Nonequilibrium Dynamics of Infinite Particle Systems with Infinite Range Interactions

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Abstract

We discuss the existence and uniqueness of non-equilibrium dynamics of infinitely many particles interacting via superstable pair interactions in one and two dimensions. The interaction is allowed to be of infinite range and of singular at the origin. Under suitable regularity conditions on the interaction potential, we show that if the potential decreases polynomially as the distance between interacting two particles increases, then the tempered solution to the system of Hamiltonian equations exists. Moreover, if the potential satisfies further that either it has a subexponential decreasing rate or it is everywhere two-times continuously differentiable, then we show that the tempered solution is unique. The results extend those of Dobrushin and Fritz obtained for finite range interactions.

Key Words: Non-equilibrium dynamics, superstable interactions, Hamiltonian equations, tempered solution.

1 Introduction

The aim of this paper is to show the existence and uniqueness of non-equilibrium dynamics of infinitely many particles interacting via (infinite-range) superstable interactions in dimensions one and two. This paper extends the results obtained by Dobrushin and Fritz in [DF,FD,Fr1] where the interaction range is assumed to be finite. By suitably modifying and extending the methods given in [DF,FD,Fr1], we remove the finite range assumption. The time evolution will be constructed in an explicitly defined set $\overline{\Omega}$ of allowed configurations characterized by a logarithmic order of energy fluctuations. The extension of the results for finite range interactions to infinite range interactions must be meaningful in the sense of practical and aesthetic points of view.

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This study may be considered as being in the extension of the line of investigation on the dynamics of infinitely many particles that has been studied by various authors [Lanf1-2,DF,FD,Fr1,MPPr,CFS,Si1-2,S-S]. The main point in this study would be to show the existence of solutions to the infinite system of Hamiltonian equations (see (1.1)) and then to investigate the detailed properties of the solution such as the uniqueness of the solution and the essential self-adjointness of the generator of the evolution semigroup of the dynamics. While various techniques for showing the existence of the solutions have been developed in the papers including the above mentioned ones (see, e.g. [Fr1] and [S-S] for further references), the method to show the energy bound of the solutions has not been fully developed for general dimensions. Only up to one and two dimensions and for finite range interactions, such an aim has been fulfilled [Lanf1, DF, FD, Fr1] for explicitly characterized initial conditions. The method of the energy bound has been also used to show the essential self-adjointness of the Liouville operator [MPPu, Fr1].

In [S-S], Siegmund-Schultze has shown that in any dimension, for a given translation invariant probability measure on the phase space such that the particle density and specific energy are finite, there is a class of configurations which carries a full measure such that the solutions with initial conditions from that class exist provided that the potential is not singular and has finite interaction range. However, the method of Siegmund-Schultze do not yet permit us to prove the uniqueness of the solution and to control the properties of the solution. See also [Si 1,2] and the references in [S-S] for earlier works.

We would like to mention the works related to the infinite system of Hamiltonian equations. In [Lang], Lang initiated to consider the system of first order differential equations for the system with additional white-noise terms [Lang, Fr2]. Recently the system of stochastic differential equations related to the particle systems has been investigated using Dirichlet form approach [AKR1-2,CPY, Osa, Sp, Yos].

Let us briefly describe the contents of this paper. We are going to consider the motion of a countable collection of identical particles of unit mass in the d-dimensional Euclidean space $\mathbb{R}^d$ with $d = 1, 2$. Let I be the index set for the particles. Configurations of the system are represented as infinite sequences $\omega = (q_k, p_k)_{k \in I}$, where $q_k \in \mathbb{R}^d$ and $p_k \in \mathbb{R}^d$ are the position and velocity of the particle labeled by $k \in I$. Sometimes we will use more informative notations $q_k = q_k(\omega), p_k = p_k(\omega)$. Only locally finite configurations are allowed, i.e., the sequence $(q_k)_{k \in I}$ of positions can not have limit points at all, but some additional restrictions are necessary too. We assume that our particles interact by a symmetric pair potential $U : \mathbb{R}^d \to (-\infty, \infty]$ and $U(x) = U(-x)$. The potential U is allowed to be of infinite range. Let $\mathrm{grad}\, U$ denote the gradient of U. The equations of motion read as

$$\frac{dp_k}{dt} = -\sum_{j \neq k} \mathrm{grad}\; U(q_k - q_j), \quad \frac{dq_k}{dt} = p_k, \quad k \in I. \tag{1.1}$$

We will impose to the potential that it is superstable in the sense of Ruelle [Ru1-2]. We allow that U is possibly singular at the origin and is of infinite range. See Section 2 for the details.

The space $\overline{\Omega}$ of allowed configurations on which the evolution of the dynamics shall be constructed is defined in the following way. Let

$$g(u) := (1 + \log(1+u))^{1/d}, \quad u \geq 0, \tag{1.2}$$

and let $H(\omega, \mu, \sigma)$ denote the total energy plus a multiple of particle numbers in the sphere with center $\mu \in \mathbb{R}^d$ and radius $\sigma > 0$, i.e.,

$$H(\omega, \mu, \sigma) := \frac{1}{2} \sum_{|q_k - \mu| \leq \sigma} [|p_k|^2 + A + \sum_{j \neq k; |q_j - \mu| \leq \sigma} U(q_k - q_j)], \tag{1.3}$$

where $A \geq 0$ is a constant appearing in the superstable condition (see (2.1)) and makes $H(\omega, \mu, \sigma)$ nonnegative numbers. Let us define

$$\overline{H}(\omega) := \sup_{\mu} \sup_{\sigma \geq g(|\mu|)} \sigma^{-d} H(\omega, \mu, \sigma). \tag{1.4}$$

$\overline{H}(\omega)$ is called the logarithmic fluctuation of energy of ω [Fr1]. The allowed configuration space is defined by

$$\overline{\Omega} := \{\omega : \; \omega \text{ is locally finite and } \; \overline{H}(\omega) < \infty\}. \tag{1.5}$$

It turns out that $\overline{\Omega}$ carries a large class of probability measures including Gibbs random fields with interaction U [DF,MPPu,CPY].

We will show that in dimensions one and two, under proper conditions on the potential, there exists a tempered solution (see Section 2 for the definition) ω_t to (1.1) with any initial configuration $\omega \in \overline{\Omega}$. This solution can be obtained as the limit of solutions to finite subsystems. Furthermore, the tempered solution is unique when the potential U is either everywhere two-times continuously differentiable or subexponentially decreasing (see Theorem 2.2) when it is singular at the origin. The main idea used in this paper is to decompose the potential U into two parts

$$U(x) = U^{(1)}(x) + U^{(2)}(x),$$

where $U^{(1)}$ is of finite range and $U^{(2)}$ is a smooth potential which decreases polynomially. See (2.2) and (2.6). For $U^{(1)}$, we use the method developed in [Fr1] with suitable modifications and for $U^{(2)}$, we utilize the decay properties of $U^{(2)}$ stated in (2.6).

Employing the method developed in section 6 and using an appropriate modification of the method of [Fr1, Section 5], it may be possible to show the (anti) self-adjointness of the corresponding Liouville operator [MPPu, Fr1]. However we do not pursue the proof here and leave it to further study.

We organize this paper as follows. In section 2, we give necessary notations, impose conditions to the potential, and give main results of this paper. In section 3, we discuss the cutoff of total energy. In section 4, we present the a priori bound which is a core in proving the results of this paper. We prove the existence and the uniqueness of tempered solution in section 5 and section 6 respectively.

2 Notations and Main Results

In this section we give conditions to the interaction and state our main results. As mentioned in Introduction, we will consider a symmetric pair potential $U : \mathbb{R}^d \to (-\infty, \infty]$, $d = 1, 2$, allowing the possibility of being of infinite range and of singularity at the origin. We need, however, some regularity conditions on the potential in order to show the existence and uniqueness of the solutions to (1.1). First, we need superstability of the interaction to control the number of particles, just as in equilibrium theory [Ru1,2]. There exist $A \geq 0$ and $B > 0$ such that for any finite collection $q_1, q_2, \cdots q_n$ of points of $\mathbb{R}^d$,

$$\sum_{k=1}^{n} \sum_{j \neq k} U(q_k - q_j) \geq -An + B \sum_{i \in \mathbb{Z}^d} n_i^2, \tag{2.1}$$

where $\mathbb{Z}^d$ is the d-dimensional integer lattice and for each site $i \in \mathbb{Z}^d$, n_i denotes the number of particles within the unit box with center i. This condition can be verified under natural assumptions, e.g., that U is not integrable near zero and integrable outside the origin [Ru2]. We assume $U(x)$ is continuously differentiable outside from the origin and suppose that U is decomposed into two parts of singular and analytic potentials;

$$U(x) = U^{(1)}(x) + U^{(2)}(x). \tag{2.2}$$

Furthermore, we assume that $U^{(1)} \geq 0$ for all x and there exists a positive number r such that

$$U^{(1)}(x) = 0, \quad \text{if } |x| \geq r. \tag{2.3}$$

It is important that singularity of U can not be too strong. Sometimes the velocity depends on the strength of the interaction and this velocity can become arbitrarily large in the extreme case. Thus, as in [Fr1], we assume that there exist positive constants a and b such that

$$|x| \, |\text{grad } U^{(1)}(x)| \leq a + b \, U^{(1)}(x). \tag{2.4}$$

From a mathematical point of view, (2.4) means that the singularity can not be stronger than that of $|x|^{-b}$. Also we need local Lipschitz condition. Let $L > 1$ and suppose that

$$|\text{grad } U^{(1)}(x) - \text{grad } U^{(1)}(y)| \leq L[L + U^{(1)}(x) + U^{(1)}(y)]^c \, |x - y| \tag{2.5}$$

holds for all $x, y \neq 0$ with some constant $c > 1$. (2.5) means that the singularity can not be too weak, logarithmic singularities are excluded. Finally we assume that the analytic potential $U^{(2)}(x)$ is everywhere two times continuously differentiable and decreases polynomially: There exist a decreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ and constants $\gamma > 2d+3$ and $0 < M$ such that

$$\psi(u) \le M(1+u)^{-\gamma}, \quad u \ge 0,$$

and

$$|U^{(2)}(x)| + |\operatorname{grad} U^{(2)}(x)| + \sum_{l,k=1}^{d} |\frac{\partial^2}{\partial x^{(l)} \partial x^{(k)}} U^{(2)}(x)| \le \psi(|x|). \tag{2.6}$$

Notice that $U^{(2)}(x) \ge -M$, and hence $U(x) \ge -M$. We refer to [Fr1] for the detailed accounting for the conditions on the potential.

We recall the notion of the logarithmic energy fluctuation $\overline{H}(\omega)$ of a given configuration ω and the possible configuration space given in (1.4)-(1.5). The configuration space $\overline{\Omega}$ is equipped with the product topology and with associated Borel structure [Fr1]. If $\omega_t^{(n)}$ is a sequence of trajectories in $\overline{\Omega}$ then convergence of $\omega_t^{(n)}$ means the uniform convergence on compact intervals of time of each of the components $q_k(\omega_t^{(n)})$ and $p_k(\omega_t^{(n)})$. This convergence need not be uniform in $k \in I$. A continuous trajectory $\omega_t : \mathbb{R} \to \overline{\Omega}$ is called a *tempered* solution to (1.1) with initial configuration $\omega \in \overline{\Omega}$ if $\omega_0 = \omega$, $\overline{H}(\omega_t)$ is bounded in bounded intervals of time, and the components $q_k(t) = q_k(\omega_t), p_k(t) = p_k(\omega_t)$ are continuously differentiable and satisfy (1.1) for all $k \in I$ and $t \in \mathbb{R}$. See [Fr1] for the details.

As usual in the theory of infinite systems, solutions to (1.1) are constructed as limits of solutions to finite subsystems of (1.1). The passage to the infinite system is based on an a priori bound expressing a local version of the law of energy conservation. Just as in [DF, FD, Fr1], energy flow will be controlled by means of a partial differential inequality formulated in terms of a spatial cutoff of total energy. Let us mention that the a priori bound can be obtained from this inequality only in dimensions one and two.

Our first main result concerns the existence of tempered solutions to (1.1).

Theorem 2.1. *Suppose that the hypotheses on the potential given in (2.1)-(2.6) are satisfied. Then, for any $\omega \in \overline{\Omega}$ there exists at least a tempered solution to (1.1) with initial configuration ω.*

The proof of the above theorem will be given in Section 5. Next, we discuss the uniqueness of the solution. When the potential is singular at the origin, the method developed in this paper demands that the decreasing rate of the potential must be stronger than that given in (2.6), at least it must be of sub-exponential, in order that the tempered solution to (1.1) is unique. Recall the constant $c > 1$ appearing in (2.5) for the Lipschitz condition of the potential.

Theorem 2.2. *Suppose that the conditions (2.1)-(2.5) for the potential hold. Suppose further that there exists a constant $\epsilon_0 > 0$ such that the function ψ in the right hand side of (2.6) satisfies $\psi(u) \leq \exp[-(\log u)^{c+\epsilon_0}]$ for large values of $u > 0$. Then the tempered solution to (1.1) exists uniquely.*

When the potential is not singular at the origin, the polynomial decreasing rate is enough to insure the uniqueness.

Theorem 2.3. *Suppose that the potential $U(x)$ satisfies (2.1) and everywhere two-times continuously differentiable with a decreasing rate given in (2.6) ($U^{(2)}(x)$ being replaced by $U(x)$). Then the tempered solution to (1.1) exists uniquely.*

The proofs of Theorem 2.2 and Theorem 2.3 will be given in Section 6.

3 Cutoff of Total Energy

In this section, we introduce a smooth version of the energy functions $H(\omega, \mu, \sigma)$ defined in (1.3). In [Fr1], exponentially decreasing functions have been used as a smooth version of the indicator functions. In order to deal with polynomially decreasing interaction potentials, we will use polynomially decreasing functions which satisfy all of desired properties (Lemma 3.1).

Let $0 < \lambda < 1$. We will choose λ sufficiently small. Choose $\varphi : \mathbb{R}^+ \to (0, 1]$, continuously differentiable such that for $p > d + 1$

(i) $\varphi(u) = (1 + \lambda\, u)^{-p}, \quad u \geq 2$

(ii) φ is concave for $u \leq 2$.

(iii) $\varphi(u) = \varphi(2) - \frac{1}{2}\varphi'(2) = (\frac{1}{1+2\lambda})^p(1 + \frac{\lambda p}{2(1+2\lambda)})$ for $u \leq 1$.

Notice that $\varphi(u) \leq (1+\lambda u)^{-p}$ and $0 \leq -\varphi'(u) \leq p\lambda(1+\lambda u)^{-(p+1)}$.

Define

$$f(x,\sigma) := \int_{\mathbb{R}^d} \varphi(\frac{|x-y|}{\sigma})(\frac{1}{1+\lambda|y|})^{p+1}\,dy, \tag{3.1}$$

where $\sigma > 0$. The definition (3.1) and the following lemma work well for $p > d$ but we need the condition $p > d+1$ for a later use (Lemma 3.3). In the proof of the a priori bound, $f(x-\mu,\sigma)$ will be used as a smooth version of the indicator function of the d-dimensional ball with center $\mu \in \mathbb{R}^d$ and radius σ. From now on we assume that $\sigma \geq 2$.

Lemma 3.1. *There exist positive constants $c_1, \overline{c_1}$, and c_2 depending only on λ and p such that the following properties hold.*

(a) $f(x-\mu,\sigma) \leq c_1(1+\lambda|x-\mu|/\sigma)^{-p}$ *and* $f(x,\sigma) \geq \overline{c_1}(1+|x|)^{-p}$ *for all* σ.

(b) $f(x-\mu,\sigma) \geq c_2 > 0 \quad$ *if* $|x-\mu| \leq \sigma$.

(c) $f(x,\sigma) \leq (1+\lambda|x-y|)^{p+1} f(y,\sigma)$.

(d) Denote by f' the derivative of f w.r.t. σ. Then $f'(x,\sigma) \leq (1+\lambda|x-y|)^{p+1} f'(y,\sigma)$.

(e) $|\operatorname{grad} f(x,\sigma)| \leq f'(x,\sigma)$.

(f) $g(|x|)\ |\operatorname{grad} f(x-\mu,\sigma)| \leq 4g(|\mu|+\sigma) f'(x-\mu,\sigma)$.

Proof. (a) We may assume $\mu = 0$. Notice that

$$(1+\frac{\lambda|x|}{\sigma})^p f(x,\sigma) \leq \int (\frac{\sigma+\lambda|x|}{\sigma+\lambda|x-y|})^p\,(\frac{1}{1+\lambda|y|})^{p+1}\,dy$$

Using the inequality $|x| \leq |x-y| + |y|$, one can check that the right hand side of the inequality is finite uniformly in x and σ. This implies the first bound.

To obtain second bound, notice $f(x,\sigma) \geq f(x,2)$. Since φ is concave for $u \leq 2$, there exists positive number $c > 0$ such that the bound

$$\begin{aligned}(1+|x|)^p f(x,2) &\geq c\,(1+|x|)^p \int \left(\frac{1}{1+\lambda\frac{|x-y|}{2}}\right)^p \left(\frac{1}{1+\lambda|y|}\right)^{p+1} dy \\ &\geq \overline{c}_1 > 0\end{aligned}$$

holds.

(b) Suppose $|x| \le \sigma$. Then there exists a constant $c > 0$ such that

$$\begin{aligned} f(x,\sigma) &\ge c \int_{|x-y|\le\sigma} \left(\frac{1}{1+\lambda|y|}\right)^{p+1} dy \\ &\ge c \int_{B_\sigma(0)\cap B_\sigma(x)} \left(\frac{1}{1+\lambda|y|}\right)^{p+1} dy, \\ &\ge c \int_D \left(\frac{1}{1+\lambda|y|}\right)^{p+1} dy =: c_2 > 0, \end{aligned}$$

where $D = B_2(0) \cap B_2(z_1)$ and $z_1 = 2x/|x|$.

(c) The bound follows from the following relations:

$$f(x,\sigma) = \int_{\mathbb{R}^d} \varphi\left(\frac{|z|}{\sigma}\right)\left(\frac{1}{1+\lambda|x-z|}\right)^{p+1} dz$$

and

$$\frac{1+\lambda|y-z|}{1+\lambda|x-z|} \le 1+\lambda|x-y|.$$

(d) Since

$$f'(x,\sigma) = -\int \varphi'\left(\frac{|x-y|}{\sigma}\right)\frac{|x-y|}{\sigma^2}\left(\frac{1}{1+\lambda|y|}\right)^{p+1} dy,$$

the result follows by the method same as that used in (c).

(e) This follows from a direct computation.

(f) We first show that

$$|x|\,|\operatorname{grad} f(x,\sigma)| \le 4\sigma f'(x,\sigma). \tag{3.2}$$

In view of the fact (e), we may assume that $|x| \ge 4\sigma$. Let $D_1 = \{y \in \mathbb{R}^d : |y| \le |x-y|\}$ and $D_2 = \mathbb{R}^d \setminus D_1$. Then $|y| \ge \frac{|x|}{2}$ if $y \in D_2$ and $-\varphi'(u) \le \frac{p\lambda}{(1+\lambda u)^{p+1}}$.

We assert that for $y \in D_1$

$$-\varphi'\left(\frac{|y|}{\sigma}\right)\left(\frac{1}{1+\lambda|x-y|}\right)^{p+1} \le -\varphi'\left(\frac{|x-y|}{\sigma}\right)\left(\frac{1}{1+\lambda|y|}\right)^{p+1}. \tag{3.3}$$

Notice that $|y-x| \ge 2\sigma$ for $|y| \le |x-y|$. One needs to show that for $y \in D_1$

$$\left(\frac{1}{1+\frac{\lambda|y|}{\sigma}}\right)\left(\frac{1}{1+\lambda|x-y|}\right) \le \left(\frac{1}{1+\frac{\lambda|x-y|}{\sigma}}\right)\left(\frac{1}{1+\lambda|y|}\right),$$

which is equivalent to

$$\frac{\sigma + \lambda|x-y|}{\sigma + \lambda|y|} \leq \frac{1 + \lambda|x-y|}{1 + \lambda|y|}.$$

Since $(\sigma + \lambda|x-y|)/(\sigma + \lambda|y|)$ is monotonic decreasing w.r.t. σ if $|y| \leq |x-y|$, this proves (3.3). Now (3.2) follows from the method used in [Fr1, page 545] and (f) follows from (3.2) by the method same as that given in [Fr1]. ■

From now on we will fix p such that $\gamma > 2p+1 > 2(d+1)+1$ in the definition of φ in (3.1). The smooth version of total energy corresponding to $H(\omega, \mu, \sigma)$ given in (1.3) is defined as

$$W(\omega, \mu, \sigma) = \sum_{k \in I} f(q_k - \mu, \sigma) W_k(\omega), \tag{3.4}$$

where $q_k = q_k(\omega), p_k = p_k(\omega)$, and

$$W_k(\omega) = 2A + |p_k|^2 + \sum_{j \neq k} U(q_k - q_j), \tag{3.5}$$

where A is the same as in (2.1). Define the logarithmic fluctuation of W by

$$\overline{W}(\omega) = \sup_{\mu} \sup_{\sigma \geq 2g(|\mu|)} \sigma^{-d} W(\omega, \mu, \sigma). \tag{3.6}$$

Recall that for any $i \in \mathbb{Z}^d$ and $\omega \in \overline{\Omega}$, $n_i(\omega)$ is the number of particles of ω in the unit box with center at i. The superstability of U implies the following:

Lemma 3.2. *There exists $0 < \lambda < 1$ such that*

$$W(\omega, \mu, \sigma) \geq \frac{B}{4} \sum_{i \in \mathbb{Z}^d} f(i - \mu, \sigma)\ n_i(\omega)^2$$

and

$$\frac{\partial}{\partial \sigma} W(\omega, \mu, \sigma) \geq \frac{B}{4} \sum_{i \in \mathbb{Z}^d} f'(i - \mu, \sigma)\ n_i(\omega)^2.$$

Proof. Let $R > 0$ be a sufficiently large natural number which will be fixed later and write the potential $U(x)$ as

$$U(x) := U^{(R,1)}(x) + U^{(R,2)}(x), \tag{3.7}$$

where $U^{(R,1)}(x) := U(x)1_{[0,R]}(|x|)$ and $U^{(R,2)}(x) := U(x) - U^{(R,1)}(x)$. We write

$$W(\omega,\mu,\sigma) := W^{(R,1)}(\omega,\mu,\sigma) + W^{(R,2)}(\omega,\mu,\sigma), \tag{3.8}$$

$$\begin{aligned} W^{(R,1)}(\omega,\mu,\sigma) &:= \sum_{k\in I} f(q_k-\mu,\sigma)[2A+|p_k|^2+\sum_{j\neq k} U^{(R,1)}(q_k-q_j)], \\ W^{(R,2)}(\omega,\mu,\sigma) &:= \sum_{k\in I} f(q_k-\mu,\sigma) \sum_{j\neq k} U^{(R,2)}(q_k-q_j). \end{aligned}$$

It is easy to check that there is a positive constant c_3 such that

$$|W^{(R,2)}(\omega,\mu,\sigma)| \le c_3\Big(\sum_{k\in\mathbb{Z}^d} n_k(\omega) f(k-\mu,\sigma)\Big)\Big(\sum_{j\in\mathbb{Z}^d;|k-j|\ge R-2} n_j(\omega)(1+|k-j|)^{-\gamma}\Big).$$

We use the fact that $n_k(\omega)n_j(\omega) \le (1/2)(n_k(\omega)^2+n_j(\omega)^2)$, Lemma 3.1(c), $\gamma > 2p+1$ and $p > d+1$ to obtain that

$$\begin{aligned} |W^{(R,2)}(\omega,\mu,\sigma)| &\le c_3[\sum_{j\in\mathbb{Z}^d:|j|\ge R-2} (1+|j|)^{(-\gamma+p+1)}][\sum_{k\in\mathbb{Z}^d} f(k-\mu,\sigma)n_k(\omega)^2] \\ &\le \frac{B}{4}\sum_{k\in\mathbb{Z}^d} f(k-\mu,\sigma)n_k(\omega)^2, \end{aligned} \tag{3.9}$$

where we have chosen R large enough.

On the other hand, by a similar method used above, it is not hard to see that if R is sufficiently large, then $U^{(R,1)}(x)$ is also superstable and satisfies that for any finite collection $q_1,\cdots,q_n$ of points in $\mathbb{R}^d$,

$$\sum_{k=1}^{n}\sum_{j\neq k} U^{(R,1)}(q_k-q_j) \ge -A\sum_{k\in\mathbb{Z}^d} n_k + \frac{3}{4}B\sum_{k\in\mathbb{Z}^d} n_k^2, \tag{3.10}$$

where $A \ge 0$ and $B > 0$ are the same constants appeared in (2.1).

In order to control $W^{(R,1)}$, we modify the method used in the proof of [Fr1, Lemma 3.1]. Introduce

$$\Lambda_u := \{x\in\mathbb{R}^d : u^{(i)} \le x^{(i)} < u^{(i)} + mR, \quad 1\le i\le d\},$$

where $x^{(i)}$ and $u^{(i)}$ are the coordinates of $x, u \in \mathbb{R}^d$, and m is a large natural number. Let P denote the set of pairs $[k,j]$ such that $|q_k - q_j| \le R$ and let P_u be the set of $[k,j]\in P$ such that $k,j\in I_u$, where I_u is the set of particles in Λ_u. Put $\mathbb{Z}_u = \mathbb{Z}^d\cap\Lambda_u$. If ρ_u denotes the minimum of $f_k = f(q_k - \mu, \sigma)$ for $k \in I_u$, and $\lambda m R\sqrt{d} < \epsilon$, i.e., the diameter of Λ_u is less than ϵ/λ, then using Lemma 3.1 (c) and $U^{(R,1)}(x) \geq -M$, we obtain for each $[k,j] \in P_u$,

$$\begin{aligned}(f_k + f_j)U^{(R,1)}(q_k - q_j) &= 2\rho_u U^{(R,1)}(q_k - q_j) + (f_k + f_j - 2\rho_u)U^{(R,1)}(q_k - q_j) \\ &\geq 2\rho_u U^{(R,1)}(q_k - q_j) - 2M\rho_u \left((1+\epsilon)^{(p+1)} - 1\right),\end{aligned}$$

and so it follows from the above bound, (3.10) and Lemma 3.1 (c) that

$$\begin{aligned}&\sum_{[k,j]\in P_u} (f_k + f_j)U^{(R,1)}(q_k - q_j) \\ &\geq -A\sum_{k\in\mathbb{Z}_u} \rho_u n_k + \frac{3}{4}B\sum_{k\in\mathbb{Z}_u} \rho_u n_k^2 - 2M\left((1+\epsilon)^{(p+1)} - 1\right)\sum_{[k,j]\in P_u} \rho_u \\ &\geq -A\sum_{k\in\mathbb{Z}_u} \rho_u n_k + \frac{3}{4}B(1+\epsilon)^{-(p+1)}\sum_{k\in\mathbb{Z}_u} f(k-\mu,\sigma)n_k^2 - M((1+\epsilon)^{(p+1)} - 1)\sum_{[k,j]\in P_u} (f_k + f_j).\end{aligned}$$

Now let $z \in \Lambda_0 \cap R\,\mathbb{Z}^d$ be given. Then $\{\Lambda_u : u \in z + Rm\mathbb{Z}^d\}$ is a partition of $\mathbb{R}^d$. Let Q_z be the union of all P_u such that $u \in z + Rm\mathbb{Z}^d$. By the above estimate and $U^{(R,1)}(x) \geq -M$, we see that

$$\begin{aligned}W^{(R,1)}(\omega, \mu.\sigma) &\geq \frac{3}{4}B(1+\epsilon)^{-(p+1)}\sum_{k\in\mathbb{Z}^d} f(k-\mu,\sigma)n_k^2 - M((1+\epsilon)^{(p+1)} - 1)\sum_{[k,j]\in Q_z} (f_k + f_j) \\ &\quad -M\sum_{[k,j]\in P\setminus Q_z} (f_k + f_j) \\ &\geq \frac{3}{4}B(1+\epsilon)^{-(p+1)}\sum_{k\in\mathbb{Z}^d} f(k-\mu,\sigma)n_k^2 - M((1+\epsilon)^{(p+1)} - 1)\sum_{[k,j]\in P} (f_k + f_j) \\ &\quad -M\sum_{[k,j]\in P\setminus Q_z} (f_k + f_j).\end{aligned} \tag{3.11}$$

We notice that

$$\sum_{[k,j]\in P} (f_k + f_j) \leq c_4 \sum_{k\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}^d;|k-j|\leq R+2} n_k\, n_j\, f(k-\mu,\sigma).$$

Since $f(k-\mu,\sigma) \leq f(j-\mu,\sigma)\ (1+\lambda(R+2))^{(p+1)}$ if $|k-j| \leq R+2$, by Lemma 3.1 (c), and the number of j such that $|k-j| \leq R+2$ has an order of $(R+2)^d$, we use Schwarz inequality to obtain

$$\sum_{[k,j]\in P} (f_k + f_j) \leq c_5\,(3+R)^{p+1+d}\sum_{k\in\mathbb{Z}^d} f(k-\mu,\sigma)\ n_k(\omega)^2. \tag{3.12}$$

Hence, inserting (3.12) into (3.11), we have

$$\begin{aligned} & W^{(R,1)}(\omega, \mu.\sigma) \\ & \geq \left(\frac{3}{4}B(1+\epsilon)^{-(p+1)} - M((1+\epsilon)^{(p+1)}-1)\, c_5(3+R)^{p+1+d}\right) \sum_{k\in\mathbb{Z}^d} f(k-\mu,\sigma)n_k^2 \\ & \quad -M \sum_{[k,j]\in P\setminus Q_z} (f_k+f_j) \end{aligned} \tag{3.13}$$

We are going to sum the both side of the inequality (3.13) over $z \in \Lambda_0 \cap R\mathbb{Z}^d$. The term in the left hand side and the first term in the right hand side of (3.13) are independent of z. Thus we consider the last term in the right hand side of (3.13). Given $[k,j] \in P$, the number of $z \in \Lambda_0 \cap R\mathbb{Z}^d$ such that $[k,j] \in Q_z$ is larger than $(m-2)^d$, thus the number of z with $[k,j] \notin Q_z$ is less than $m^d - (m-2)^d \leq 2dm^{(d-1)}$. Consequently, summing both sides of (3.13) over $z \in \Lambda_0 \cap R\,\mathbb{Z}^d$ we have

$$\begin{aligned} & m^d\, W^{(R,1)}(\omega, \mu.\sigma) \\ & \geq \left[m^d\big(\frac{3}{4}B(1+\epsilon)^{-(p+1)} - M((1+\epsilon)^{(p+1)}-1)\, c_5(3+R)^{p+1+d}\big) \sum_{k\in\mathbb{Z}^d} f(k-\mu,\sigma)n_k^2\right] \\ & \quad -\left[M2dm^{(d-1)} \sum_{[k,j]\in P} (f_k+f_j)\right]. \end{aligned} \tag{3.14}$$

Therefore, inserting (3.12) into (3.14) and dividing both sides of (3.14) by m^d, we get

$$W^{(R,1)}(\omega,\mu,\sigma) \geq C(R,\epsilon,m) \sum_{k\in\mathbb{Z}^d} f(k-\mu,\sigma)n_k(\omega)^2, \tag{3.15}$$

where

$$\begin{aligned} C(R,\epsilon,m) \quad = \quad & \left(\frac{3}{4}B(1+\epsilon)^{-(p+1)} - M((1+\epsilon)^{(p+1)}-1)\, c_5(3+R)^{p+1+d}\right) \\ & -c_5\frac{M2d}{m}(3+R)^{(p+1+d)}. \end{aligned}$$

Now we choose and fix a sufficiently large R so that (3.9) and (3.10) hold. Then fix $\epsilon > 0$ to be sufficiently small and fix m sufficiently large so that

$$C(R,\epsilon,m) \geq \frac{1}{2}B. \tag{3.16}$$

Finally we take and fix $0 < \lambda < 1$ so small that $\lambda m R\sqrt{d} < \epsilon$. By (3.8) - (3.9), (3.15)-(3.16), the first statement in the Lemma follows directly. Using Lemma 3.1 (d) instead of Lemma 3.1(c), we obtain the second inequality in the same way. ■

In the rest of this paper $\lambda > 0$ will be fixed such that Lemma 3.2. holds. Finally, let us remark that H and W are equivalent in the following sense.

Lemma 3.3. *There exist positive constants M_1 and M_2 such that*

$$M_1 \overline{H}(\omega) \le \overline{W}(\omega) \le M_2 \overline{H}(\omega)$$

for all $\omega \in \overline{\Omega}$.

Proof. To prove the first part it is enough to show that

$$H(\omega, \mu, \sigma) \le c\, W(\omega, \mu, \sigma) \tag{3.17}$$

for some constant $c > 0$. By (2.6), $U(q_k - q_j) \ge -M(1 + |q_k - q_j|)^{-\gamma}$ for all k, j. It follows from Lemma 3.1 (b) that

$$W(\omega, \mu, \sigma) \ge c_2 H(\omega, \mu, \sigma) - M \sum_k \sum_{j \ne k} f(q_k - \mu, \sigma)(1 + |q_k - q_j|)^{-\gamma}.$$

By a simple computation and Lemma 3.2, we see that

$$\begin{aligned} \sum_k \sum_{j \ne k} f(q_k - \mu, \sigma)(1 + |q_k - q_j|)^{-\gamma} &\le c_6 \sum_{k \in \mathbb{Z}^d} f(k - \mu, \sigma) n_k(\omega)^2 \\ &\le c_6' \ W(\omega, \mu, \sigma). \end{aligned}$$

Thus (3.17) follows from the above bounds.

Let us consider the second inequality. By the superstability of potential, one has that

$$\sum_{|j-\mu| \le \sigma} n_j(\omega)^2 \le \frac{2}{B} H(\omega, \mu, \sigma + 1). \tag{3.18}$$

We use the decomposition $U(x) = U^{(1)}(x) + U^{(2)}(x)$ in (2.2), so that $U^{(1)}(x) \ge 0$ and $U^{(1)}(x) = 0$ if $|x| \ge r$, and $|U^{(2)}(x)| \le M\ (1 + |x|)^{-\gamma}$. We write

$$\begin{aligned} &W(\omega, \mu, \sigma) \\ &= \sum_k f(q_k - \mu, \sigma)\Big(2A + |p_k|^2 + \sum_{j \ne k} U^{(1)}(q_k - q_j)\Big) + \sum_k f(q_k - \mu, \sigma) \sum_{j \ne k} U^{(2)}(q_k - q_j) \\ &\equiv W_1 + W_2. \end{aligned}$$

Using Lemma 3.1 (a) and (c), and (3.18), we have that

$$
\begin{aligned}
|W_2| &= |\sum_k f(q_k-\mu,\sigma)\sum_{j\neq k} U^{(2)}(q_k-q_j)| \\
&\leq c_7 \sum_{k,j\in\mathbb{Z}^d;k\neq j} f(k-\mu,\sigma)\ n_k(\omega)\ n_j(\omega)\ (1+|k-j|)^{-\gamma} \\
&\leq c_7' \sum_{k\in\mathbb{Z}^d} f(k-\mu,\sigma)\ n_k(\omega)^2 \\
&\leq \tilde{c}\sum_{n=1}^{\infty} n^{-p} \sum_{k\in\mathbb{Z}^d;|k-\mu|\leq n\sigma} n_k(\omega)^2 \\
&\leq \tilde{c}_1 \sum_{n=1}^{\infty} n^{-p} H(\omega,\mu,n\sigma+1). \qquad (3.19)
\end{aligned}
$$

On the other hand, since $U^{(1)}(q_k-q_j)\geq 0$ and equals 0 for $|q_k-q_j|\geq r$, and since $f(q_k-\mu,\sigma)\leq f(q_j-\mu,\sigma)(1+\lambda|q_k-q_j|)^{(p+1)}$, we use Lemma 3.1 (a) to see that

$$
W_1 \leq c_8\sum_{n=1}^{\infty} n^{-p}\Big(\sum_{|q_k-\mu|\leq n\sigma}(A+|p_k|^2+\sum_{|q_j-\mu|\leq n\sigma} U^{(1)}(q_k-q_j))\Big).
$$

By adding and subtracting $U^{(2)}(q_k-q_j)$ to the last summand, we get that

$$
W_1 \leq c_8\sum_{n=1}^{\infty} n^{-p}\Big(H(\omega,\mu,n\sigma)+\sum_{|q_k-\mu|\leq n\sigma,|q_j-\mu|\leq n\sigma} |U^{(2)}(q_k-q_j)|\Big). \qquad (3.20)
$$

Observe that

$$
\begin{aligned}
\sum_{|q_k-\mu|\leq n\sigma,|q_j-\mu|\leq n\sigma} |U^{(2)}(q_k-q_j)| &\leq c_9 \sum_{k,j\in\mathbb{Z}^d;|k-\mu|\leq n\sigma+1,|j-\mu|\leq n\sigma+1} n_k(\omega)\ n_j(\omega)(1+|k-j|)^{-\gamma} \\
&\leq c_9' \sum_{|k-\mu|\leq n\sigma+1} n_k(\omega)^2 \leq c_9''\ H(\omega,\mu,n\sigma+2). \qquad (3.21)
\end{aligned}
$$

It follows from (3.19)-(3.21) that

$$
\begin{aligned}
W(\omega,\mu,\sigma) &\leq c_{10}\sum_{n=1}^{\infty} n^{-p}\ H(\omega,\mu,n\sigma+2) \\
&\leq c_{10}'\ \sigma^d(\sum_{n=1}^{\infty} n^{-(p-d)})\ \overline{H}(\omega). \qquad (3.22)
\end{aligned}
$$

Dividing both sides of (3.22) by σ^d and using the fact that $p-d>1$, we get the desired results. ■

4 A priori Bound

Recall that a continuous trajectory $\omega_t : \mathbb{R} \to \overline{\Omega}$ is called a tempered solution to (1.1) with initial configuration ω if $\omega_0 = \omega$ and the components $q_k(t) = q_k(w_t)$, $p_k(t) = p_k(w_t)$ are continuously differentiable and satisfy (1.1) for all $k \in I$ and $t \in \mathbb{R}$. We assume the dimension $d \leq 2$. The following is a local version of the law of energy conservation.

Proposition 4.1. *There exists a constant $K > 0$ such that along any tempered solution ω_t to (1.1) we have*

$$\frac{\partial}{\partial t} W(\omega_t, \mu, \sigma) \leq K \; g(|\mu| + \sigma)[\overline{W}(\omega_t)]^{1/2} \frac{\partial}{\partial \sigma} W(\omega_t, \mu, \sigma)$$

for all $t \in \mathbb{R}, \mu \in \mathbb{R}^d$, and $\sigma \geq 2$.

Proof. In order to prove the proposition, we modify and extend the method used in the proof of [Fr1, Proposition 4.1]. Recall the decomposition $U(x) = U^{(1)}(x) + U^{(2)}(x)$ in (2.2) and the inequality (2.4). A direct computation gives

$$\begin{aligned} \frac{\partial W}{\partial t} &= \sum_{k \in I} < \text{grad } f_k, p_k > W_k \\ &+ \frac{1}{2} \sum_k \sum_{j \neq k; |q_k - q_j| \leq r} (f_j - f_k) < \text{grad } U^{(1)}(q_k - q_j), p_k + p_j > \\ &+ \frac{1}{2} \sum_k \sum_{j \neq k; |q_k - q_j| \geq r} (f_j - f_k) < \text{grad } U^{(2)}(q_k - q_j), p_k + p_j > \\ &\equiv D_1 + D_2 + D_3 \end{aligned} \tag{4.1}$$

where $f_k = f(q_k - \mu, \sigma)$.

For D_1, it is not hard to see that $|W_k| \leq W_k + M \; \sum_{j \neq k} (1 + |q_k - q_j|)^{-\gamma}$. Thus

$$|D_1| \leq \sum_{k \in I} |\text{ grad } f_k| \; |p_k| \, [W_k + M \; \sum_{j \neq k} (1 + |q_k - q_j|)^{-\gamma}].$$

Since $|p_k| \leq K_1 \; g(|q_k|) \; [\overline{W}(\omega_t)]^{1/2}$, we use Lemma 3.1 (f) to get

$$|D_1| \leq 4K_1 \; g(|\mu| + \sigma)[\overline{W}(\omega_t)]^{1/2} \sum_k f'_k \; [W_k(\omega_t) + M \; \sum_{j \neq k} (1 + |q_k - q_j|)^{-\gamma}]. \tag{4.2}$$

By Lemma 3.1 (d), $f'_k \le (1+\lambda|q_k-q_j|)^{p+1}\, f'_j$. Thus using Schwarz inequality and the fact that $\gamma > 2p$, we get

$$\sum_{k\in I} f'_k \sum_{j\neq k}(1+|q_k-q_j|)^{-\gamma} \le K_2 \sum_{k\in\mathbb{Z}^d} f'(k-\mu,\sigma)\; n_k(\omega)^2. \tag{4.3}$$

From (4.2)-(4.3) and Lemma 3.2, one obtains

$$|D_1| \le K_3\, g(|\mu|+\sigma)\,[\overline{W}(\omega_t)]^{1/2}\, \frac{\partial}{\partial\sigma}W(\omega_t,\mu,\sigma). \tag{4.4}$$

For D_3, since $|\operatorname{grad} U^{(2)}(q_k-q_j)| \le M\ (1+|q_k-q_j|)^{-\gamma}$, we have

$$|D_3| \le K_4 \sum_k\sum_{j\neq k} |\operatorname{grad}\, f(z_{kj}-\mu,\sigma)|\,|q_k-q_j|\,(1+|q_k-q_j|)^{-\gamma}\ (g(|q_k|)+g(|q_j|))[\overline{W}(\omega_t)]^{1/2},$$

where z_{kj} is a some point between q_k and q_j. Note that, by Lemma 3.1 (f),

$$|\operatorname{grad}\ f(z_{kj}-\mu,\sigma)| \le \frac{1}{g(|z_{kj}|)}\ 4\ g(|\mu|+\sigma)\, f'(z_{kj}-\mu,\sigma).$$

We note that $g(x+y)\le g(x)g(y)$ for any $x,y\ge 0$, and so $g(|q_k|)+g(|g_j|)\le 2g(|z_{kj}|)g(|q_k-q_j|)$. Using Schwarz inequality, Lemma 3.1 (d) and Lemma 3.2, we obtain

$$\begin{aligned}
&|D_3| \le K_5\ g(|\mu|+\sigma)[\overline{W}(\omega_t)]^{1/2}\\
&\quad\times\sum_{k\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}^d}[n_k(\omega)\ n_j(\omega)\ f'(l_{kj}-\mu,\sigma)|k-j|(1+|k-j|)^{-\gamma}\ (1+\log(1+|k-j|))]\\
&\le K'_5\ g(|\mu|+\sigma)[\overline{W}(\omega_t)]^{1/2}\\
&\quad\times\sum_{k\in\mathbb{Z}^d} f'_k\ n_k^2(\omega)\ [\sum_{j\in\mathbb{Z}^d}|k-j|\,(1+|k-j|)^{-\gamma}\ (1+\log(1+|k-j|)\ (1+\lambda|k-j|)^{p+1}]\\
&\le K''_5\ g(|\mu|+\sigma)\,[\overline{W}(\omega_t)]^{1/2}\frac{\partial}{\partial\sigma}W(\omega_t,\mu,\sigma),
\end{aligned}\tag{4.5}$$

where l_{kj} is a point between k and j. Here we have used $\gamma > 2p+1$ to get the last inequality.

Finally for D_2, we use (2.4) to obtain the following bound:

$$\begin{aligned}
|D_2| &\le K_6 \sum_k \sum_{j\neq k;|q_k-q_j|\le r} |\operatorname{grad}\ f(z_{kj}-\mu,\sigma)|\,(a+b\ U^{(1)}(q_k-q_j))\ g(|q_k|)\ [\overline{W}(\omega_t)]^{1/2}\\
&\le K'_6\ g(|\mu|+\sigma)[\overline{W}(\omega_t)]^{1/2}\ \sum_k\ \sum_{j\neq k;|q_k-q_j|\le r} f'(z_{kj}-\mu,\sigma)\ (a+b\ U^{(1)}(q_k-q_j|))\frac{g(|q_k|)}{g(|z_{kj}|)}.
\end{aligned}$$

Notice that if $|q_k - q_j| < r$, $g(|q_k|) \le K_7\ g(|z_{kj}|)$. By Lemma 3.1 (d),

$$\begin{aligned} \sum_k \sum_{j\neq k; |q_k-q_j|\le r} f'(z_{kj}-\mu,\sigma) &\le K_8 \sum_{k\in\mathbb{Z}^d} \sum_{|j-k|\le r} n_k\ n_j\ f'_k \\ &\le K'_8 \sum_{k\in\mathbb{Z}^d} f'_k\ n_k^2. \end{aligned} \tag{4.6}$$

Moreover, a direct computation yields

$$\begin{aligned} &\sum_k \sum_{j\neq k; |q_k-q_j|\le r} f'(z_{kj}-\mu,\sigma) U^{(1)}(q_k-q_j) \\ &\le K_9 \sum_k f'_k \sum_{j\neq k; |q_k-q_j|\le r} U^{(1)}(q_k-q_j) \\ &\le K_9 \sum_k f'_k\ [2A + |p_k|^2 + \sum_{j\neq k} U(q_k-q_j)] + K_{10} \sum_k f'_k \sum_{j\neq k; |q_k-q_j|\le r} (1+|q_k-q_j|)^{-\gamma} \\ &\le K_9 \sum_k f'_k\ W_k(\omega_t) + K_{11} \sum_{k\in\mathbb{Z}^d} f'(k-\mu,\sigma)\ n_k^2(\omega). \end{aligned} \tag{4.7}$$

Using (4.6)-(4.7) and Lemma 3.2(b), we see that

$$|D_2| \le K_{12}\ g(|\mu|+\sigma)\ [\overline{W}(\omega_t)]^{1/2} \frac{\partial}{\partial\sigma} W(\omega_t,\mu,\sigma). \tag{4.8}$$

Thus the proof of the proposition follows from (4.4), (4.5) and (4.8). ■

By the method of Fritz [Fr1, Proposition 4.2] and Lemma 3.3 one can prove the following proposition:

Proposition 4.2. *For each $h > 0$ and $T > 0$ there exists a finite $\overline{h} = \overline{h}(h,T)$ such that $\overline{H}(\omega_0) \le h$ implies $\overline{H}(\omega_t) \le \overline{h}$ for all $|t| \le T$ provided that ω_t is a tempered solution to (1.1).*

As a consequence we have the following localization of particles:

Proposition 4.3. *If ω_t is a tempered solution to (1.1) and $\overline{H}(\omega_0) < h$, then there is a constant $c(h,T)$ such that*

$$|q_k(\omega_{t_1}) - q_k(\omega_{t_2})| \le c(h,T)\, g(|q_k(\omega_{t_1})|),$$

for all $k \in I$, $\quad |t_1| \le T, \quad |t_2| \le T$.

Proof. Let us simply write $q_k(\omega_t) := q_k(t)$. By Proposition 4.2, we have

$$|\frac{dq_k}{dt}| \leq g(|q_k|)\ c_1(h,T). \tag{4.9}$$

Define

$$q_{max} := \max_{|t|\leq T} |q_k(t)| \quad \text{and} \quad q_{min} := \min_{|t|\leq T} |q_k(t)|.$$

It follows from (4.9) that

$$|q_k(t_2) - q_k(t_1)| \leq 2\,T\,c_1(h,T)\ g(q_{max}). \tag{4.10}$$

Similarly we have also

$$q_{min} \geq q_{max} - 2\,T\,c_1(h,T)\ g(q_{max}). \tag{4.11}$$

(4.11) implies the existence of $c_2(h,T)$ such that

$$\frac{g(q_{max})}{g(q_{min})} \leq c_2(h,T) \tag{4.12}$$

Inserting (4.12) into (4.10) and using the fact that $g(u)$ is an increasing function, we get the desired result. ■

5 Existence

We are going to show the existence of tempered solutions to (1.1). For that purpose we have to investigate a kind of uniform boundedness. Let $(r_n)_{n=1}^{\infty}$ be any increasing sequence of positive numbers such that $r_n \to \infty$ as $n \to \infty$. Let $\omega_0 \in \overline{\Omega}$ be any initial configuration. For each $n \geq 1$, let $\omega_0^{(n)}$ be the configuration obtained by discarding the particles outside the r_n-ball with center the origin. Let $\omega^{(n)}(t) = (q_k^{(n)}(t), p_k^{(n)}(t))$ be the (finite system) solution of (1.1) with initial condition $\omega^{(n)}(0) = \omega_0^{(n)}$. We start with the following lemma saying that the energy densities of cutoff sub-systems are uniformly bounded.

Lemma 5.1. *Let $\omega_0 \in \overline{\Omega}$ and $\omega_0^{(n)}$, $\quad n \geq 1$, be defined as above. There exists a constant $c_0 > 0$ independent of ω_0 such that the bounds*

$$\overline{H}(\omega_0^{(n)}) \leq c_0\ \overline{H}(\omega_0)$$

hold uniformly in $n \in \mathbb{N}$.

Proof. For any $\mu \in \mathbb{R}^d$ and $\sigma > 0$, let us compare $H(\omega_0^{(n)}, \mu, \sigma)$ to $H(\omega_0, \mu, \sigma)$. For our purpose it is sufficient to consider the cases where the σ-ball $B_\sigma(\mu)$ touches the boundary of r_n-ball $B_{r_n}(0)$. In that case we see that

$$H(\omega_0^{(n)}, \mu, \sigma) \le H(\omega_0, \mu, \sigma) - \sum^{*} U(q_k - q_l), \tag{5.1}$$

where $q_k = q_k(\omega_0)$, $p_k = p_k(\omega_0)$, and

$$\sum^{*} U(q_k - q_l) = \frac{1}{2}\left(\sum_{k\in\Lambda_1}\sum_{l\in\Lambda_2} U(q_k - q_l) + \sum_{k,l\in\Lambda_2} U(q_k - q_l)\right), \tag{5.2}$$

where $\Lambda_1 := B_{r_n}(0) \cap B_\sigma(\mu)$, $\Lambda_2 := B_\sigma(\mu) \setminus \Lambda_1$. By the superstability of the potential, we see that

$$\begin{aligned}
\sum_{k,l\in\Lambda_2} U(q_k - q_l) &\ge -A \sum_{l\in\Lambda_2\cap\mathbb{Z}^d} n_l(\omega_0) \\
&\ge -A \sum_{l\in B_\sigma(\mu)\cap\mathbb{Z}^d} n_l(\omega_0)^2 \\
&\ge -\frac{2A}{B} H(\omega_0, \mu, \sigma + 1).
\end{aligned} \tag{5.3}$$

Here we have used (3.18) to get the last inequality. On the other hand, by using the decomposition of $U = U^{(1)} + U^{(2)}$ in (2.2), let us put

$$\sum_{k\in\Lambda_1}\sum_{l\in\Lambda_2} U(q_k - q_l) = \sum_{k\in\Lambda_1}\sum_{l\in\Lambda_2} U^{(1)}(q_k - q_l) + \sum_{k\in\Lambda_1}\sum_{l\in\Lambda_2} U^{(2)}(q_k - q_l). \tag{5.4}$$

Since $U^{(1)} \ge 0$ and $U^{(1)}(x) \ge -M(1 + |x|)^\gamma$, it is not hard to see that

$$\begin{aligned}
\sum_{k\in\Lambda_1}\sum_{l\in\Lambda_2} U(q_k - q_l) &\ge -c_{11} \sum_{k\in B_\sigma(\mu)\cap\mathbb{Z}^d} n_k(\omega_0)^2 \\
&\ge -c'_{11} H(\omega_0, \mu, \sigma + 1).
\end{aligned} \tag{5.5}$$

Inserting (5.3) and (5.5) into (5.1), we proved the lemma. ■

When $q_k(0)$ is outside of the r_n-ball, define $q_k^{(n)}(t) \equiv q_k(0)$ and $p_k^{(n)}(t) \equiv 0$ for all $t \ge 0$.

Lemma 5.2. *For each $k \in I$, and $T > 0$, the sequence of continuous functions $(q_k^{(n)}(\cdot))_{n=1}^\infty$ and $(p_k^{(n)}(\cdot))_{n=1}^\infty$ on $[0, T]$ are equi-continuous.*

Proof. For any $-T \le t_1 \le t_2 \le T$, it follows from Lemma 5.1, Proposition 4.2, and Proposition 4.3 that

$$\begin{aligned} |q_k^{(n)}(t_2) - q_k^{(n)}(t_1)| &\le \int_{t_1}^{t_2} |p_k^{(n)}(s)|\, ds \\ &\le c_{12}\, g(|q_k(\omega_0)|)\ |t_2 - t_1|. \end{aligned} \tag{5.6}$$

On the other hand

$$|p_k^{(n)}(t_2) - p_k^{(n)}(t_1)| \le \sum_{j \ne k} \int_{t_1}^{t_2} |\operatorname{grad} U(q_j^{(n)}(s) - q_k^{(n)}(s))|\, ds.$$

We again decompose $U = U^{(1)} + U^{(2)}$ as in (2.2). By (2.5), we may assume that there exist $c > 1$ and $L > 1$ such that

$$|\operatorname{grad} U(x)| \le L(L + U^{(1)}(x))^c. \tag{5.7}$$

Thus we have

$$\begin{aligned} \sum_{l \ne k} |\operatorname{grad} U(q_j^{(n)} - q_k^{(n)})| &\le \sum_{j \ne k} L[L + U^{(1)}(q_j^{(n)} - q_k^{(n)})]^c \\ &\le L\Big(\sum_{j \ne k : |q_j^{(n)} - q_k^{(n)}| \le r} [L + U^{(1)}(q_j^{(n)} - q_k^{(n)})] \Big)^c \\ &\le L\Big(\sum_{j,l \in I_{k,n}} [L + U^{(1)}(q_j^{(n)} - q_l^{(n)})] \Big)^c, \end{aligned} \tag{5.8}$$

where $I_{k,n} = \{ j \in I : |q_j^{(n)} - q_k^{(n)}| \le r \}$. By direct computations, we have that

$$\sum_{j,l \in I_{k,n}} L \le L\, r^d \sum_{j \in \mathbb{Z}^d \cap B_r(q_k^{(n)})} (1 + n_j^2) \le c_{13}\, L r^d H(\omega^{(n)}, q_k^{(n)}, r), \tag{5.9}$$

and

$$\begin{aligned} &\sum_{j,l:} U^{(1)}(q_j^{(n)} - q_l^{(n)}) \\ &\le \sum_{j : |q_j^{(n)} - q_k^{(n)}| \le r} \Big(|p_j^{(n)}|^2 + A + \sum_{l : |q_l^{(n)} - q_k^{(n)}| \le r} U(q_j^{(n)} - q_l^{(n)}) \Big) + c_{14} \sum_{j,l \in I_{k,n}} (1 + |q_j^{(n)} - q_l^{(n)}|)^{-\gamma} \\ &\le c'_{14}\, H(\omega^{(n)}, q_k^{(n)}, r). \end{aligned} \tag{5.10}$$

Inserting (5.9) and (5.10) into (5.8) and using Lemma 5.1, Proposition 4.2, and Proposition 4.3, we have that

$$|p_k^{(n)}(t_2) - p_k^{(n)}(t_1)| \le c_{15}\, g(|q_k(\omega_0)|)^{cd}\ |t_2 - t_1|. \tag{5.11}$$

Thus the lemma follows from (5.6) and (5.11). ∎

We are now ready to prove the existence of tempered solutions to (1.1). The method of the proof is given in [Fr1, Theorem 5.1], but we provide it for completeness.

Proof of Theorem 2.1. For each $n = 1, 2, \cdots$, let $\omega^{(n)}(t)$ be the tempered solution corresponding to the cutoff finite subsystem introduced above. By Lemma 5.2, for any finite $T > 0$ and $k \in I$, the sequences $(q_k^{(n)}(t))_{n=1}^{\infty}$ and $(p_k^{(n)}(t))_{n=1}^{\infty}$ are equi-continuous. Thus by Arzela-Ascoli theorem, after a diagonal procedure, we see that there exists a subsequence $n_i, \quad i = 1, 2, \cdots$, such that for all $k \in I$, the sequences $(q_k^{(n_i)}(t))_{i=1}^{\infty}$ and $(p_k^{(n_i)}(t))_{i=1}^{\infty}$ uniformly converge on $[0, T]$ and define continuous functions $q_k(t)$ and $p_k(t)$. Since $T > 0$ is arbitrary we can extend these functions to all $t \in \mathbb{R}$. Since $\overline{H}$ is lower semicontinuous, the a priori bounds remain in force for $\omega_t := (q_k, p_k)_{k \in I}$. Exploiting the continuity of the r.h.s. of the integral version of (1.1), we conclude that ω_t really satisfies (1.1). ∎

6 Uniqueness

In order to prove uniqueness of tempered solution, we use an iteration method which turns out to be much complicated compared to that used in [Fr1, Section 5]. Consider a sequence $\delta(t, m)$, $m = 0, 1, \ldots$, of nonnegative and continuous functions on $[0, T]$ such that the bounds

$$\delta(t, m) \leq \delta(0, m) + L_1 a_m \int_0^t \delta(s, m+1)\, ds + L_2 b_m \tag{6.1}$$

hold for all $t \in [0, T]$ and m with some constants L_1 and L_2 (independent of m) and positive, increasing sequences $(a_m)_{m \geq 0}$ and $(b_m)_{m \geq 0}$. If

$$\delta(t, m) \leq Q_T \exp(m Q_T) \tag{6.2}$$

for $t \leq T$ and for all $m \geq 0$, and furthermore if the infinite series

$$\sum_{m=0}^{\infty} \frac{1}{m!} [(a_m)^m \, b_m]^{\alpha} < \infty \tag{6.3}$$

converges for any $\alpha \geq 1$, then (6.1) can be iterated infinitely many times to obtain

$$\delta(t,0) \leq \sum_{m=0}^{\infty} \delta(0,m) \frac{(L_1 t)^m}{m!} (a_m)^m + \sum_{m=0}^{\infty} \frac{(L_1 t)^m}{m!} (a_m)^m \ (L_2 b_m). \tag{6.4}$$

Recall the decomposition $U(x) = U^{(1)}(x) + U^{(2)}(x)$ in (2.2). $U^{(1)}(x)$ satisfies (2.3)-(2.5) and $U^{(2)}(x)$ satisfies (2.6).

Proof of Theorem 2.2. Let $f_0 : \mathbb{R} \to [0,1]$ be a continuously differentiable nonincreasing function such that $f_0(u) = 1$ if $u \leq 1$, $f_0(u) = 0$ if $u \geq 2$, and $-f_0'(u) \leq f_0(u-1)$ for all $u \in \mathbb{R}$. Recall that the singular part of the interaction has finite range $r < \infty$. For each $R \geq r+1$ let $f_R(u) := f_0(\frac{u}{R}), \quad u \in \mathbb{R}$. Then $f_R(u)$ also satisfies

$$f_R(u) = 1 \text{ if } u \leq R, \quad f_R(u) = 0 \text{ if } u \geq 2R, \quad \text{and} \quad -f_R'(u) \leq f_R(u-R). \tag{6.5}$$

Suppose now that $\omega_t = (q_k(t), p_k(t))_{k \in I}$ and $\overline{\omega_t} = (\overline{q}_k(t), \overline{p}_k(t))_{k \in I}$ are two tempered solutions of (1.1) with the same initial condition $\omega_0 = (q_k(0), p_k(0)) \in \overline{\Omega}$. For each $R \geq r+1$, let us define

$$\begin{aligned} d_R(\omega_t, \overline{\omega}_t, m) \ &:= \ \sum_{k \in I} f_R\big(|q_k| - 2\tilde{c}R(1+mg(m))\big) \ f_R\big(|\overline{q}_k| - 2\tilde{c}R(1+mg(m))\big) \\ &\qquad \times [|q_k - \overline{q}_k|_1 + |p_k - \overline{p}_k|_1], \end{aligned} \tag{6.6}$$

where $\tilde{c} \geq 1$ is a sufficiently large fixed number that will be characterized later. We have used also the norm $|x|_1 := |x^{(1)}| + \cdots + |x^{(d)}|$ for each $x = (x^{(1)}, \cdots, x^{(d)}) \in \mathbb{R}^d$. We would like to show that for each $R \geq r+1$ the sequence of functions $\delta_R(t,m) := d_R(\omega_t, \overline{\omega}_t, m)$ satisfies (6.1)-(6.3) with suitably chosen sequences $(a_m)_{m \geq 0}$, $(b_m)_{m \geq 0}$ and constants $L_1(R), L_2(R)$. Given $R \geq r+1$ and $T > 0$, the r.h.s. of (6.6) is an absolutely continuous function of $t \in [0,T]$, thus we can differentiate it almost everywhere. By a notational convenience, let us drop the subscript R in $f_R(\cdot)$; $f(\cdot) \equiv f_R(\cdot)$ (cf. $f(x,\sigma)$ in (3.1)) and denote $f(t,k,m) := f(|q_k| - 2\tilde{c}R(1+mg(m)))$, $\overline{f}(t,k,m) := f(|\overline{q}_k| - 2\tilde{c}R(1+mg(m)))$. By differentiating the right hand side of (6.6), we have

$$\frac{d}{dt}(\delta_R(t,m)) = A^{(1)}(t,m) + A^{(2)}(t,m) + A^{(3)}(t,m) + A^{(4)}(t,m) \tag{6.7}$$

where

$$
\begin{aligned}
A^{(1)}(t,m) &= \sum_{k\in I}\Big(\frac{d}{dt}f(t,k,m)\Big)\overline{f}(t,k,m)\Big(|q_k-\overline{q}_k|_1+|p_k-\overline{p}_k|_1\Big), \qquad (6.8)\\
A^{(2)}(t,m) &= \sum_{k\in I}f(t,k,m)\Big(\frac{d}{dt}\overline{f}(t,k,m)\Big)\Big(|q_k-\overline{q}_k|_1+|p_k-\overline{p}_k|_1\Big),\\
A^{(3)}(t,m) &= \sum_{k\in I}f(t,k,m)\overline{f}(t,k,m)\frac{d}{dt}|q_k-\overline{q}_k|_1,\\
A^{(4)}(t,m) &= \sum_{k\in I}f(t,k,m)\overline{f}(t,k,m)\frac{d}{dt}|p_k-\overline{p}_k|_1.
\end{aligned}
$$

In the below, we will estimate each term in (6.8).

Since the bound

$$|p_k| \le H(\omega, q_k, g(|q_k|))^{1/2} \le c_{16}\, g(|q_k|)^{d/2} \qquad (6.9)$$

holds for any $k\in I$, we use (6.5) to see that

$$|\frac{d}{dt}f(t,k,m)| \le c_{17}\, f(t,k,m+1)\ (g(R)\,g(m))^{d/2}, \qquad (6.10)$$

and

$$|\frac{d}{dt}\overline{f}(t,k,m)| \le c_{17}\ \overline{f}(t,k,m+1)\ (g(R)\,g(m))^{d/2}. \qquad (6.11)$$

Obviously,

$$|\frac{d}{dt}|q_k-\overline{q}_k|_1\,| \le |p_k-\overline{p}_k|_1. \qquad (6.12)$$

Thus, it follows from (6.9)-(6.12) that

$$|A^{(1)}(t,m)+A^{(2)}(t,m)+A^{(3)}(t,m)| \le c_{18}(g(R)g(m))^{d/2}\delta_R(t,m+1). \qquad (6.13)$$

Finally, let us consider the last term $A^{(4)}(t,m)$ in (6.8). Observe that

$$
\begin{aligned}
|\frac{d}{dt}|p_k-\overline{p}_k|_1\,| \ &\le \ \sum_{l\ne k}|\ \mathrm{grad}\ U^{(1)}(q_l-q_k)-\mathrm{grad}\ U^{(1)}(\overline{q}_l-\overline{q}_k)|\\
&\quad +\sum_{l\ne k}|\ \mathrm{grad}\ U^{(2)}(q_l-q_k)-\mathrm{grad}\ U^{(2)}(\overline{q}_l-\overline{q}_k)|. \qquad (6.14)
\end{aligned}
$$

Thus

$$|A^{(4)}(t,m)| \le Q_1(t,m)+Q_2(t,m), \qquad (6.15)$$

where

$$Q_1(t,m) \tag{6.16}$$
$$= \sum_{k\in I}\left(f(t,k,m)\overline{f}(t,k,m)\sum_{l\neq k}\Big(|\text{ grad } U^{(1)}(q_l-q_k) - \text{grad } U^{(1)}(\overline{q}_l-\overline{q}_k)|\Big)\right),$$
$$Q_2(t,m)$$
$$= \sum_{k\in I}\left(f(t,k,m)\overline{f}(t,k,m)\sum_{l\neq k}\Big(|\text{ grad } U^{(2)}(q_l-q_k) - \text{grad } U^{(2)}(\overline{q}_l-\overline{q}_k)|\Big)\right).$$

In order to control the first term $Q_1(t,m)$, we use (2.5) to see that

$$\begin{aligned}&\sum_{l\neq k}|\text{ grad } U^{(1)}(q_l-q_k) - \text{grad } U^{(1)}(\overline{q}_l-\overline{q}_k)|\\ &\le L\sum_{l\neq k}[\,L+U^{(1)}(q_l-q_k)+U^{(1)}(\overline{q}_l-\overline{q}_k)\,]^c\,[\,|q_l-\overline{q}_l|_1+|q_k-\overline{q}_k|_1\,].\end{aligned} \tag{6.17}$$

Following the method similar to that used to obtain (5.11), one can check that

$$\begin{aligned}&\sum_{l\neq k}[\,L+U^{(1)}(q_l-q_k)+U^{(1)}(\overline{q}_l-\overline{q}_k)\,]^c\;|q_k-\overline{q}_k|_1\\ &\le c_{19}\;[\,H(\omega_t,q_k,r)+H(\overline{\omega}_t,\overline{q}_k,r)\,]^c\;|q_k-\overline{q}_k|_1\\ &\le c'_{19}\;[\,g(|q_k|)^{cd}+g(|\overline{q}_k|)^{cd}\,]\;|q_k-\overline{q}_k|_1\\ &\le c''_{19}\;g(R)^{cd}\;g(m)^{cd}\;|q_k-\overline{q}_k|_1.\end{aligned} \tag{6.18}$$

On the other hand, since $U^{(1)}$ has interaction range $r < R$, we see that

$$\begin{aligned}&\sum_{k\in I} f\big(|q_k|-2\tilde{c}R(1+mg(m))\big)\;f\big(|\overline{q}_k|-2\tilde{c}R(1+mg(m))\big)\\ &\quad\times\sum_{l\neq k}[L+U^{(1)}(q_l-q_k)+U^{(1)}(\overline{q}_l-\overline{q}_k)]^c\;|q_l-\overline{q}_l|_1\\ &\le \sum_{l\in I} f\big(|q_l|-2\tilde{c}R(1+(m+1)g(m+1))\big)\;f\big(|\overline{q}_l|-2\tilde{c}R(1+(m+1)g(m+1))\big)\\ &\quad\times|q_l-\overline{q}_l|_1\;\sum_{k\neq l}[L+U^{(1)}(q_l-q_k)+U^{(1)}(\overline{q}_l-\overline{q}_k)]^c\\ &\le c_{20}\;g(R)^{cd}\;g(m)^{cd}\;\delta_R(t,m+1).\end{aligned} \tag{6.19}$$

Hence by (6.16)-(6.19), we obtain that

$$Q_1(t,m) \le c_{21}\,g(R)^{cd}g(m)^{cd}\delta_R(t,m+1) \tag{6.20}$$

Let us estimate the second term $Q_2(t,m)$ in (6.16). We use (2.6) to obtain

$$\begin{aligned}
&\sum_{l\neq k} |\ \mathrm{grad}\ U^{(2)}(q_l-q_k) - \mathrm{grad}\ U^{(2)}(\overline{q}_l-\overline{q}_k)\ | \\
&\le \sum_{l\neq k}(\psi(|q_l-q_k|)+\psi(|\overline{q}_l-\overline{q}_k|))\ [|q_l-\overline{q}_l|_1+|q_k-\overline{q}_k|_1] \\
&= \Big(\sum_{l\neq k}\psi(|q_l-q_k|)+\psi(|\overline{q}_l-\overline{q}_k|)\Big)|q_k-\overline{q}_k|_1 + \sum_{l\neq k}(\psi(|q_l-q_k|)+\psi(|\overline{q}_l-\overline{q}_k|))|q_l-\overline{q}_l|_1 \\
&\equiv S_1(t,k)+S_2(t,k). \qquad (6.21)
\end{aligned}$$

Thus, by the definition of $Q_2(t,m)$ in (6.16), we have

$$\begin{aligned}
Q_2(t,m) &\le \sum_{k\in I} f(t,k,m)\overline{f}(t,k,m)S_1(t,k) + \sum_{k\in I} f(t,k,m)\overline{f}(t,k,m)\, S_2(t,k) \\
&\equiv Q_2^{(1)}(t,m)+Q_2^{(2)}(t,m). \qquad (6.22)
\end{aligned}$$

In order to control $Q_2^{(1)}(t,m)$, we notice that by Lemma 3.1 (a) and (2.6), the bound

$$\psi(|q_l-q_k|) \le c_{22}\, f(q_l-q_k, g(|q_k|)) \qquad (6.23)$$

holds. We then use the above bound, Schwarz inequality and Lemma 3.2 to see that

$$\sum_{l\neq k}\psi(|q_l-q_k|) \le c_{23}\, W(\omega_t, q_k, g(|q_k|))^{d/2}. \qquad (6.24)$$

Repeating the same calculation after a change ω_t by $\overline{\omega}_t$, we conclude that

$$S_1(t,k) \le c_{24}\,(g(R)\,g(m))^{d/2}\ |q_k-\overline{q}_k|_1,$$

and so

$$Q_2^{(1)}(t,m) \le c_{24}\,(g(R)g(m))^{1/2}\delta_R(t,m+1). \qquad (6.25)$$

Next, we consider $Q_2^{(2)}(t,m)$. We write

$$\begin{aligned}
S_2(t,k) &= \sum_{l\neq k}\psi(|q_l-q_k|)\ |q_l-\overline{q}_l|_1 + \sum_{l\neq k}\psi(|\overline{q}_l-\overline{q}_k|)\ |q_l-\overline{q}_l|_1 \\
&\equiv S_{2,1}(t,k)+S_{2.2}(t,k), \qquad (6.26)
\end{aligned}$$

and

$$\begin{aligned}
Q_2^{(2)}(t,m) &= \sum_{k\in I} f(t,k,m)\overline{f}(t,k,m)\, S_{2,1}(t,k) + \sum_{k\in I} f(t,k,m)\overline{f}(t,k,m)\, S_{2,2}(t,k) \\
&\equiv Q_2^{(2,1)}(t,m)+Q_2^{(2,2)}(t,m). \qquad (6.27)
\end{aligned}$$

By symmetry, it is enough to control only $Q_2^{(2,1)}(t,m)$. For that purpose, let us further decompose $S_{2,1}$ as follows:

$$S_{2,1}(t,k) = \sum_{l\neq k: |q_l-q_k|\leq R} \psi(|q_l-q_k|)\ |q_l-\overline{q}_l|_1 + \sum_{l:\,|q_l-q_k|>R} \psi(|q_l-q_k|)\ |q_l-\overline{q}_l|_1.$$

Substituting the above decomposition into the definition of $Q_2^{(2,1)}(t,m)$ in (6.27), one has that

$$\begin{aligned} Q_2^{(2,1)}(t,m) &= \sum_{k\in I} f(t,k,m)\overline{f}(t,k,m) \sum_{l\neq k:|q_l-q_k|\leq R} \psi(|q_l-q_k|)\ |q_l-\overline{q}_l|_1 \\ &\quad + \sum_{k\in I} f(t,k,m)\overline{f}(t,k,m) \sum_{l:|q_l-q_k|>R} \psi(|q_l-q_k|)\ |q_l-\overline{q}_l|_1 \\ &\equiv D_1(t,m) + D_2(t,m). \end{aligned} \tag{6.28}$$

By Proposition 4.3, it is not hard (and very important) to see that if $f\big(|q_k| - 2\tilde{c}R(1+mg(m))\big) \neq 0$, $f\big(|\overline{q}_k| - 2\tilde{c}R(1+mg(m))\big) \neq 0$, and $|q_l-q_k| \leq R$, then $f\big(|q_l| - 2\tilde{c}R(1+(m+1)g(m+1))\big) = 1$ and $f\big(|\overline{q}_l| - 2\tilde{c}R(1+(m+1)g(m+1))\big) = 1$ when $\tilde{c} \geq 1$ is a large enough constant. Thus, by interchanging the order of summation and using the argument used in deriving (6.24), we have the bound for $D_1(t,m)$ in (6.28):

$$D_1(t,m) \leq c_{25}\,(g(R)\,g(m))^{d/2}\ \delta_R(t,m+1). \tag{6.29}$$

Finally, let us consider $D_2(t,m)$. If $f\big(|q_k| - 2\tilde{c}R(1+mg(m))\big) \neq 0$, it follows from Proposition 4.3 that

$$|q_l-\overline{q}_l|_1 \leq c_{26}\,g(|q_l|) \leq c_{26}'\,g(|q_k|)\ g(|q_l-q_k|) \leq c_{26}''\,g(R)\ g(m)\ g(|q_l-q_k|).$$

Thus, using (6.23) and Lemma 3.2, we obtain that

$$\begin{aligned} &\sum_{l:|q_l-q_k|>R} \psi(|q_l-q_k|)\ |q_l-\overline{q}_l|_1 \\ &\leq c_{27}\,g(R)g(m) \sum_{l\in\mathbb{Z}^d:|l-q_k|\geq R} n_l(\omega_t)\ \psi(|l-q_k|)\ g(|l-q_k|) \\ &\leq c_{27}\,g(R)g(m)\Big(\sum_{l:|l-q_k|\geq R} \psi(|l-q_k|)g(|l-q_k|)^2\Big)^{1/2}\Big(\sum_{l\in\mathbb{Z}^d} n_l(\omega_t)^2 f(l-q_k, g(|q_k|))\Big)^{1/2} \\ &\leq c_{27}'\,g(R)g(m)[\psi(R)R^{d+\epsilon_1}]^{1/2}[g(R)g(m)]^{d/2}, \end{aligned}$$

where $0 < \epsilon_1 < 1$ is a small constant. Substituting the above bound into the expression of $D_2(t,m)$ in (6.28), we conclude that

$$\begin{aligned} D_2(t,m) &\leq c'_{28}\, g(R)^2 g(m)^2 [\psi(R) R^{d+\epsilon_1}]^{1/2} \sum_{l\in\mathbb{Z}^d:|l|\leq 2\tilde{c}R(1+mg(m))+3R} n_l(\omega_t) \\ &\leq c_{28}\, g(R)^2 g(m)^2 [\psi(R) R^{d+\epsilon_1}]^{1/2} [Rmg(m)]^d. \end{aligned} \tag{6.30}$$

Here we have used superstability to get the last inequality (see (3.18)). Thus it follows from (6.22), (6.25) and (6.28) - (6.30) that

$$\begin{aligned} |Q_2(t,m)| &\leq c_{29}\, (g(R)g(m))^{d/2}\, \delta_R(t,m+1) \\ &\quad + c_{28}\, g(R)^2 g(m)^2\, [\psi(R) R^{d+\epsilon_1}]^{1/2} [Rmg(m)]^d. \end{aligned} \tag{6.31}$$

Therefore from (6.13), (6.15), (6.20), (6.25), and (6.31) we conclude that

$$\begin{aligned} |\frac{d}{dt}\delta_R(t,m)| &\leq \overline{C_1}[g(R)^{cd} g(m)^{cd} + (g(R)g(m))^{d/2}]\, \delta_R(t,m+1) \\ &\quad + \overline{C}_2\, g(R)^2 g(m)^2\, [\psi(R) R^{d+\epsilon_1}]^{1/2}\, [Rmg(m)]^d \\ &\leq C_1 (g(R)^{cd}\ g(m)^{cd})\ \delta_R(t,m+1) + C_2\, m^{2d}\ [\psi(R)]^{1/2} R^{d+\frac{2}{3}d} \end{aligned} \tag{6.32}$$

where $\overline{C_1}, C_1$, $\overline{C}_2$ and C_2 are positive constants.

To use the iteration method, we see that for all $0 \leq t \leq T$, the sequence $\delta_R(t,m) := d_R(\omega_t, \overline{\omega}_t, m)$ satisfies the relation (6.1) with $L_1 = C_1\, g(R)^{cd}$, $L_2 = T\ C_2\, [\psi(R)]^{1/2} R^{d+\frac{2}{3}d}$, $a_m = g(m)^{cd}$, and $b_m = m^{2d}$. It is easy to check that the conditions in (6.2)-(6.3) are fulfilled. Thus by (6.4) we have

$$\begin{aligned} \delta_R(\omega_t, \overline{\omega}_t, m) &\leq \sum_{m=0}^{\infty} \delta_R(\omega_0, \overline{\omega}_0, m) \frac{1}{m!} (C_1\, g(R)^{cd} t)^m\ g(m)^{cdm} \\ &\quad + C_3\, [\psi(R)]^{1/2} R^{d+\frac{2}{3}d} \sum_{m=0}^{\infty} \frac{1}{m!} (C_1\ g(R)^{cd} t)^m\ g(m)^{cdm} m^{2d} \end{aligned} \tag{6.33}$$

where C_3 is a positive constant. Notice that $\delta_R(\omega_0, \overline{\omega}_0, m) = 0$ for all $m = 0, 1, \cdots$. By using Hölder's inequality with conjugate pair $p = (1+\epsilon)$ and $p' = \frac{1}{\epsilon} p$ with small $\epsilon > 0$, we obtain

$$\begin{aligned} \delta_R(\omega_t, \overline{\omega}_t, m) \leq \quad & C_3\ [\psi(R)]^{1/2} R^{d+\frac{2}{3}d} \left(\exp[(C_1 t)^{1+\epsilon} g(R)^{cd(1+\epsilon)}]\right)^{\frac{1}{1+\epsilon}} \\ & \times \left(\sum_{m=0}^{\infty} \frac{1}{m!} [g(m)^{cdm} m^{2d}]^{p'} \right)^{\frac{1}{p'}}. \end{aligned} \tag{6.34}$$

Notice that the last term converges to a constant. We check here that when R is large, $g(R)$ is similar to $(\log R)^{1/d}$ (see (1.2)). On the other hand, when the potential is singular at the origin, then $c > 1$ (see (2.5)). Thus the exponential term in the r.h.s. of (6.34) increases faster than any polynomial order, but slower than any exponential order as $R \to \infty$. But, since we are assuming that $|U^{(2)}(x)| \le \psi(|x|) \le \exp[-(\log |x|)^{c+\epsilon_0}]$ for some $\epsilon_0 > 0$ and since the function $R \to \delta_R(\omega_t, \overline{\omega}_t, m)$ is increasing as R increases, by taking ϵ such that $c(1+\epsilon) < c + \epsilon_0$ and letting R go to infinity, we conclude from (6.34) that

$$\delta_{R_0}(\omega_t, \overline{\omega}_t, 0) = 0, \quad \text{for all} \quad R_0 \ge r + 1. \tag{6.35}$$

Since $R_0 \ge r+1$ can be taken arbitrarily large, (6.35) concludes the uniqueness. ∎

When the potential is not singular at the origin, the proof of uniqueness under the condition of polynomial decreasing rate follows as a simple corollary to the method given in the above.

Proof of Theorem 2.3. Since the potential $U(x)$ is everywhere two-times continuously differentiable, i.e., $U^{(1)}(x) \equiv 0$, the terms concerning to $U^{(1)}$ disappear in the proof of Theorem 2.2. For example, we may discard the terms of (6.20). Thus, observing the derivation of (6.32), we have the bound

$$|\frac{d}{dt}\delta_R(t,m)| \le C_1\left(g(R)g(m)\right)^{d/2}\delta_R(t,m+1) + C_2[\psi(R)]^{1/2}R^{d+\frac{2}{3}d}m^{2d}. \tag{6.36}$$

That is, we may replace the constant cd by $d/2$ in (6.34). Notice again that $g(R)$ is similar to $(\log R)^{1/d}$ for large values of R. Since $\exp[(C_1T)^{1+\epsilon}(\log R)^{\frac{1+\epsilon}{2}}]$ increases slower than $(1+R)^\alpha$ for any $\alpha > 0$ as $R \to \infty$, we see that the uniqueness holds with $\psi(R) = (1+R)^{-\gamma}$, the appropriate rate used to derive the a priori bound. ∎

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