IV we have to add the Aharonov–Bohm scattering amplitude f_{AB} in order to obtain the differential cross–section (see (IV.8)). A numerical example is displayed in figure 4. For different values of the statistics parameter α one obtains very different angular distributions of the scattered particles. One observes the interpolation between the bosonic ($\alpha = 0$) and the fermionic ($\alpha = 1$) cross section at energy $E = k^2 = 0.25$. Note that the singularities at the scattering angles $\theta = 0$ and $\theta = \pi$ for $\alpha \notin \mathbb{Z}$ enter due to the long range nature of the statistics. In fact, according to (IV.8) they result from adding the Aharonov-Bohm amplitude given in (II.16). For fermions ($\alpha = 1$) there is no scattering under an angle of $\theta = \pi/2$ as is well known. The figure for $\alpha = 0.5$ shows that a similar effect can take place for fractional statistics.

According to our discussion in section V the zeros of the Jost function F lie on the positive imaginary axis and determine the bound states. Thus, plotting the solutions of

$$F(ik, \mu) = 0, \quad k > 0$$

we get so called (real) *Regge trajectories* (see e.g. 43) displaying the μ dependence of the discrete spectrum of h_μ. Figure 5 provides a numerical example. In contrast to three dimensional scattering theory here every point on the Regge trajectories has a specific physical meaning because the statistics parameter α interpolates between the integer valued angular momentum channels.

VIII. CONCLUSION

We have demonstrated that it is possible to formulate nonrelativistic quantum scattering theories for two particles obeying anyon statistics. In particular we have proven the existence of the wave operators under certain assumptions concerning the interaction. For spherically symmetric potentials we gave criteria for completeness, i.e. unitarity of the S–matrix. These criteria turned out to be the same as in three dimensional Schrödinger theory. It remains an open problem to show completeness in the general case in a way comparable to Enss' method. To generalize the latter, we believe it is crucial to obtain a better analytic control of the propagator. (There is an extensive literature on a similar problem, namely to obtain a closed form of the Aharonov–Bohm propagator see e.g. 19.)

We extended the notion of differential cross–section to two dimensions and showed that in case of anyon statistics, the corresponding scattering amplitude consists of two parts. The first encodes the anyon statistics and resembles the Aharonov–Bohm amplitude, while the second is relevant to spectral analysis of the perturbed Hamiltonian. For spherically symmetric potentials we carried out this analysis by introducing Jost functions and showed how fractional statistics is equivalent to fractional angular momentum. We also showed that Levinson's theorem holds in the conventional case and gave the modifications necessary in the presence of a zero energy resonance. In the latter case we found that for positive angular momentum $\mu < 1$ the statistics parameter – independently of the detailed form of the short range potential – can be determined from the scattering phase, namely (compare (V.26))

$$\alpha \equiv \frac{1}{\pi}\Big(\delta(k=0,\alpha) - \delta(k=\infty,\alpha)\Big) \mod \mathbb{Z}.$$

We then applied our results to the following two examples. First we considered the δ–potential, where we extracted information regarding the point spectrum of the perturbed Hamiltonian from its resolvent and showed that the above relation between statistics parameter and scattering phase is also valid in this context(compare (VI.8)). Second we considered the square well potential, where we obtained similar information from the corresponding Jost function. In both situations we numerically evaluated the differential cross–section for non–integer values of the statistics parameter. It then became evident that fractional statistics is fundamentally different from Bose and Fermi statistics: The statistical gauge forces produce singularities of the differential cross–section in forward direction, showing the long range nature of these forces. Due to this fact the angular dependence of the statistical interaction will invariably show up in the scattering data. In particular this might cause a differential cross–section which is constant for bosons or fermions, but which becomes angular dependent for intermediate statistics. Our discussion of the δ–potential provides an example.

The results presented in this paper may also be relevant to the investigation of bulk properties of anyon matter, via the relation between the virial coefficients and the scattering data[9]. This we intend to discuss in a forthcoming publication.

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APPENDIX A: THE BUNDLE THEORETIC FORMULATION

In section II we used the physical picture of particles carrying flux–tubes in order to describe the anyon model. Anyons, however, are but a special case in a more general class of particles, called plektons, obeying neither Bose nor Fermi statistics. In this appendix we give a short review of plektonic systems, i.e. systems of n particles whose statistics is determined by a finite dimensional unitary representation of the braid group B_n. The appropriate mathematical tool to describe such particles is the concept of vector bundles. This will in particular provide the Hilbert space and a canonical "free" dynamics described by a certain Laplace operator. Our presentation will closely follow the one given in 40, 41, where, however, emphasis was mainly given to relativistic formulations using momentum space considerations. For other references on the bundle theoretic formulation see e.g. 25, 26. We continue to use $\mathbb{C}$ to describe the one–particle configuration space. Let $\mathbb{C}^{\times n}$ denote the n–fold product, viewed as the configuration space for n distinguishable particles. The set D_n in $\mathbb{C}^{\times n}$ is the set of all points $z = (z_1, z_2, \ldots, z_n)$ with $z_i = z_j$ for at least one pair of different indices. Any element σ of the permutation group S_n acts in an obvious way on $\mathbb{C}^{\times n}$ via $(z\sigma)_i = z_{\sigma(i)}$. We define the configuration space of n identical particles in two dimensions to be

$$^n\mathbb{C} := (\mathbb{C}^{\times n} \setminus D_n)/S_n \ ,$$

with points in it denoted by $\underline{z}$. Let $^n\tilde{\mathbb{C}}$ denote the universal covering space of $^n\mathbb{C}$ with points there being written as $\tilde{z}$ and be $\pi : {}^n\tilde{\mathbb{C}} \to {}^n\mathbb{C}, \tilde{z} \mapsto \pi(\tilde{z}) = \underline{z}$ the associated projection mapping.

It is well–known that the fundamental group of $^n\mathbb{C}$, denoted by $\pi_1(^n\mathbb{C})$, is isomorphic to the braid group B_n and that any element $b \in B_n$ acts in a standard way from the right $\tilde{z} \mapsto \tilde{z}b$ on the manifold $^n\tilde{\mathbb{C}}$. Furthermore, the universal covering $^n\tilde{\mathbb{C}} \mapsto {}^n\mathbb{C}$ can be viewed as a principal fiber bundle over $^n\mathbb{C}$ with structure group $\pi_1(^n\mathbb{C})$ and fiber $\pi^{-1}(\underline{z})$ for any $\underline{z} \in {}^n\mathbb{C}$. Given any finite dimensional unitary representation $b \mapsto \rho(b)$ of B_n in a Hilbert space F with scalar product $\langle \cdot \mid \cdot \rangle$ there is an associated hermitian vector bundle. This vector bundle $\mathcal{F}$ with base space $^n\mathbb{C}$ is given by $^n\tilde{\mathbb{C}} \times_{\rho, B_n} F$, which by definition is the set of orbits in $^n\tilde{\mathbb{C}} \times F$ under the following action of B_n on this space

$$b : (\tilde{z}, f) \longmapsto \big(\tilde{z}b, \rho(b^{-1})f\big) \ , \qquad \tilde{z} \in {}^n\tilde{\mathbb{C}},\ f \in F. \tag{A.A1}$$

$\mathcal{F}$ is a smooth fibered space with base $^n\mathbb{C}$ and fibers isomorphic to F. On the fiber over $\underline{z}$ there is a natural scalar product denoted by $\langle \cdot \mid \cdot \rangle_{\underline{z}}$. Furthermore there is a canonical measure $d\mu(\underline{z})$ on $^n\mathbb{C}$ inherited from the Lebesgue measure on $\mathbb{C}^{\times n}$. This defines a scalar product on the space $\Gamma_c(\mathcal{F})$ of smooth sections in $\mathcal{F}$ with compact support via

$$\langle \psi \mid \phi \rangle := \int\limits_{^n\mathbb{C}} \langle \psi(\underline{z}) \mid \phi(\underline{z}) \rangle_{\underline{z}}\, d\mu(\underline{z}) \ . \tag{A.A2}$$

By $\mathcal{L}^2(\mathcal{F})$ we denote the resulting Hilbert space completion. There is another Hilbert space equally well suited and canonically isomorphic to this space. Consider the set of maps

$\Psi : {}^n\tilde{\mathbb{C}} \mapsto F$ which are smooth with $\pi(\operatorname{supp}\Psi) \subset {}^n\mathbb{C}$ being compact and which satisfy the equivariance property

$$\Psi(\tilde{z}b) = \rho(b^{-1})\Psi(\tilde{z}), \qquad \forall\ \tilde{z} \in {}^n\tilde{\mathbb{C}},\ b \in B_n\ . \tag{A.A3}$$

For any two such functions Φ and Ψ we therefore have

$$\langle \Psi(\tilde{z}b) \mid \Phi(\tilde{z}b) \rangle = \langle \Psi(\tilde{z}) \mid \Phi(\tilde{z}) \rangle \qquad \forall\ \tilde{z} \in {}^n\tilde{\mathbb{C}},\ b \in B_n\ . \tag{A.A4}$$

Hence this expression depends on $\underline{z} = \pi(\tilde{z})$ only and so the integral over the base space makes sense and we may define a scalar product by

$$\langle \Psi \mid \Phi \rangle := \int\limits_{{}^n\mathbb{C}} \langle \Psi(\tilde{z}) \mid \Phi(\tilde{z}) \rangle \, d\mu(\underline{z}). \tag{A.A5}$$

The resulting Hilbert space obtained again by completion is denoted by $\mathcal{L}^2_{eq}({}^n\tilde{\mathbb{C}}, F)$ and there is a canonical isomorphism between $\mathcal{L}^2(\mathcal{F})$ and $\mathcal{L}^2_{eq}({}^n\tilde{\mathbb{C}}, F)$ (see e.g. 40, 41). Furthermore the canonical (flat) Levi–Civita connection on $\mathbb{C}^{\times n}$ induces a (flat) connection on ${}^n\mathbb{C}$ which in turn defines a hermitian connection ∇ on $\mathcal{F}$. The associated Bochner or generalized Laplacean (see e.g. 8) $\Delta = \nabla \circ \nabla$ on $\mathcal{L}^2(\mathcal{F})$ is what defines a free Hamiltonian $H_0 = -\Delta/2m$ for a system of n plektons, if $m > 0$ is taken to be the mass of one particle.

When the unitary representation ρ of B_n is one–dimensional one speaks of anyons. Any such representation is obviously abelian and can be shown to be of the form $b_k \mapsto \exp(i\alpha\pi)$ in terms of the generators $b_1, \dots b_{n-1} \in B_n$ for a fixed $\alpha \in [0, 2)$. This follows easily from the observation that all the b_k's are conjugate to each other. We denote the resulting line bundle by $\mathcal{F}_\alpha$. Another consequence of F being one–dimensional is that all anyonic line bundles are actually trivial. This observation was first made by J. S. Dowker[12] based on V. I. Arnold's result that $H^2({}^n\mathbb{C}, \mathbb{Z}) = 0$[6] and the classification theorem of Cartan, Kostant, Souriau and Isham. It was rediscovered by M. Gaberdiel[18] and is implicitly contained in 59 and 17. Another proof is given in 40, 41.

Let us now consider in more detail the Hilbert spaces constructed above when $n = 2$. We first introduce relative coordinates by considering the following transformation of $\mathbb{C}^{\times 2}$:

$$(z_1, z_2) \longmapsto \big(2^{-1}(z_1 + z_1), z_1 - z_2\big) = (z_{cen}, z_{rel}) \tag{A.A6}$$

The transposition in $S_2 \cong \mathbb{Z}_2$ obviously maps (z_{cen}, z_{rel}) into $(z_{cen}, -z_{rel})$. Therefore ${}^2\mathbb{C}$ is diffeomorphic to $\mathbb{C} \times \mathbb{C}^\star/\mathbb{Z}_2$. The first factor is the configuration space for the center of mass motion. The corresponding quantum mechanical discussion is analogous to ordinary multi particle systems, since it is not affected by the statistics. Therefore we will concentrate on the second factor and the associated quantum mechanical description. In the same way as $\mathbb{R}$ is the universal covering space of S^1, the universal covering space of $\mathbb{C}^\star/\mathbb{Z}_2 \cong \mathbb{R}^+ \times S^1$ is given by $\mathbb{R}^+ \times \mathbb{R}$ with the projection mapping

$$\pi : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}^+ \times S^1 \quad , \text{with} \qquad (r, \theta) \longmapsto (r, e^{2i\theta})\ . \tag{A.A7}$$

The action of B_2 on the universal covering is given in terms of its only generator b_1 as

$$(r,\theta) \longmapsto (r,\theta)b_1 := (r,\theta+\pi) \ . \tag{A.A8}$$

Choosing the representation $b_1 \mapsto \exp(-i\pi\alpha)$ for a fixed $\alpha \in [0,1]$ we are prepared to construct the line bundle $\mathcal{F}_\alpha$ and the Hilbert space $\mathcal{L}^2(\mathcal{F}_\alpha)$ or equivalently $\mathcal{L}^2_{eq}({}^n\tilde{\mathbb{C}}, F)$. The fact that the bundle $\mathcal{F}_\alpha$ is trivial is now reflected in the possibility to "pull–down" the theory from either of those $\mathcal{L}^2$ spaces onto the space of square integrable functions on ${}^n\mathbb{C}$ itself, denoted by $\mathcal{L}^2({}^n\mathbb{C})$. Explicitly there is the following unitary map from $\mathcal{L}^2({}^n\mathbb{C})$ onto $\mathcal{L}^2_{eq}({}^n\tilde{\mathbb{C}})$ given as

$$\psi \in \mathcal{L}^2({}^n\mathbb{C}) \longmapsto \Psi \in \mathcal{L}^2_{eq}({}^n\tilde{\mathbb{C}}) \qquad \text{with} \qquad \Psi(\tilde{z}) = e^{i\pi\alpha\cdot\theta(\tilde{z})}\cdot\psi\left(\pi(\tilde{z})\right) \ ,$$

where $\theta(\tilde{z})$ is a real valued, continuous function of $\tilde{z}$ induced by the polar angle. With the help of this unitary mapping, we can pull–down the Bochner Laplacean from $\mathcal{L}^2(\mathcal{F}_\alpha)$ to obtain an expression for the free Hamiltonian on $\mathcal{L}^2({}^n\mathbb{C})$. The result of this procedure is precisely the Hamiltonian (II) we introduced in section II on physical grounds.

Let it be noted that α was introduced as a parameter fixing the representation $B_2 \ni b_1 \mapsto \exp(-i\pi\alpha)$ and is hence determined up to an additive even integer, only. Therefore we loose no generality if we restrict α to the interval $[0,2)$. In fact, we have $H_0(\alpha+2) = \exp(-i2\theta)H_0(\alpha)\exp(i2\theta)$, which is a gauge transformation reflecting the arbitrariness we have. On the other hand, time inversion causes α to change sign, i.e. if T denotes the anti–unitary operator of time inversion, we have $H_0(-\alpha) = TH_0(\alpha)T$. However $H_0(-\alpha)$ is in turn equivalent to $H_0(2-\alpha)$ and consequently the anyon models with $\alpha \in [0,1]$ are connected to those with $\alpha \in [1,2]$ by time reversal. This justifies the restriction of α to the interval $[0,1]$.

Furthermore we remark that $\mathcal{L}^2(\mathbb{C}^\star/\mathbb{Z}_2, rdrd\theta)$ is unitary equivalent to the square integrable *symmetric* functions on the punctured plane $P_+\mathcal{L}^2(\mathbb{C}^\star, \frac{1}{2}rdrd\theta)$ and therefore the case $\alpha = 0$ corresponds to the bosonic case (since $H_0(\alpha=0) = -\triangle$). If $\alpha = 1$ our model is equivalent to $H_0(\alpha=1)$ acting on $P_+\mathcal{L}^2(\mathbb{C}^*)$, which is in turn unitary equivalent to $-\triangle$ on $P_-\mathcal{L}^2(\mathbb{C}^*, \frac{1}{2}rdrd\theta)$, the Hilbert space of square integrable, *anti–symmetric* functions on the punctured plane. Hence we call anyons obeying statistics corresponding to $\alpha = 1$ fermions.

There is also a geometric description of the universal covering space essentially as the Riemann surface of the function $\log(z)$, which is useful for the understanding of the Hilbert space constructions made above and which goes as follows. Take an infinite set of upper and lower half planes in $\mathbb{C}$ and glue them together alternatively along the positive and negative axes respectively. In fact, consider an element f in $C^\infty_{0,\alpha}(\mathbb{H}\setminus\{0\})$. Then it is easy to see that this f uniquely defines a function Ψ in $C^\infty_{0,eq}({}^n\tilde{\mathbb{C}}, F_\alpha)$ and conversely any function Ψ in $C^\infty_{0,eq}({}^n\tilde{\mathbb{C}}, F_\alpha)$ defines a unique element f in $C^\infty_{0,\alpha}(\mathbb{H}\setminus\{0\})$. Furthermore this correspondence is linear and the scalar products are the same under this correspondence. This establishes the desired isomorphism between $\mathcal{H}_\alpha$ and $\mathcal{L}^2_{eq}({}^n\tilde{\mathbb{C}}, F_\alpha)$ and hence also $\mathcal{L}^2(\mathcal{F}_\alpha)$. This isomorphism also extends to the various Laplace operators. Indeed, note first that locally ${}^n\tilde{\mathbb{C}}$ looks like $\mathbb{C}^{\times n}$, so there is a natural Laplace operator $\tilde{\Delta}$ on $C^\infty({}^n\tilde{\mathbb{C}})$, the space of smooth functions on ${}^n\tilde{\mathbb{C}}$. This operator is easily seen to map $C^\infty_{0,eq}({}^n\tilde{\mathbb{C}}, F_\alpha)$ into itself and hence defines an operator on $\mathcal{L}^2_{eq}({}^n\tilde{\mathbb{C}}, F_\alpha)$. Furthermore this operator also corresponds to the Laplace operator on

$C_0^\infty(\mathbb{H}\backslash\{0\})$ under the above correspondence $f \leftrightarrow \Psi$. Finally it is easy to see that under the canonical isomorphism between $\mathcal{L}^2_{eq}({}^n\tilde{\mathbb{C}}, F_\alpha)$ and $\mathcal{L}^2(\mathcal{F}_\alpha)$ referred to above $\tilde{\Delta}$ corresponds to the Bochner Laplacean $\Delta = \nabla \circ \nabla$.

APPENDIX B: SOME ESTIMATES

This appendix is devoted to a proof of the lemmas III.2 and III.3.

Proof of lemma III.2: We start with a proof of lemma III.2. For the moment let $0 < \epsilon < 1$ be arbitrary. Since $|e^{ix} - 1| \leq \min(2, |\, x\,|)$, we have $|e^{ix} - 1| < 2\,|x|^{\,\epsilon}$ for all $x \in \mathbb{R}$. This gives

$$
\begin{aligned}
|I_\alpha(\rho,\chi) - I_\alpha(0,\chi)| &\leq \left| \int_{-\infty}^{+\infty} dy\, \left(e^{i\rho\cosh y} - 1\right) \frac{e^{-\alpha y}}{1 - e^{-2y-2i\chi}} \right| \\
&\leq 2\rho^\epsilon \int_0^\infty dy\, \cosh^\epsilon y \left| \frac{e^{-\alpha y}}{1 - e^{-2y-2i\chi}} + \frac{e^{\alpha y}}{1 - e^{2y-2i\chi}} \right| \ .
\end{aligned}
$$

To proceed further,we split the integral into a part from 0 to 1 and the rest. This gives

$$
|I_\alpha(\rho,\chi) - I_\alpha(0,\chi)| \leq 2\rho^\epsilon \left(\cosh(1)\cdot I_1(\chi) + I_2(\chi)\right) \ .
$$

For I_1 we have an expression involving the sum of three integrals $I_{1,1}$, $I_{1,2}$ and $I_{1,3}$, given by

$$
\begin{aligned}
I_1(\chi) &= I_{1,1}(\chi) + I_{1,2}(\chi) + I_{1,3}(\chi) = \\
&= \int_0^1 dy \left| \frac{e^{-\alpha y} - 1}{1 - e^{-2y-2i\chi}} \right| + \int_0^1 dy \left| \frac{e^{\alpha y} - 1}{1 - e^{2y-2i\chi}} \right| + \\
&\quad + \int_0^1 dy \left| \frac{1}{1 - e^{-2y-2i\chi}} + \frac{1}{1 - e^{2y-2i\chi}} \right| \ .
\end{aligned}
$$

For I_2 we have the expression

$$
I_2(\chi) = \int_1^\infty dy\, \cosh^\epsilon y \left(\left| \frac{e^{-\alpha y}}{1 - e^{-2y-2i\chi}} \right| + \left| \frac{e^{\alpha y}}{1 - e^{2y-2i\chi}} \right| \right) \ .
$$

The aim is to estimate these quantities for $\chi \in [-\pi, +\pi]$. To estimate $I_{1,1}(\chi)$ we use the following little estimates:

$$
\left|1 - e^{-\alpha y}\right| \leq 2y \qquad \text{for } y > 0 \text{ and } \alpha \in (0,1): \qquad \text{(B.B1)}
$$

$$
\left|1 - e^{-2y-2i\chi}\right| \geq \left|1 - e^{-2y}\right| \qquad \text{for } y \in \mathbb{R}: \qquad \text{(B.B2)}
$$

This gives

$$
I_{1,1}(\chi) \leq \int_0^1 dy\, \frac{\alpha y}{1 - e^{-2y}} < C \ .
$$

$I_{1,2}(\chi)$ is estimated similarly, if one replaces the estimate (B.B1) by

$$
0 \leq e^{\alpha y} - 1 \leq 2e^2 y\ , \qquad \text{for } 0 \leq y \leq 1 \text{ and } \alpha \in (0,1)\ .
$$

To estimate $I_{1,3}(\chi)$ for $\chi \in [-\pi, +\pi]$ we first observe that for $\min(|\chi|, \pi - \chi, \pi + \chi) \geq \frac{\pi}{4}$ it is obviously bounded. Hence it suffices to estimate the three remaining cases $|\chi| \leq \frac{\pi}{4}$, $\frac{3}{4}\pi \leq \chi \leq \pi$ or $-\pi \leq \chi \leq -\frac{3}{4}\pi$. We only consider the first case, since the other two cases may be discussed with similar arguments. So let $|\chi| \leq \frac{1}{4}\pi$ and add and subtract $(\pm 2y + 2i\chi)^{-1}$ in the integrand of $I_{1,3}(\chi)$. This way we can estimate $I_{1,3}$ by a sum of three integrals, which we will denote by $\hat{I}_i$.

$$
\begin{aligned}
I_{1,3}(\chi) \leq \hat{I}_1(\chi) + \hat{I}_2(\chi) + \hat{I}_3(\chi) = \\
= \int_0^1 dy \left| \frac{1}{1 - e^{-2y-2i\chi}} - \frac{1}{2y + 2i\chi} \right| \qquad & + \int_0^1 dy \left| \frac{1}{1 - e^{2y-2i\chi}} - \frac{1}{-2y + 2i\chi} \right| \\
& + \int_0^1 dy \left| \frac{1}{2y + 2i\chi} + \frac{1}{-2y + 2i\chi} \right|
\end{aligned}
$$

To estimate $\hat{I}_1(\chi)$ and $\hat{I}_2(\chi)$ we note that the function

$$
G(\zeta) = \frac{1}{1 - e^{-\zeta}} - \frac{1}{\zeta}
$$

is obviously analytic in $\left\{ \zeta \in \mathbb{C} \mid |\mathrm{Im}(\zeta)| \leq \frac{\pi}{2} \right\}$ except possibly at the origin $\zeta = 0$. However, $G(\zeta = 0) = 1$ and hence by Riemann's theorem, $G(\zeta)$ is an analytic function in that domain. In particular $G(\zeta)$ is bounded on every compact subset and this gives the boundedness of $\hat{I}_1(\chi)$ and $\hat{I}_2(\chi)$ for $\chi \in [-\frac{1}{4}\pi, \frac{1}{4}\pi]$. $\hat{I}_3(\chi)$ can be estimated as follows

$$
\hat{I}_3(\chi) \leq \int_0^1 dy \, \frac{|\chi|}{\chi^2 + y^2} \leq \int_0^\infty d\eta \, \frac{1}{1 + \eta^2} < \infty \; .
$$

This concludes the estimate for $I_1(\chi)$ and it remains to estimate $I_2(\chi)$. By (B.B2) we have

$$
I_2(\chi) \; \leq \; \int_1^\infty dy \, 2 \, e^{\epsilon y} \left(\frac{e^{-\alpha y}}{1 - e^{-2y}} + \frac{e^{\alpha y}}{e^{2y} - 1} \right) \; .
$$

Now this integral is finite whenever $\epsilon < \min(\alpha, 1 - \alpha)$. □

Proof of lemma III.3: We will consider the third partial derivatives w.r.t. E only, since this is actually the case we will need and since the other cases may be discussed similarly. By interchanging integration and differentiation we formally have

$$
\frac{\partial^n}{\partial E^n} \, v_\alpha^{m,m'}(E, E') = \int\limits_{\mathbb{H} \setminus \{0\}} \frac{\partial^n}{\partial E^n} \left[\overline{\phi_{\alpha;m,E}(z)} \, V(z)^2 \, \phi_{\alpha;m',E'}(z) \right] d\mu(z) \tag{B.B3}
$$

for $n = 1, 2, 3$. This is permitted provided the integrand is a measurable function in z, E and E', bounded in the sense of the modulus by a function which is in $\mathcal{L}^1$ w.r.t z and with uniform bounds in E and E' in compact sets in $(0, \infty)$. To prove this claim it obviously suffices to replace $\frac{\partial}{\partial E}$ by $\frac{\partial}{\partial k}$ with $E = k^2$ in (B.B3). By the explicit form of $\phi_{\alpha;m,E}$ given by (II), for fixed α, m and m' we therefore have to estimate the product

$$
|\partial_k^n \, J_\mu(kr)| \cdot |J_\mu(k'r)| \cdot |V(r, \theta)|^2 \; , \qquad \text{for } n = 1, 2, 3 \; .
$$

Using the formula to be found in[54] or[37]

$$\frac{d}{dk} J_\sigma(kr) \;=\; \frac{r}{2}\,\left(J_{\sigma-1}(kr) - J_{\sigma+1}(kr)\right) \tag{B.B4}$$

it follows by iteration that it suffices to estimate

$$r^n \,|J_{\mu+l}(kr)| \,\cdot\, |J_{\mu'}(k'r)| \,\cdot\, |V(r,\theta)|^{\,2} \tag{B.B5}$$

for $-n \le l \le n$ with $0 \le n \le 3$. Let the compact set in question be given by $E_1 \le E, E' \le E_2$ such that $k_1 \le k,\, k' \le k_2$. Using the well known estimates

$$\begin{array}{lll} a) & |J_\sigma(kr)| \,\le C(\sigma) & \text{for } |kr| \,\le 1 \\ b) & |J_\sigma(kr)| \,\le C(\sigma)(kr)^{-\frac{1}{2}} & \text{for } |kr| \,\ge 1 \end{array}$$

we see that (B.B5) with $k_1 \le k,\, k' \le k_2$ is bounded by

$$C(\mu,\mu')\cdot r^n \,|V(r,\theta)|^{\,2} \le C(\mu,\mu')\cdot \max(1,k_2^{-1})^3 \,|V(r,\theta)|^{\,2}$$

for $r \le k_2^{-1}$ and by

$$C(\mu,\mu')\cdot r^{n-1}k_1^{-1}\,|V(r,\theta)|^{\,2} \le C(\mu,\mu')\cdot \max(k_1^2,k_1^{-1})\cdot r^2\,|V(r,\theta)|^{\,2}$$

for $r \ge k_1^{-1}$. Obviously (B.B5) is bounded for r in the interval (k_2^{-1}, k_1^{-1}) uniformly for $k_1 \le k, k' \le k_2$. This proves the claim and by our previous remark this concludes the proof. □

APPENDIX C: BOUNDS FOR JOST FUNCTIONS

In this section we will prove the estimates (V.11b),(V.17), (V.11a) and (V.16) for the functions ϕ and $\chi^\pm$ and their power series coefficients ϕ^n and $\chi^{\pm,n}$. Also we will show that the analyticity properties for ϕ and $\chi^\pm$ also extend to their derivatives with respect to r. We start with two preparations. Observe first that (V.11b) and (V.17) hold when $V = 0$, i.e. for $\phi = \phi_0$ and $\chi^\pm = \chi_0^\pm$ respectively. Indeed for ϕ_0 (V.11b) follows from the relations (see e.g.[54,37])

$$J_\mu(z) \underset{z\to 0}{\longrightarrow} C(\mu) z^\mu \quad \text{and} \quad J_\mu(z) \underset{|z|\to\infty}{\longrightarrow} \frac{\cos(z-\frac{\pi}{2}(2\mu+1))}{\sqrt{\frac{\pi}{2}z}}.$$

For $\chi_0^\pm$ the equation (V.17) follows from the following asymptotic behavior

$$H_\mu^\pm(z) \underset{z\to 0}{\longrightarrow} C(\mu) z^\mu \quad \text{and} \quad H_\mu^\pm(z) \underset{z\to\infty}{\longrightarrow} \frac{e^{\pm i(z-\frac{\pi}{2}\mu-\frac{1}{4})}}{\sqrt{\frac{\pi}{2}z}}.$$

Secondly for g_0 (see (V.7)) one establishes analogously

$$0 \le r' \le r\ :\quad |g_0(r,r';k,\mu)| \le C(\mu)\, e^{|\,\mathrm{Im}(k)|(r-r')} \left(\tfrac{r}{1+|k|r}\right)^{\mu+\frac{1}{2}} \left(\tfrac{r'}{1+|k|r'}\right)^{-\mu+\frac{1}{2}}, \tag{C.C1}$$

$$0 \le r \le r'\ :\quad |g_0(r,r';k,\mu)| \le C(\mu)\, e^{|\,\mathrm{Im}(k)|r-\mathrm{Im}(kr')} \left(\tfrac{r}{1+|k|r}\right)^{\mu+\frac{1}{2}} \left(\tfrac{r'}{1+|k|r'}\right)^{-\mu+\frac{1}{2}}. \tag{C.C2}$$

Now (V.11a) is proved by induction on n as follows. Define A_n by

$$|\phi_n(r;k,\mu)| \;=\; e^{|\,\mathrm{Im}(k)|r}\,\left(\tfrac{r}{1+|k|r}\right)^{\mu+\frac{1}{2}}\,A_n(r;k,\mu)$$

and it suffices to show that

$$A_n(r;k,\mu) \le |k|^{\mu+\frac{1}{2}}\,\frac{C(\mu)^{n+1}}{n!}\left[\int_0^r dr'\,\frac{r'|V(r')|}{1+|k|r'}\right]^n,$$

where $C(\mu)$ is the maximum of the $C(\mu)$ in (V.11b) and the one in (C.C1). By the preparatory remarks (V.11a) holds for $n=0$, i.e. for $\phi_0=\phi_{n=0}$. To perform the induction step, by construction we have

$$\phi_n(r;k,\mu) = \int_0^r dr'\,g_0(r,r';k,\mu)V(r')\phi_{n-1}(r';k,\mu),$$

which gives the inequality

$$A_n(r;k,\mu) \le C(\mu)\int_0^r dr'\,\frac{r'|V(r')|}{1+|k|r'}\,A_{n-1}(r';k,\mu),$$

from which the induction step follows easily. This completes the proof of (V.11a) and (V.11b) follows from it by summation.
The proof of the bounds (V.16) and (V.17) is similar and we will consider the case χ^- only. Now write

$$|\chi_n^-(r;k,\mu)| = e^{\mathrm{Im}(kr)}\left(\tfrac{r}{1+|k|r}\right)^{-\mu+\frac{1}{2}}B_n(r;k,\mu),$$

such that it suffices to prove

$$B_n(r;k,\mu) \le \frac{C(\mu)^{n+1}}{n!}\,|k|^{-\mu+\frac{1}{2}}\left(\int_r^\infty dr'\,\frac{r'|V(r')|}{1+|k|r'}\right)^n.$$

By the preparatory remarks, this inequality is valid for $n=0$. To make the induction step, we note that by construction

$$\chi_n^-(r;k,\mu) = \int_r^\infty dr'\,g_0(r,r';k,\mu)\,\chi_{n-1}^-(r';k,\mu).$$

Hence one has the estimate

$$B_n(r;k,\mu) \le C(\mu)\int_r^\infty dr'\,\frac{r'|V(r')|}{1+|k|r'}\,B_{n-1}(r';k,\mu),$$

from which the induction step easily follows. This completes the proof of (V.16) and (V.17) follows by summation.

APPENDIX D: THE RESOLVENT OF THE δ–POTENTIAL

Proof of theorem VI.2: For any $\eta \in \mathcal{L}^2(\mathbb{R}^+)$ define

$$\psi := \frac{1}{h_{0,\alpha}(s) - k^2}\,\eta\ ,$$

using the integral kernel given by (VI.2). It is easily seen that ψ is well–defined. In order to prove the theorem we shall show that (VI.2) formally defines a resolvent, i.e. we have

$$\left(-\frac{\partial^2}{\partial r} - \frac{\alpha^2 - \frac{1}{4}}{r^2}\right)\psi - k^2\psi = \eta\ . \tag{D.D1}$$

That this "resolvent property" is formally satisfied, can easily be verified with the aid of the well–known formulae for the first derivatives of Bessel functions. In particular one uses the relations:

$$C'_\alpha = \frac{\alpha}{z}C_\alpha - C_{\alpha+1} \qquad C'_{\alpha+1} = C_\alpha - \frac{\alpha+1}{z}C_{\alpha+1} \qquad \text{for}\quad C_\alpha = J_\alpha \text{ or } H_\alpha.$$

Furthermore we shall show that ψ satisfies the boundary condition in (VI.1), i.e.

$$\lim_{r\downarrow 0} W(\psi_{\alpha,s},\psi)(r) = 0\ . \tag{D.D2}$$

For convenience we introduce the following notation:

$$I_<(r) := \int_0^r dr'\,\sqrt{r'}J_\alpha(kr')\eta(r') \qquad\qquad I := \int_0^\infty dr'\,\sqrt{r'}H^{(1)}_\alpha(kr')\eta(r')$$
$$I_>(r) := \int_r^\infty dr'\,\sqrt{r'}H^{(1)}_\alpha(kr')\eta(r')$$

ψ can now be cast into the form

$$\psi(r) = \sqrt{r}\left[\frac{i\pi}{2}H^{(1)}_\alpha(kr)I_<(r) + \frac{i\pi}{2}J_\alpha(kr)I_>(r) - A(k,\alpha;s)H^{(1)}(kr)I\right]\ . \tag{D.D3}$$

To verify the boundary condition (D.D2) we note that

$$\psi_{\alpha,s}(r) = \frac{1}{2\alpha}\cdot r^{\frac{1}{2}-\alpha} + \tilde{s}\cdot r^{\frac{1}{2}+\alpha} \qquad \text{with} \qquad \tilde{s} = s - \frac{1}{2\alpha}$$

$$\text{giving} \qquad \psi'_{\alpha,s}(r) = \left(\frac{1}{4\alpha} - \frac{1}{2}\right)\cdot r^{-\frac{1}{2}-\alpha} + \left(\frac{\tilde{s}}{2} + \tilde{s}\alpha\right) r^{-\frac{1}{2}+\alpha}$$

$$\text{and hence} \qquad W(\psi_{\alpha,s},\psi)(r) = B'(r)\left(\frac{1}{2\alpha}\cdot r^{1-\alpha} + \tilde{s}r^{1+\alpha}\right) + B(r)\left(\frac{1}{2}\cdot r^{-\alpha} - \alpha\tilde{s}r^{\alpha}\right)\ ,$$

where $B(r)$ denotes the quantity in the square brackets in (D.D3). In order to take the limit, we remark that $z^{-\alpha}J_\alpha(z)$ is an analytic function for any value of $\alpha \in \mathbb{R}$. From its power series expansion at $z = 0$ (see e.g.[37]) one obtains the following asymptotic relations when $\alpha > 0$:

$$\begin{aligned}
\lim_{r\downarrow 0} r^{-\alpha} J_\alpha(kr) &= \iota_\alpha(k) := \left(\frac{k}{2}\right)^{\alpha} \frac{1}{\Gamma(1+\alpha)} \\
\lim_{r\downarrow 0} \quad r^{\alpha} Y_\alpha(kr) &= \gamma_\alpha(k) := -\left(\frac{2}{k}\right)^{\alpha} \frac{1}{\sin\pi\alpha} \frac{1}{\Gamma(1-\alpha)} \\
\lim_{r\downarrow 0} r^{\alpha} Y_{-\alpha}(kr) &= \gamma_{-\alpha}(k) := -\left(\frac{k}{2}\right)^{-\alpha} \frac{1}{\tan\pi\alpha} \frac{1}{\Gamma(1-\alpha)}
\end{aligned}$$

For the definition of the integral kernel (VI.2) we restricted the Bessel functions to the sheet given by $0 < \arg(k) < \pi$. Since the irrational powers of k in the above expressions stem from the asymptotic behavior of the Bessel functions, they have to be evaluated on the same sheet. Bearing this in mind, we are prepared to take the limit $r \to 0^+$, which will make most of the terms in (D.D3) disappear. Since $|I_<(r)| \sim C \cdot r^{\alpha+1}$ and $I_> \sim I$ for $r \ll 1$, the surviving terms are:

$$\begin{aligned}
\lim_{r\downarrow 0} W(\psi_{\alpha,s}, \psi)(r) = I \cdot \lim_{r\downarrow 0} \Bigg[&\frac{i\pi}{2} r^{-\alpha} J_\alpha(kr) + \tilde{s} \cdot k \cdot A(k,\alpha;s) r^{1+\alpha} H^{(1)}_{\alpha+1}(kr) \\
&+ \frac{k}{2\alpha} A(k,\alpha;s) r^{1-\alpha} \left(H^{(1)}_{\alpha+1}(kr) - \frac{2\alpha}{kr} H^{(1)}_{\alpha}(kr) \right) \Bigg]
\end{aligned}$$

Now, by a standard theorem for Bessel functions, we have $H^{(1)}_{\alpha+1}(kr) - \frac{2\alpha}{kr} H^{(1)}_\alpha(kr) = -H^{(1)}_{\alpha-1}(kr)$. The boundary condition (D.D2) therefore implies

$$A(k,\alpha;s) = \frac{i\pi}{2} \iota_\alpha(k) \left[\iota_\alpha(k) - 2i\tilde{s}\alpha \cdot \gamma_\alpha + \frac{ik}{2\alpha} \gamma_{\alpha-1} \right]^{-1} .$$

Substituting $\tilde{s} = s - \frac{1}{2\alpha}$ and the expressions for ι_α, γ_α and $\gamma_{\alpha-1}$ we see that $A(k,\alpha;s)$ is given by (VI.3), thus completing the proof of theorem VI.2. □

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APPENDIX: List of Figures

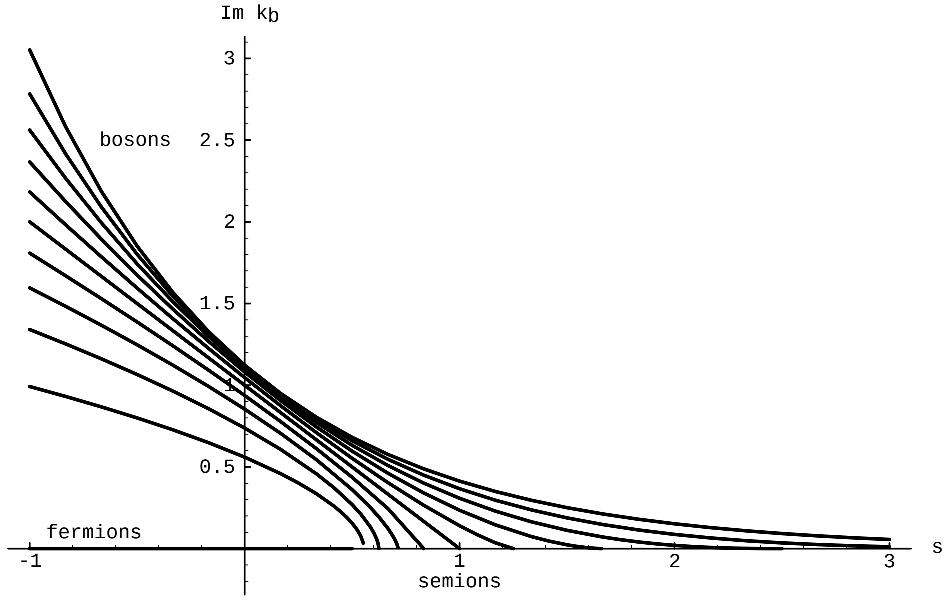

FIG. 1. The diagram shows the location of the square root of the bound state energy k_b on the imaginary axis as a function of the coupling parameter s for ten different, equispaced values of the statistical parameter α. "bosons" labels the graph corresponding to $\alpha = 0$, "semions" corresponds to $\alpha = \frac{1}{2}$, the graph intersecting the s–axis at 1 and "fermions" corresponds to $\alpha = 1$. The latter case does not correspond to a bound state.

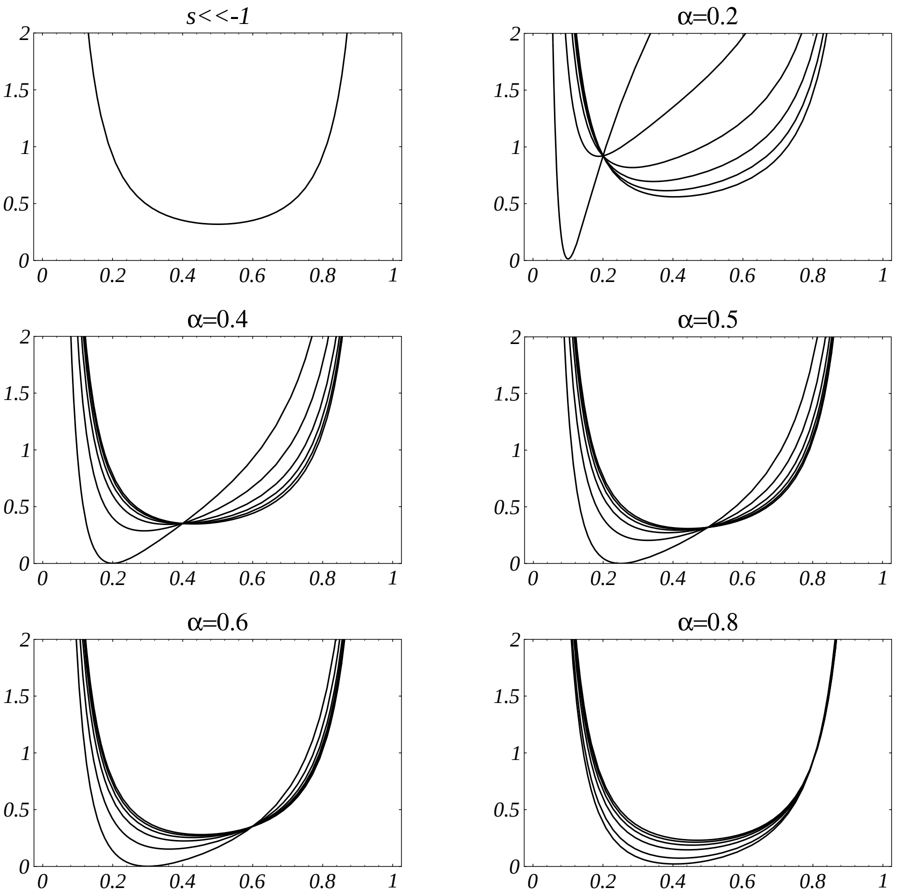

FIG. 2. The above figures show the “normalized” differential cross–section $\frac{1}{\sin^2 \pi\alpha}\frac{d\sigma}{d\theta}$ as a function of the scattering angle θ in units of π. The normalization factor is chosen such that the asymptote in the limit $s \to -\infty$ (i.e. when the interaction is turned off, giving the Friedrich's extension of $h_{0,\alpha}$) becomes independent of α. The latter is shown in the first plot (top left) and corresponds to Aharonov–Bohm scattering. The other five plots show a family of graphs corresponding to the following coupling strengths $s = -10, -8, \ldots, 0$, $k = 1$ and the given values for α. For small values of θ the graphs for different s are ordered from left to right starting with $s = 0$.

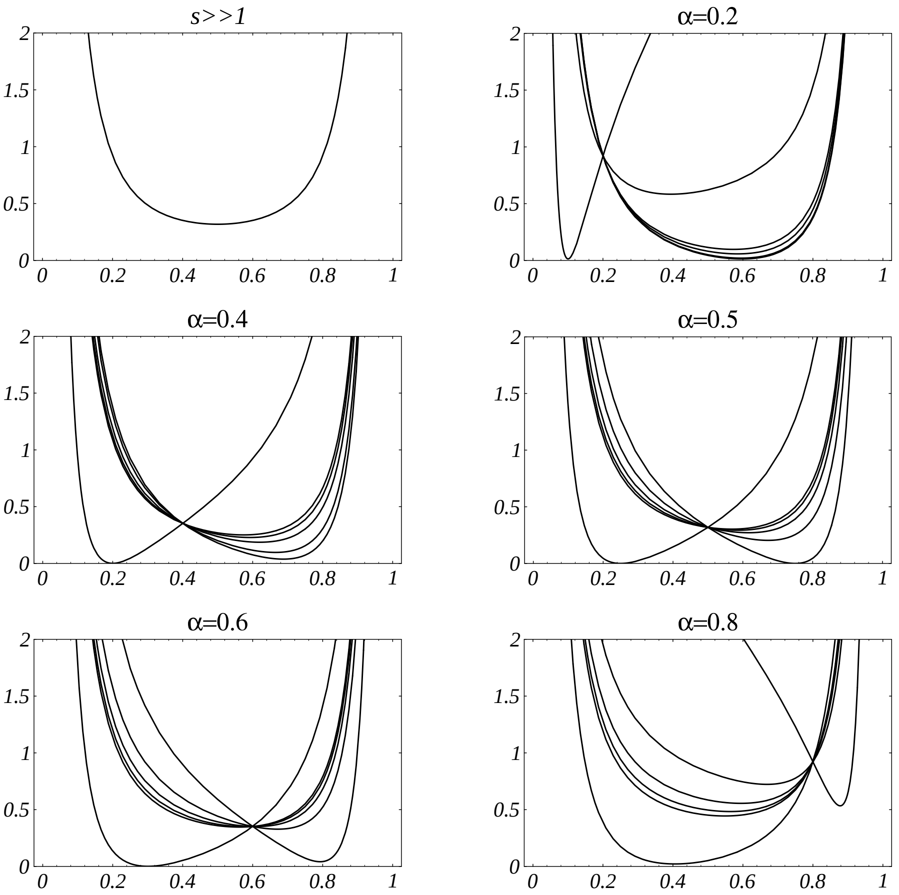

FIG. 3. Similar to figure 2 the above plots show the differential cross–section $\frac{1}{\sin^2 \pi\alpha}\frac{d\sigma}{d\theta}$ as a function of the scattering angle θ in units of π. But now only positive s values are considered. The first plot (top left) shows the limiting case $s \to \infty$ corresponding again to Aharonov–Bohm scattering. The other five plots display a family of graphs with coupling strengths $s = 0, 2, \ldots, 10$, $k = 1$ and the given values for α. Here the ordering of graphs is different from figure 2. For small values of θ the leftmost graph always corresponds to $s = 0$ but the ordering is reversed once at a certain value $s = s(\alpha)$.

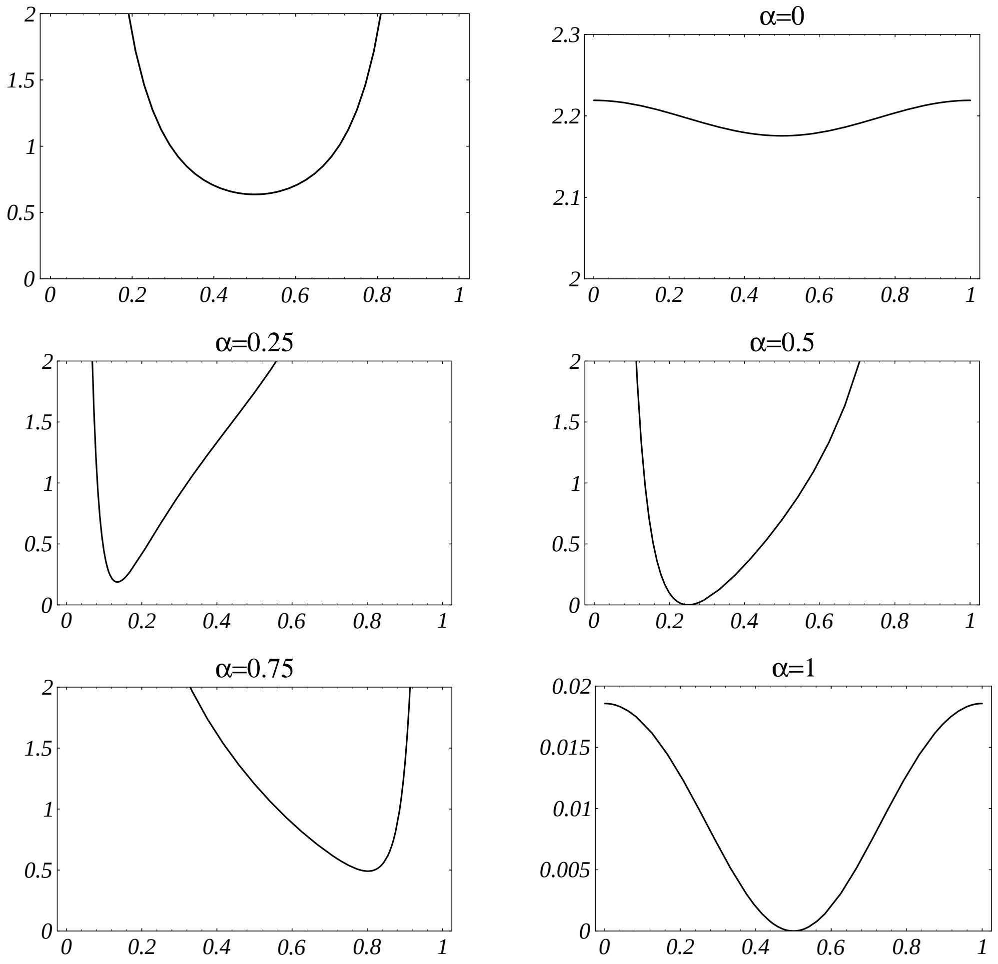

FIG. 4. Displayed are the differential cross sections $\frac{d\sigma}{d\theta}$ per unit length (y-axis) in dependence of the scattering angle θ in units of π (x-axis). The first figure (top left) shows the symmetrized Aharonov-Bohm cross-section divided by $\sin^2 \pi\alpha$ for non-integral values of α. The other graphs depict the differential cross-section when the statistical interaction as well as the square well potential are present. The depth of the square well is chosen to be $V_0 = 25$ and the radius is set to $d = 1$.

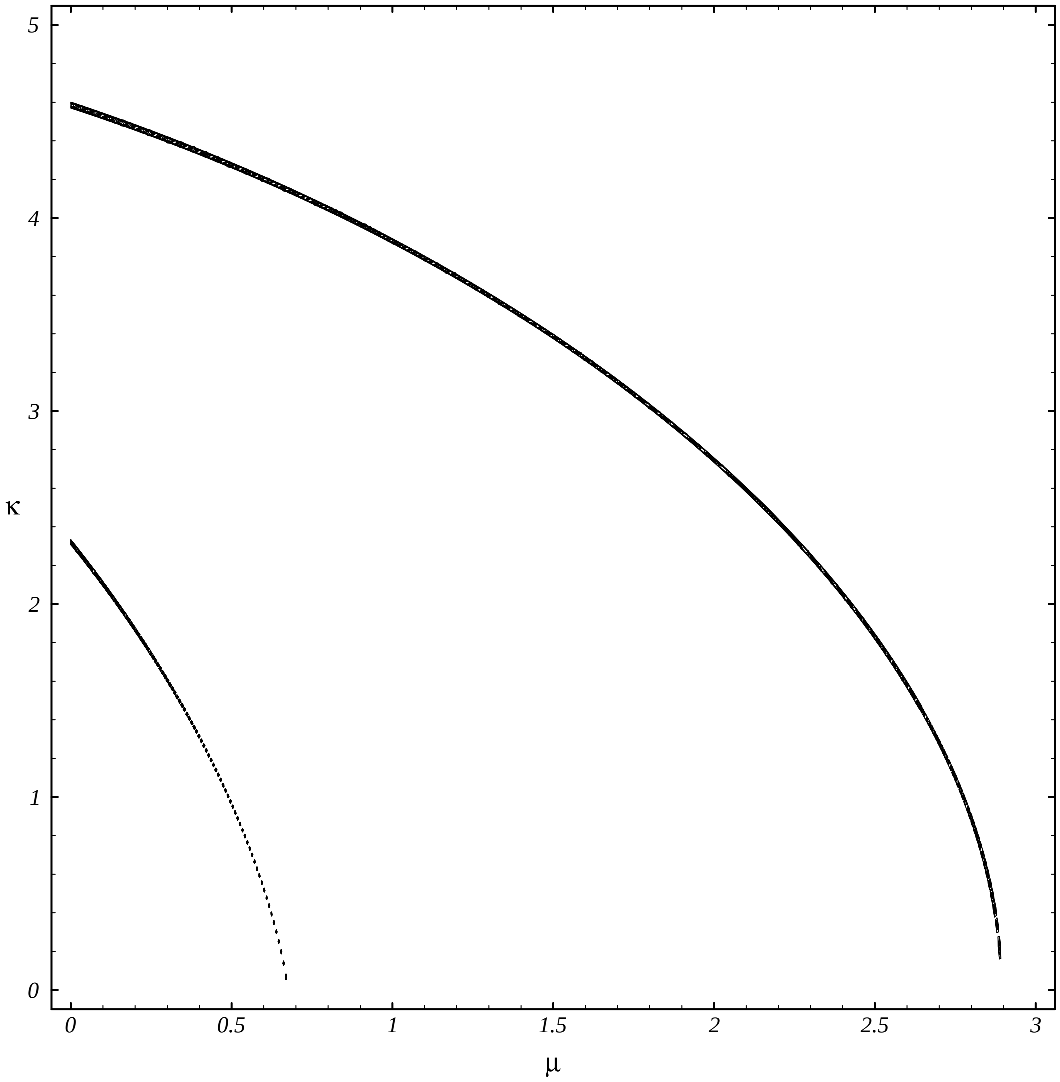

FIG. 5. The graphics shows the zeroes of the Jost function F in dependence of the momentum $k = i\kappa, \kappa > 0$ and the angular momentum μ for the square well potential. The line broadening is due to numerical error. The chosen numerical values are $d = 1, V_0 = 25$. One observes two bound states the first of which disappears at $\mu = 0.674$ and the second at $\mu = 2.893$.