# Localization of Surface Spectra*

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**Abstract**

We study spectral properties of the discrete Laplacian $H$ on the half-space $\mathbf{Z}_+^{d+1} = \mathbf{Z}^d \times \mathbf{Z}_+$ with random boundary condition $\psi(n,-1) = \lambda V(n)\psi(n,0)$; the $V(n)$ are independent random variables on a probability space $(\Omega, \mathcal{F}, P)$ and $\lambda$ is the coupling constant. It is known that if the $V(n)$ have densities, then on the interval $[-2(d+1), 2(d+1)]$ $(= \sigma(H_0)$, the spectrum of the Dirichlet Laplacian) the spectrum of $H$ is $P$-a.s. absolutely continuous for all $\lambda$ [JL1]. Here we show that if the random potential $V$ satisfies the assumption of Aizenman-Molchanov [AM], then there are constants $\lambda_d$ and $\Lambda_d$ such that for $|\lambda| < \lambda_d$ and $|\lambda| > \Lambda_d$ the spectrum of $H$ outside $\sigma(H_0)$ is $P$-a.s. pure point with exponentially decaying eigenfunctions.

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# 1 Introduction

This paper deals with the spectral theory of the discrete Laplacian on a half-space with a random boundary condition. The history of this problem and its physical aspects are discussed in [JMP, KP]. For some recent rigorous work on the subject we refer the reader to [AM, BS, Gri, JM1, JM2, JMP, JL1, JL2, KP, M1, P].

In this section we introduce the model, review some known results and state our theorems. At the end of the section we will briefly explain the basic ideas of our proofs and discuss some open problems.

## 1.1 The model

Let $d \geq 1$ be given and let $\mathbf{Z}_+^{d+1} := \mathbf{Z}^d \times \mathbf{Z}_+$, where $\mathbf{Z}_+ = \{0, 1, \ldots\}$. We denote the points in $\mathbf{Z}_+^{d+1}$ by $(n, x)$, for $n \in \mathbf{Z}^d$ and $x \in \mathbf{Z}_+$. Let $H$ be the discrete Laplacian on $l^2(\mathbf{Z}_+^{d+1})$ with boundary condition $\psi(n, -1) = V(n)\psi(n, 0)$. When $V = 0$ the operator $H$ reduces to the Dirichlet Laplacian which we denote by $H_0$. The operator $H$ acts as

$$(H\psi)(n, x) = \begin{cases} \sum_{|n-n'|_+ + |x-x'|=1} \psi(n', x') & \text{if } x > 0, \\ \psi(n, 1) + \sum_{|n-n'|_+=1} \psi(n', 0) + V(n)\psi(n, 0) & \text{if } x = 0, \end{cases}$$

where $|n|_+ = \sum_{i=1}^d |n_i|$. Note that the operator $H$ can be viewed as Schrödinger operator

$$H = H_0 + V, \tag{1.1}$$

where the potential $V$ acts only along the boundary $\partial \mathbf{Z}_+^{d+1} = \mathbf{Z}^d$, that is, $(V\psi)(n, x) = 0$ if $x > 0$ and $(V\psi)(n, 0) = V(n)\psi(n, 0)$. For many purposes, it is convenient to adopt this point of view and we will do so in the sequel. Since $H_0$ is bounded, the operator $H$ is properly defined as a self-adjoint operator. We recall that the spectrum of $H_0$ is purely absolutely continuous and that

$$\sigma(H_0) = [-2(d + 1), 2(d + 1)].$$

We are interested in the spectral results which hold for "almost every" boundary potential $V$. More precisely, let $\Omega$ be the set of all boundary potentials, that is, the functions $V : \mathbf{Z}^d \mapsto \mathbf{R}$. The set $\Omega$ can be identified with

$$\Omega = \mathbf{R}^{\mathbf{Z}^d} = \bigtimes_{\mathbf{Z}^d} \mathbf{R}.$$

Let $\mathcal{F}$ be the $\sigma$-algebra in $\Omega$ generated by the cylinder sets $\{V : V(n_1) \in B_1, \ldots, V(n_k) \in B_k\}$, where $B_1, \ldots, B_k$ are Borel subsets of $\mathbf{R}$. For each $n \in \mathbf{Z}^d$ let $\mu_n$ be a probability measure on $\mathbf{R}$, and let $P$ be a measure on $(\Omega, \mathcal{F})$ defined by

$$P := \bigtimes_{n \in \mathbf{Z}^d} \mu_n.$$

Note that $\mu_n$ is the probability distribution of the random variable $\Omega \ni V \mapsto V(n)$. We say that the random variable $V(n)$ has a density if the measure $\mu_n$ is absolutely continuous with respect to the Lebesgue measure. Obviously, the random variables $\{V(n)\}$ are independent[1], and we say that they are i.i.d. if all the measures $\mu_n$ are equal to $\mu$. Recall that the topological support of $\mu$, $\mathrm{supp}\mu$, is the complement of the largest open set $B$ such that $\mu(B) = 0$.

Let $U_0$ be a given background boundary potential on $\mathbf{Z}^d$. We will always assume that $U_0$ is bounded. In this paper we will study the operators

$$H = H_0 + U_0 + \lambda V, \qquad V \in \Omega. \tag{1.2}$$

Here, $\lambda$ is a real constant which measures the strength of the disorder. As usual in the theory of random Schrödinger operators, we are interested in the spectral properties of $H$ which hold $P$-a.s., that is, for a set of $V$'s of $P$-measure 1. For additional information about random Schrödinger operators we refer the reader to [CL, CFKS].

Let us briefly summarize the known results about the model (1.2).

**(1)** For any $V$, the wave operators

$$W^{\pm} = \mathrm{s} - \lim_{t \to \mp\infty} \mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{-\mathrm{i}tH_0}$$

exist. In particular, $\sigma(H_0) \subset \sigma_{\mathrm{ac}}(H)$. Moreover, if the random variables $V(n)$ have densities, then the spectrum of $H$ in $\sigma(H_0)$ is $P$-a.s. purely absolutely continuous. These results are proven in [JL1, JL2]. We emphasize that the first result is deterministic while the second is random – there are examples of potentials $V$ (which even satisfy $\lim_{|n|\to\infty} V(n) = 0$) such that $H_0 + V$ has embedded eigenvalues in $\sigma(H_0)$ [MV].

**(2)** If $U_0 =$ const. and the random variables $\{V(n)\}$ are i.i.d. with distribution $\mu$, then (1.2) is an ergodic family of random operators. In particular, it follows from the standard argument that there exists a set $\Sigma$ such that $\sigma(H) = \Sigma$ $P$-a.s. The set $\Sigma$ can be computed (see [JL1]). We set $\lambda = 1$ and absorb $U_0$ in $V$. Let

$$\mathcal{S} := \left\{ E + a + a^{-1} : E \in [-2d, 2d],\ a \in \mathrm{supp}\mu \text{ and } |a| \geq 1 \right\}.$$

Then $\Sigma = \sigma(H_0) \cup \mathcal{S}$. Note that whenever $\mathrm{supp}\mu \cap (\mathbf{R} \setminus [-1, 1]) \neq \emptyset$, the set $\Sigma$ has parts lying outside $\sigma(H_0)$.

**(3)** Assume that $d = 1$, $U_0 =$ const. and that the random variables $\{V(n)\}$ are i.i.d. with distribution $\mu$. Assume that $\mathrm{d}\mu = p(x)\mathrm{d}x$, that $p \in L^\infty(\mathbf{R})$ and that the topological

[1] We remark that the method of Aizenman-Molchanov (and therefore of our paper) easily allows for correlated random variables. For notational simplicity, however, we will deal only with independent random variables.

boundary of supp$\mu$ is a discrete set. Under these assumptions it was shown in [JM1] that for any $\lambda$ the spectrum of $H$ outside $\sigma(H_0)$ is $P$-a.s. pure point and that the corresponding eigenfunctions decay faster than any polynomial in the $n$-variable, and exponentially fast in the $x$-variable. Unfortunately, the techniques of [JM1] are sensitive to addition of (even periodic) background potentials $U_0$. If however supp$\mu$ is an unbounded set and $p \in L^\infty(\mathbf{R})$, then for any bounded background potential $U_0$ and all $\lambda$, the spectrum of $H = H_0 + U_0 + \lambda V$ outside $\sigma(H_0)$ is $P$-a.s. pure point and the corresponding eigenfunctions decay as above. Although this last result was not explicitly stated in [JM1], it is an easy consequence of the results proven in [JM1, JM4].

**(4)** In [AM] and [Gri] it is shown that for arbitrary dimensions we have localization away from the edges of $\sigma(H_0)$, that is, $\forall\lambda$ there exist $\delta(\lambda) > 0$ such that the spectrum of $H$ in the set

$$\{E : |E| > 2(d+1) + \delta(\lambda)\}$$

is $P$-a.s. pure point with exponentially decaying eigenfunctions. Moreover, $\delta(\lambda) \downarrow 0$ as $|\lambda| \uparrow \infty$. Similar results hold for fixed $\lambda$ and large $|E|$. The assumption on the $\mu_n$'s under which this result is proven in [AM] is Hypothesis $\mathcal{B}(d)$ below (which should hold for *some* $0 < s < 1$). In [Gri], the assumption on the $\mu_n$'s is the usual assumption of multiscale analysis.

This work grew from our attempts to extend the results of (3) to $d > 1$ and thus improve the results of [AM] and [Gri]. More precisely, we are seeking under which conditions on $\lambda$ and $\mu_n$'s, the operator $H$ has $P$-a.s. only pure point spectrum outside $\sigma(H_0)$. Such a result and (1) would yield that $P$-a.s.

$$\sigma_{\rm ac}(H) = \sigma(H_0), \qquad \sigma_{\rm pp}(H) = \overline{\sigma(H) \setminus \sigma(H_0)}, \qquad \sigma_{\rm sc}(H) = \emptyset. \tag{1.3}$$

For $d = 1$, (1.3) follows from (1) and (3) above.

## 1.2 The results

For $0 < s < 1$ we set

$$\begin{aligned} k_s(n) &:= \inf_{\alpha,\beta\in\mathbf{C}} \frac{\displaystyle\int (|x-\alpha|^s/|x-\beta|^s)\mathrm{d}\mu_n(x)}{\displaystyle\int (1/|x-\beta|^s)\mathrm{d}\mu_n(x)}, \\ K_s(n) &:= \sup_{\beta\in\mathbf{C}} \frac{\displaystyle\int (|x|^s/|x-\beta|^s)\mathrm{d}\mu_n(x)}{\displaystyle\int (1/|x-\beta|^s)\mathrm{d}\mu_n(x)}, \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} k_s &:= \liminf_{n\to\infty} k_s(n), \\ K_s &:= \limsup_{n\to\infty} K_s(n). \end{aligned} \tag{1.5}$$

We will use the conventions $0^{-1}=\infty$, $\infty^{-1}=0$.

Certain positive constants $c_d(s)$ will play an important role in this paper. These constants are defined at the end of Section 1.4 by Relation (1.22). We mention only that $c_d(s)$ is defined for $s > d/(d+1)$ and that $]d/(d+1),\infty[\ni s \mapsto c_d(s)$ is a strictly decreasing $C^\infty$ function with $c_d(1)=1$. We set

$$\begin{aligned} \Lambda_d &:= \inf\left\{\lambda : \lambda > \left[(c_d(s)+2d)k_s^{-1}\right]^{\frac{1}{s}} \text{ for some } s \in ]d/(d+1),1[\right\}, \\ \lambda_d &:= \sup\left\{\lambda : \lambda < [c_d(s)K_s]^{-\frac{1}{s}} \text{ for some } s \in ]d/(d+1),1[\right\} \end{aligned} \tag{1.6}$$

where we use the convention $\inf\emptyset = \infty$.

We make the following hypotheses:

Hypothesis $\mathcal{A}$. For all $n$, the measure $\mu_n$ is absolutely continuous with respect to the Lebesgue measure.

Hypothesis $\mathcal{B}(d)$. $k_s > 0$ for some $s \in ]d/(d+1),1[$.

Hypothesis $\mathcal{C}(d)$. $K_s < \infty$ for some $s \in ]d/(d+1),1[$.

Hypotheses $\mathcal{B}(d)$ and $\mathcal{C}(d)$ ensure that $\lambda_d$ and $\Lambda_d$ are finite positive numbers. Note that these hypotheses require that $k_s > 0$ and $K_s < \infty$ for values of $s$ close to 1. In this respect, our results differ from the localization results in [A, AM].

Various conditions under which Hypotheses $\mathcal{B}(d)$ and $\mathcal{C}(d)$ hold are discussed in [A, AM, Gra, M1]. For example, they hold if the random variables $\{V(n)\}$ are i.i.d. with any of the following distributions:
(a) the uniform distribution in some interval,
(b) the Gaussian distribution,
(c) the Cauchy distribution.
Hypotheses $\mathcal{B}(d)$ and $\mathcal{C}(d)$ also allow for random potentials such that $V$ or $V^{-1}$ vanish at infinity in a suitable probabilistic sense.

We will discuss Hypotheses $\mathcal{B}(d)$ and $\mathcal{C}(d)$ in more detail in Section 1.3.

Our main result is

**Theorem 1.1** *Assume that Hypotheses $\mathcal{A}$ and $\mathcal{B}(d)$ hold. Let $U_0$ be an arbitrary bounded background potential and $H = H_0 + U_0 + \lambda V$, $V \in \Omega$. Then for any $|\lambda| > \Lambda_d$ the operator $H$ has $P$-a.s. only pure point spectrum outside $\sigma(H_0)$ with exponentially decaying eigenfunctions.*

As we will explain in Section 1.5, it is not likely that Theorem 1.1 holds for arbitrary $\lambda$ if the dimension $d+1$ is sufficiently high. However, if the background potential is equal to zero, we can deal with the weak coupling regime.

**Theorem 1.2** *Assume that Hypotheses $\mathcal{A}$ and $\mathcal{C}(d)$ hold and let $H = H_0 + \lambda V$, $V \in \Omega$. Then for $|\lambda| < \lambda_d$ the operator $H$ has $P$-a.s. only pure point spectrum outside $\sigma(H_0)$ with exponentially decaying eigenfunctions.*

**Remark 1.** If $\lambda_d \|V\| \leq 1$ $P$-a.s. then for $|\lambda| \leq \lambda_d$, the operator $H$ has $P$-a.s. no spectrum outside $\sigma(H_0)$. Thus, for bounded random variables, the above theorem could be an empty statement. Using densities of the form

$$\alpha p(x) + (1-\alpha)\ell^{-1} p(\ell^{-1}x), \qquad \alpha \in ]0,1[, \ \ \ell > 0,$$

one can construct a large class of i.i.d. bounded random variables for which $\lambda_d \|V\|_\infty > 1$. In this case, for $\|V\|_\infty^{-1} < |\lambda| < \lambda_d$ the operator $H$ has some essential spectrum outside $\sigma(H_0)$, and Theorem 1.2 asserts that this spectrum is $P$-a.s. pure point with exponentially decaying eigenfunctions.

**Remark 2.** If the random variables $\{V(n)\}$ are i.i.d. and unbounded, then for all $\lambda \neq 0$ the operator $H$ has $P$-a.s. some essential spectrum outside $\sigma(H_0)$. For example, if the random variables $\{V(n)\}$ are i.i.d. with the Gaussian or Cauchy distribution, then for all $\lambda \neq 0$ $\sigma(H) = \mathbf{R}$ $P$-a.s., and the theorem asserts that for $\lambda$ sufficiently small the spectrum of $H$ in $\mathbf{R} \setminus \sigma(H_0)$ is $P$-a.s. pure point with exponentially decaying eigenfunctions.

**Remark 3.** We will discuss below some non-i.i.d. examples for which Theorems 1.1 and 1.2 hold for all $\lambda \neq 0$.

## 1.3 Examples

We first consider the case where the $\{V(n)\}$ are i.i.d. random variables with distribution $\mathrm{d}\mu = p(x)\mathrm{d}x$. In this case the constants in (1.4) are equal respectively to $k_s$ and $K_s$. In this section we will use the shorthand $\langle x \rangle = \sqrt{1+x^2}$.

Hypothesis $\mathcal{B}(d)$ holds for all $d$ if $p \in L^\infty(\mathbf{R})$. Moreover, there are explicit constants $c_s$, which depend on $s$ only, such that

$$k_s \geq c_s \|p\|_\infty^{-s}$$

(for the proof see [Gra]).

If

$$\int |x|^\gamma p(x)\mathrm{d}x < \infty, \tag{1.7}$$

and $p$ is piecewise continuous and strictly monotone for large $|x|$, then $K_s < \infty$ for $s < \min(1, \gamma/2)$ (see [AM]). Thus, if in addition $\gamma > 2d/(d+1)$, $\mathcal{C}(d)$ holds. In particular, for the Gaussian distribution, $\mathcal{C}(d)$ holds for all $d$. The above criterion fails for the Cauchy distribution even if $d = 1$.

If

$$p(x) \leq C\langle x\rangle^{-1-\alpha}$$

for some $\alpha > 0$, then $K_s < \infty$ for $s < \min(1, \alpha/2)$. The proof of this result is elementary and we will skip it. Thus, if in addition $\alpha > 2d/(d+1)$, $\mathcal{C}(d)$ holds. In particular, for the Cauchy distribution, $\mathcal{C}(d)$ holds for all $d$.

We remark that for the Cauchy distribution the integrals in (1.4) can be explicitly evaluated (see [M2]) and one can take

$$K_s = 1/\cos(s\pi/2),$$

irrespectively of the parameters of the distribution.

A different condition under which $K_s < \infty$ has been discussed in [A], Appendix I. The condition of Aizenman, however, requires that $s < 1/3$, and is not applicable in our case.

An interesting class of non-i.i.d. examples arises as follows. Let $\{a_n\}_{n\in\mathbf{Z}^d}$ be a real sequence with $a_n \neq 0$ and let $\{W(n)\}$ be i.i.d. random variables with distribution $\mathrm{d}\mu = p(x)\mathrm{d}x$. We denote the constants (1.4) associated to $W$ by $k_{s,w}$ and $K_{s,w}$. Let

$$V(n) := a_n W(n). \tag{1.8}$$

Then the distribution of $V(n)$ is $\mathrm{d}\mu_n(x) = |a_n|^{-1}p(a_n^{-1}x)$ and

$$k_s(n) = |a_n|^s k_{s,w}, \qquad K_s(n) = |a_n|^s K_{s,w}.$$

In particular, if $\mathcal{B}(d)$ holds for $\{W(n)\}$ and $\lim |a_n| = \infty$, then Theorem 1.1 holds for all $\lambda \neq 0$. If $\mathcal{C}(d)$ holds for $\{W(n)\}$ and $\lim |a_n| = 0$ then Theorem 1.2 holds for all $\lambda$.

To illustrate these results with a concrete example, take $a_n = \langle n\rangle^\beta$ and assume that $\{W(n)\}$ has either the Cauchy or Gaussian distribution. Let $V$ be given by (1.8) and $H = H_0 + V$. Then it follows from Theorems 1.1 and 1.2 that for any $\beta \neq 0$ the operator $H$ has $P$-a.s. only pure point spectrum outside $\sigma(H_0)$ with exponentially decaying eigenfunctions. One can show that in the case of the Cauchy distribution, $\sigma_{\text{ess}}(H) = \mathbf{R}$ $P$-a.s. if $\beta \in [-d, d]$, and that $\sigma_{\text{ess}}(H) = \sigma(H_0)$ $P$-a.s. if $\beta \notin [-d, d]$. In the case of the Gaussian distribution, $\sigma_{\text{ess}}(H) = \mathbf{R}$ $P$-a.s. if $\beta \in [0, d]$, and $\sigma_{\text{ess}}(H) = \sigma(H_0)$ $P$-a.s. if $\beta \notin [0, d]$. In all the above cases, the spectrum of $H$ in $\sigma(H_0)$ is purely absolutely continuous $P$-a.s. [JL1].

The spectral properties of the Anderson model with decaying randomness have been discussed recently in [KKO].

## 1.4 About the proofs

In this section we sketch some of the ideas involved in our proofs.

The first idea, which has been used in practically all work on the spectral theory of operators (1.1), is to "integrate" the $x$-variable and reduce the $d+1$-dimensional spectral problem to a non-linear $d$-dimensional spectral problem. The details of the argument are given in [JM1] and here we summarize the results which we will need in the sequel.

Let $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ be the circle and $\mathbf{T}^d$ the $d$-dimensional torus. We denote the points in $\mathbf{T}^d$ by $\phi = (\phi_1, \ldots, \phi_d)$ and by $\mathrm{d}\phi$ the usual Lebesgue measure. We set

$$\Phi(\phi) := 2\sum_{i=1}^{d} \cos\phi_i.$$

For $z \in \mathbf{C} \setminus \sigma(H_0)$, let $\lambda(\phi, z)$ be such that

$$\lambda(\phi, z) + \frac{1}{\lambda(z, E)} + \Phi(\phi) = z, \qquad |\lambda(\phi, z)| < 1. \tag{1.9}$$

We set[2]

$$\hat{j}(\phi, z) = \lambda(\phi, z) + \Phi(\phi), \qquad j(n, z) = (2\pi)^{-d} \int_{\mathbf{T}^d} \hat{j}(\phi, z)\mathrm{e}^{-\mathrm{i}n\phi}\mathrm{d}\phi. \tag{1.10}$$

One can show that the function $j(n, z)$ decays exponentially in the variable $n$. Let $h_0(z)$ be the operator on $l^2(\mathbf{Z}^d)$ defined by

$$(h_0(z)\psi)(n) = \sum_{k\in\mathbf{Z}^d} j(n - k, z)\psi(k).$$

We define a one-parameter family of random operators on $l^2(\mathbf{Z}^d)$ by

$$h(z) = h_0(z) + U_0 + \lambda V, \qquad z \in \mathbf{C} \setminus \sigma(H_0),\ V \in \Omega. \tag{1.11}$$

The key property of these operators is that $\forall m, n \in \mathbf{Z}^d$,

$$(\delta_{(m,0)}|(H - z)^{-1}\delta_{(n,0)}) = (\delta_m|(h(z) - z)^{-1}\delta_n) \tag{1.12}$$

(for the proof see [JM1] or [JL1]). Since the set of vectors $\{\delta_{(n,0)}\}_{n\in\mathbf{Z}^d}$ is cyclic for $H$ (see [JL1]), the spectral properties of $H$ are encoded by the family $h(z)$. In particular, it follows from the Simon-Wolff theorem (see Section 2.1 for details) that Theorems 1.1 and 1.2 follow from a suitable estimate on the matrix elements

$$(\delta_m|(h(E) - E - \mathrm{i}\varepsilon)\delta_n), \qquad E \in \mathbf{R} \setminus \sigma(H_0). \tag{1.13}$$

In comparison with the usual theory of random Schrödinger operators, the difficulties in estimating the matrix elements (1.13) stem from the fact that $h_0(E)$ is a long-range Laplacian which depends on the energy. To study the resolvent $(h(E) - E - \mathrm{i}\varepsilon)^{-1}$ with the standard techniques one needs efficient estimates on the kernel $j(n, E)$ for $E \in \mathbf{R} \setminus \sigma(H_0)$.

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[2] There are typographical errors in similar formulas in [JM1] (Relation (1.5)) and [JM2], where the factor $(2\pi)^{-d}$ is missing in the front of the integral.

Let us describe the estimates previously used in the literature and the estimate we will use in this paper.

We set

$$t(n,E) := (2\pi)^{-d}\int_{\mathbf{T}^d} \lambda(\phi,E)\mathrm{e}^{-\mathrm{i}n\phi}\mathrm{d}\phi. \tag{1.14}$$

In the Fourier representation $(h_0(E)-E)^{-1}$ acts as multiplication by $-\lambda(\phi,E)$ and for any $p,q\in\mathbf{Z}^d$,

$$(\delta_p|(h_0(E)-E)^{-1}\delta_q) = -t(p-q,E) = -t(q-p,E) \tag{1.15}$$

(these relations will be used in Section 4). From the definition of $j(n,E)$ it follows that

$$j(n,E) = t(n,E) + \delta_{1|n|_+}, \tag{1.16}$$

where $\delta_{ij}$ stands for the Kronecker symbol. To estimate $t(n,E)$, it is useful to note that (see [JM1] or [JL1])

$$t(n,E) = (\delta_{(0,0)}|(E-H_0)^{-1}\delta_{(n,0)}). \tag{1.17}$$

From this identity one easily gets the estimate (see e.g. Lemma III.4 in [S])

$$|t(n,E)| \leq C_E \mathrm{e}^{-d_E|n|_+}, \tag{1.18}$$

where

$$C_E = (|E|-2(d+1))^{-1}, \qquad d_E = \ln\left(\frac{2(d+1)}{|E|}\right).$$

A better estimate can be obtained using (1.14) and the analyticity properties of $\lambda(\phi,E)$ (see Proposition 2.2 in [JM1]):

$$|t(n,E)| \leq \mathrm{e}^{-a(E)|n|_+}, \tag{1.19}$$

where[3]

$$a(E) = \ln\gamma_E, \qquad \text{and} \qquad \gamma_E + \gamma_E^{-1} = (|E|-2)/d. \tag{1.20}$$

Either of the estimates (1.18), (1.19) suffices for the arguments in [AM] and [Gri]. However, the estimate (1.18) blows up as $E$ approaches $\pm 2(d+1)$ while (1.19) gives the useless bound $|t(n,E)| \leq 1$. Therefore, these estimates are not useful near the edges of $\sigma(H_0)$. In fact one can easily show that a uniform exponential estimate of $t(n,E)$ near $\pm 2(d+1)$ is not possible – otherwise, the function $\lambda(\phi, \pm 2(d+1))$ would be analytic in $\phi$, which is not the case. We will derive an useful bound near the edges of $\sigma(H_0)$ from the following observations:

(i) $t(n,E) = (-1)^{|n|_+} t(n,-E)$.

[3] There is another unfortunate typographical error in [JM1], where in the second formula in (1.20) the factor $d$ is replaced with $2d$.

(ii) The function $E \mapsto t(n,E)$ is positive and strictly decreasing on $[2(d+1),\infty[$.

(iii) For some $C$, $|t(n,2(d+1))| \leq C\prod_{i=1}^{d}(1+|n_i|)^{-\frac{d+1}{d}}$.

From (i)-(iii) it follows that for $s > d/(d+1)$,

$$\begin{aligned} \sup_{E\notin\sigma(H_0)} \sum_{n\in\mathbf{Z}^d} |t(n,E)|^s &\leq c_d(s), \\ \sup_{E\notin\sigma(H_0)} \sum_{n\in\mathbf{Z}^d} |j(n,E)|^s &\leq c_d(s) + 2d, \end{aligned} \tag{1.21}$$

where

$$c_d(s) := \sum_{n\in\mathbf{Z}^d} |t(n,2(d+1))|^s. \tag{1.22}$$

These estimates are sufficient to employ the method of Aizenman-Molchanov. We will prove Theorem 1.1 using the second relation in (1.21) and by following an elegant presentation of Aizenman-Molchanov theory in [S]. In the proof of Theorem 1.2, which deals with the weak coupling regime, we use the first relation (1.21) and essentially follow the argument of Aizenman [A].

## 1.5 Some remarks

First, we would like to remark that Theorems 1.1 and 1.2 are not simply extensions of the results in [JM1] to higher dimension. Theorem 1.1 allows for a background potential, which is important in physical applications. The above two theorems also establish exponential decay of the eigenfunctions. The method of the proof allows for correlated random variables and can be used to prove dynamical localization outside $\sigma(H_0)$ (see [A, RJLS, GD]). None of these is covered by the method of [JM1]. Moreover, the proofs of Theorems 1.1 and 1.2 follow relatively easily from the Aizenman-Molchanov theory, while the arguments in [JM1] are quite elaborate. On the other hand, if $d = 1$, the techniques of [JM1] yield localization for all $\lambda$ and do not require that random variables are unbounded if $\lambda$ is small. Theorems 1.1 and 1.2 do not yield such a result. This brings us to our second remark. We believe that in many cases Theorem 1.2 holds for all $\lambda$ and $d$. It would be interesting to exhibit at least some classes of distributions for which this result holds.

We finish this section with a brief explanation of why we do not expect that Theorem 1.1 will hold for small $\lambda$'s and arbitrary $U_0$. Let $U_0$ be a large constant (it suffices that $|U_0| > 4d+2$). Then, the spectrum of $H_0+U_0$ is purely absolutely continuous, and consists of two disjoint components, $\sigma(H_0)$ and $[-2d,2d]+U_0+U_0^{-1}$. If physicists expectations about the Anderson model are correct, one may expect that for $d \geq 3$ and $\lambda$ small, the operator $H$ will have some absolutely continuous spectrum on the second branch $[-2d,2d]+U_0+U_0^{-1}$ (note however that since the dimension of our half-space is $d+1$,

$d \geq 3$ corresponds to the unphysical $d+1 \geq 4$). This absolutely continuous spectrum would have an interesting property – the corresponding generalized eigenfunctions would decay exponentially fast in the $x$-variable and would be extended in the $n$-variable. Such generalized eigenfunctions describe propagating surface states (surface waves), see [JMP] and [KP] for discussion. It is an interesting question as to whether propagating surface states exist in the random models studied here. Theorems 1.1 and 1.2 yield that in many situations all the propagating surface states with energies $E \notin \sigma(H_0)$ (which exist if the boundary potential is constant or periodic) are exponentially localized by the random fluctuations of the boundary. This is physically the most interesting consequence of our results. Finally, we remark that although it is known that the spectrum of $H$ in $\sigma(H_0)$ is $P$-a.s. purely absolutely continuous, the structure of the generalized eigenfunctions is not known, and in particular it is not known whether surface states with energies in $\sigma(H_0)$ exist.

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# 2 Preliminaries

## 2.1 Simon-Wolff criterion

As we have already remarked, our proofs of Theorems 1.1 and 1.2 are based on a suitable variant of Simon-Wolff theorem. In this section we describe this variant and collect some related technical results which will be used in the sequel.

In this section $I = ]a, b[$ is a fixed open interval outside $\sigma(H_0)$. We denote by $\mathfrak{m}$ the Lebesgue measure on $\mathbf{R}$ (the symbol a.e. without qualification will always mean with respect to Lebesgue measure).

Consider the following conditions:

Condition C(1). $\forall m \in \mathbf{Z}^d$ and for $P \times \mathfrak{m}$ - a.e. $(V, E) \in \Omega \times I$,

$$\lim_{\varepsilon \downarrow 0} \|(H - E - \mathrm{i}\varepsilon)^{-1}\delta_{(m,0)}\| < \infty. \tag{2.23}$$

Condition C(2). $\forall m, n \in \mathbf{Z}^d$ and for $P \times \mathfrak{m}$ - a.e. $(V, E) \in \Omega \times I$,

$$\lim_{\varepsilon \downarrow 0} |(\delta_{(m,0)}|(H - E - \mathrm{i}\varepsilon)^{-1})\delta_{(n,0)}| \leq C_{V,E,m}\mathrm{e}^{-a(E)|n|_+}, \tag{2.24}$$

for some $a(E) > 0$.

The existence of the limit (2.23) follows from monotonicity. The existence and finiteness of the limit (2.23) for $P \times \mathfrak{m}$ - a.e. $(V, E)$ follows from Fubini's theorem and the well-known property of Herglotz functions.

The estimate (2.24) implies that for all $x \geq 0$,

$$\lim_{\varepsilon \downarrow 0} |(\delta_{(m,0)}|(H - E - \mathrm{i}\varepsilon)^{-1})\delta_{(n,x)}| \leq C_{V,E,m} \mathrm{e}^{-a(E)|n|_+ - b(E)x}, \tag{2.25}$$

where $b(E) = \sup_{\phi \in \mathbf{T}^d} |\ln \lambda(\phi, E)|$ ($\lambda(\phi, E)$ is given by (1.9)). See Section 2 in [JM1] for details.

Consider the following statements:

Statement S(1): The spectrum of $H$ in $I$ is $P$-a.s. pure point.

Statement S(2): The spectrum of $H$ in $I$ is $P$-a.s. pure point with exponentially decaying eigenfunctions.

**Theorem 2.1** *Assume that Hypothesis $\mathcal{A}$ holds. Then* C(1) $\Leftrightarrow$ S(1) *and* C(1) + C(2) $\Rightarrow$ S(2).

This result follows from the Simon-Wolff theorem [SW] and the fact that the set of vectors $\{\delta_{(m,0)}\}_{m \in \mathbf{Z}^d}$ is cyclic for $H$.

Our next lemma shows that changing the distributions within a finite box does not affect the condition C(2).

**Lemma 2.2** *Assume that $P_1$ and $P_2$ are measures on $(\Omega, \mathcal{F})$ of the form*

$$P_1 = \bigtimes_{n \in \mathbf{Z}^d} \mu_n^{(1)}, \qquad P_2 = \bigtimes_{n \in \mathbf{Z}^d} \mu_n^{(2)},$$

*that $\mu_n^{(1)} = \mu_n^{(2)}$ for $|n|_+ > l$, and that the conditions* C(1) *and* C(2) *hold for the measure $P_1$. Then these conditions also hold for the measure $P_2$.*

**Proof.** We will deal with the condition C(2). A similar argument applies to the condition C(1).

Let $B_l = \{n \in \mathbf{Z}^d : |n|_+ \leq l\}$, $B_{\bar{l}} = \{n \in \mathbf{Z}^d : |n|_+ > l\}$,

$$\Omega_l = \mathbf{R}^{B_l}, \qquad \Omega_{\bar{l}} = \mathbf{R}^{B_{\bar{l}}},$$

and for $i = 1, 2$, let

$$P_i^l = \bigtimes_{n \in B_l} \mu_n^{(i)}, \qquad P_i^{\bar{l}} = \bigtimes_{n \in B_{\bar{l}}} \mu_n^{(i)}.$$

Obviously,

$$\Omega = \Omega_l \times \Omega_{\bar{l}}, \qquad P_i = P_i^l \times P_i^{\bar{l}},$$

and by the assumption,

$$P_1^{\bar{l}} = P_2^{\bar{l}}. \tag{2.26}$$

In what follows we view the points in $\Omega$ as the pairs $V = (V_l, V_{\bar{l}})$, $V_l \in \Omega_l$, $V_{\bar{l}} \in \Omega_{\bar{l}}$.

Since the condition C(2) holds for the measure $P_1$, for $P_1^l \times P_1^{\bar{l}} \times \mathfrak{m}$ a.e. $(V_l, V_{\bar{l}}, E) \in \Omega_l \times \Omega_{\bar{l}} \times I$ the estimate

$$\lim_{\varepsilon \downarrow 0} |(\delta_{(m,0)}|(H - E - \mathrm{i}\varepsilon)^{-1}\delta_{(n,0)})| \leq C_{V,E,m}\mathrm{e}^{-a(E)|n|_+} \tag{2.27}$$

holds for all $m, n \in \mathbf{Z}^d$. By Fubini's theorem, there exists a set $\tilde{\Omega}_{\bar{l}} \subset \Omega_{\bar{l}}$ of full $P_1^{\bar{l}}$ measure such that, for $V_{\bar{l}} \in \tilde{\Omega}_{\bar{l}}$, the estimate (2.27) holds for $P_1^l \times \mathfrak{m}$ a.e. $(V_l, E) \in \Omega_l \times I$. Now fix $V_{\bar{l}} \in \Omega_{\bar{l}}$. By Fubini's theorem there exists a ($V_{\bar{l}}$-dependent) set $\tilde{\Omega}_l \subset \Omega_l$ of full $P_1^l$ measure such that, for $V_l \in \tilde{\Omega}_l$, the estimate (2.27) holds for a.e. $E \in I$. We now fix $V_l \in \tilde{\Omega}_l$ and set $V = (V_l, V_{\bar{l}})$. Let $W \in \Omega_l$ be arbitrary and

$$H_W = H + W.$$

Then

$$\begin{aligned}&(\delta_{(m,0)}|(H_W - E - \mathrm{i}\varepsilon)^{-1}\delta_{(n,0)}) = (\delta_{(m,0)}|(H - E - \mathrm{i}\varepsilon)^{-1}\delta_{n,0})\\ &-\lambda \textstyle\sum_{p \in B_l} W(p)(\delta_{(m,0)}|(H_W - E - \mathrm{i}\varepsilon)^{-1}\delta_{(p,0)})(\delta_{(p,0)}|(H - E - \mathrm{i}\varepsilon)^{-1}\delta_{(n,0)}).\end{aligned} \tag{2.28}$$

Since for a.e. $E$ the limits

$$\lim_{\varepsilon \downarrow 0} |(\delta_{(m,0)}|(H_W - E - \mathrm{i}\varepsilon)^{-1}\delta_{(p,0)})|$$

exist and are finite, we derive from (2.28) that the estimate (2.27) holds for $(V_l + W, V_{\bar{l}})$ and a.e. $E \in I$. Therefore, for $V_{\bar{l}} \in \tilde{\Omega}_{\bar{l}}$ and *all* $V_l \in \Omega_l$, the estimate (2.27) holds for a.e. $E \in I$. By Fubini's theorem and (2.26) this estimate then also holds for $P_2 \times \mathfrak{m}$ a.e. $(V, E) \in \Omega \times I$, and the condition C(2) holds for the measure $P_2$. □

We now introduce a new condition. Recall that the operators $h(E)$ are defined by (1.11).

Condition C(3). $\forall m$ and for $P \times \mathfrak{m}$ - a.e. $(V, E) \in \Omega \times I$,

$$\lim_{\varepsilon \downarrow 0} \|(h(E) - E - \mathrm{i}\varepsilon)^{-1}\delta_m\| < \infty. \tag{2.29}$$

**Lemma 2.3**
(i) C(1) $\Leftrightarrow$ C(3).
(ii) *If* C(3) *holds then* $\forall m, n \in \mathbf{Z}^d$ *and for* $P \times \mathfrak{m}$ *a.e.* $(V, E) \in \Omega \times I$,

$$\lim_{\varepsilon \downarrow 0}(\delta_{(m,0)}|(H - E - \mathrm{i}\varepsilon)^{-1}\delta_{(n,0)}) = \lim_{\varepsilon \downarrow 0}(\delta_m|(h(E) - E - \mathrm{i}\varepsilon)\delta_n).$$

**Proof.** Part (i) of this lemma is proven in [JM1] (Lemma 2.1). In fact, a stronger result holds: for all $(V, E) \in \Omega \times I$, the limit (2.23) is finite iff the limit (2.29) is finite.

To prove Part (ii) we will use the relation (1.12). The resolvent identity yields that

$$\begin{aligned}&\left|(\delta_m|(h(E+\mathrm{i}\varepsilon)-E-\mathrm{i}\varepsilon)^{-1}\delta_n)-(\delta_m|(h(E)-E-\mathrm{i}\varepsilon)^{-1}\delta_n)\right|\\ &\leq \|h_0(E+\mathrm{i}\varepsilon)-h_0(E)\|\,\|(h(E+\mathrm{i}\varepsilon)-E-\mathrm{i}\varepsilon)^{-1}\delta_m\|\,\|(h(E)-E-\mathrm{i}\varepsilon)^{-1}\delta_n\|,\end{aligned}$$

and the result follows from the estimate

$$\|(h_0(E+\mathrm{i}\varepsilon)-h_0(E)\| = \sup_{\phi\in\mathbf{T}^d}|\lambda(\phi, E+\mathrm{i}\varepsilon)-\lambda(\phi,E)| = O(\varepsilon).$$

□

Our last condition is

Condition C(4). $\forall m, n \in \mathbf{Z}^d$ and for $P \times \mathfrak{m}$ - a.e. $(V, E) \in \Omega \times I$,

$$\lim_{\varepsilon\downarrow 0}|(\delta_m|(h(E)-E-\mathrm{i}\varepsilon)^{-1}\delta_n)| \leq C_{V,E,m}\mathrm{e}^{-a(E)|n|_+}, \tag{2.30}$$

for some $a(E) > 0$.

We can not guarantee a priori the existence of the limits (2.30). However, by Lemma 2.3, if C(3) holds then the limits (2.30) exist and C(3) + C(4) $\Rightarrow$ C(1) + C(2).

Before we state our final criterion under which the statement S(2) holds, we need

**Lemma 2.4** *Let* $\{f_n\}_{n\in\mathbf{Z}^d}$ *be a sequence of random variables on the probability space* $(\Omega, \mathcal{F}, P)$ *such that for some* $0 < s < 1$ *and* $\forall n$,

$$\mathbf{E}(|f_n|^s) \leq C\mathrm{e}^{-d|n|_+}.$$

($\mathbf{E}$ *stands for the expectation). Then there are finite constants* $D_V$ *such that*

$$|f_n(V)| \leq D_V\mathrm{e}^{-d|n|_+} \qquad P - a.s.$$

**Proof.** Let

$$A_n = \left\{V \in \Omega : |f_n(V)| > \mathrm{e}^{-d|n|_+}\right\}.$$

By Chebyshev's inequality,

$$P(A_n) \leq \mathrm{e}^{sd|n|_+}\mathbf{E}(|f_n|^s) \leq C\mathrm{e}^{-(1-s)d|n|_+}.$$

Thus, $\sum P(A_n) < \infty$, and the statement follows from the Borel-Cantelli lemma. □

**Lemma 2.5** *Assume that for some* $0 < s < 1$, $\varepsilon_0 > 0$ *and* $a(E) > 0$ *the relation*

$$\sup_{0<\varepsilon<\varepsilon_0} \mathbf{E}\left(|(\delta_m|(h(E) - E - \mathrm{i}\varepsilon)^{-1}\delta_n)|^s\right) \leq C_E \mathrm{e}^{-a(E)|n-m|_+}, \tag{2.31}$$

*holds for all* $E \in I$ *and* $m, n \in \mathbf{Z}^d$. *Then the conditions* C(3) *and* C(4) *hold. In particular, the statement* S(2) *holds.*

**Proof.** We first establish C(3). Let $m$ be fixed. Then,

$$\|(h(E) - E - \mathrm{i}\varepsilon)\delta_m\|^2 = \sum_n |(\delta_m|(h(E) - E - \mathrm{i}\varepsilon)^{-1}\delta_n)|^2. \tag{2.32}$$

Since for any $0 < q \leq 1$ and any sequence of complex numbers $x_k$ we have

$$\left|\sum x_k\right|^q \leq \sum |x_k|^q,$$

(2.32) yield (take $q = s/2$)

$$\mathbf{E}(\|(h(E) - E - \mathrm{i}\varepsilon)\delta_m\|^s) \leq \sum_n \mathbf{E}(|(\delta_m|(h(E) - E - \mathrm{i}\varepsilon)^{-1}\delta_n)|^s).$$

It follows from (2.31) that for any $E \in I$,

$$\lim_{\varepsilon\downarrow 0} \mathbf{E}(\|(h(E) - E - \mathrm{i}\varepsilon)\delta_m\|^s) < \infty,$$

and by the Monotone Convergence Theorem, that

$$\mathbf{E}\left(\lim_{\varepsilon\downarrow 0} \|(h(E) - E - \mathrm{i}\varepsilon)\delta_m\|^s\right) < \infty.$$

This estimate and Fubini's theorem yield C(3).

Since C(3) holds, by Part (ii) of Lemma 2.3, for $P \times \mathfrak{m}$ -a.e $(V, E) \in \Omega \times I$ the limits

$$\lim_{\varepsilon\downarrow 0}(\delta_m|(h(E) - E - \mathrm{i}\varepsilon)^{-1}\delta_n)$$

exist and are finite. Therefore, by Fatou's Lemma, for a.e. $E \in I$,

$$\begin{aligned}\mathbf{E}\left(\lim_{\varepsilon\downarrow 0} |(\delta_m|(h(E) - E - \mathrm{i}\varepsilon)^{-1}\delta_n)|^s\right) &\leq \liminf_{\varepsilon\downarrow 0} \mathbf{E}(|(\delta_m|(h(E) - E - \mathrm{i}\varepsilon)^{-1}\delta_n)|^s) \\ &\leq C_E \mathrm{e}^{-a(E)|n-m|_+}.\end{aligned}$$

The condition C(4) now follows from Lemma 2.4 and Fubini's theorem. □

## 2.2 The key estimates

In this section we collect some technical results which we will need for our proofs.

First, we need a lemma about the Dirichlet Laplacian $H_0$. Recall that $|n|_+ = \sum_{i=1}^d |n_i|$.

**Lemma 2.6** *Let $a(n,k) := (\delta_{(0,0)}|H_0^k \delta_{(n,0)})$, where $k \geq 0$. Then $a(n,k) = 0$ if $k < |n|_+$ or $k - |n|_+$ is odd, and $a(n,k) > 0$ if $k - |n|_+$ is even.*

**Proof.** An elementary induction. □

We now prove the properties of the sequence $t(n,E)$ described in Section 1.4. At the boundary of $\sigma(H_0)$ ($E = \pm 2(d+1)$) we define $\lambda(\phi, \pm 2(d+1))$ by the equation (1.9) and the condition $|\lambda(\phi, \pm 2(d+1)| \leq 1$. The sequences $t(n, \pm 2(d+1))$ are defined by (1.14). It follows easily from (1.14) that for all $n$, $E \mapsto t(n,E)$ is a continuous function on $\mathbf{R} \setminus \operatorname{int} \sigma(H_0)$.

**Lemma 2.7** *For $E \geq 2(d+1)$, $t(n,E) = (-1)^{|n|_+} t(n,-E)$.*

**Proof.** For $E > 2(d+1)$ it follows from (1.17) and Lemma 2.6 that

$$t(n,E) = \sum_{p=0}^{\infty} \frac{1}{E^{2p+1+|n|_+}} (\delta_{(0,0)}|H_0^{2p+|n|_+} \delta_{(n,0)}), \tag{2.33}$$

and

$$t(n,-E) = \sum_{p=0}^{\infty} \frac{(-1)^{|n|_+}}{E^{2p+1+|n|_+}} (\delta_{(0,0)}|H_0^{2p+|n|_+} \delta_{(n,0)}). \tag{2.34}$$

Clearly, these relations yield the statement for $E > 2(d+1)$. By the continuity of $t(n,E)$, the statement also holds for $E = 2(d+1)$. □

**Lemma 2.8** *The function $E \mapsto t(n,E)$ is positive and strictly decreasing on $[2(d+1), \infty)$.*

**Proof.** It follows from Lemma 2.6 and (2.33) that for $E > 2(d+1)$,

$$t(n,E) > 0, \qquad \frac{\mathrm{d}}{\mathrm{d}E} t(n,E) < 0.$$

These two observations yield the result. □

**Lemma 2.9** *There exists a constant $C$ such that*

$$|t(n, 2(d+1))| \leq C \prod_{i=1}^{d} (1 + |n_i|)^{-\frac{d+1}{d}} \tag{2.35}$$

**Proof.** Let $n = (n_1, \ldots, n_d)$. For notational simplicity, we assume that $n_i > 0$. Since $E = 2(d+1)$ is fixed, in the sequel we write $\lambda(\phi)$ for $\lambda(\phi, 2(d+1))$, etc. We recall that

$$t(n) = (2\pi)^{-d} \int_{\mathbf{T}^d} \lambda(\phi) \mathrm{e}^{-in\phi} \mathrm{d}\phi,$$

where

$$\lambda(\phi) = \frac{1}{2}\left(2(d+1) - \Phi(\phi) - \sqrt{(2(d+1) - \Phi(\phi))^2 - 4}\right).$$

Since $\Phi(\phi) = 2\sum_{i=1}^{d} \cos\phi_i$, we can write $\lambda(\phi)$ as

$$\lambda(\phi) = \Psi_1(\phi)\Psi_2(\phi) + \Psi_3(\phi),$$

where $\Psi_2$ and $\Psi_3$ are $C^\infty$ functions on $\mathbf{T}^d$ and

$$\Psi_1(\phi) = \left(\sum_{i=1}^{d} \sin^2 \frac{\phi_i}{2}\right)^{\frac{1}{2}}.$$

Clearly, $\Psi_1$ is $C^\infty$ away from the point $\phi = 0$, and it is a simple exercise to verify that the function

$$\partial_{\phi_1}^{\alpha_1} \ldots \partial_{\phi_d}^{\alpha_d} \Psi_1(\phi), \qquad \alpha_i \geq 0,\ \sum \alpha_i \leq d+1,$$

is in $L^1(\mathbf{T}^d)$. Integration by parts yields that for all $j$ and some $C > 0$,

$$|t(n)| \leq C|n_j|^{-1}\left(\prod_{i=1}^{d} n_i\right)^{-1}.$$

Multiplying these relations we derive (2.35). $\square$

We are now ready to prove the key properties of the sequences $t(n, E)$ and $j(n, E)$. Recall that the constant $c_d(s)$ is defined by (1.22).

**Lemma 2.10** *If $s \in ]d/(d+1), 1]$ and $|E| \geq 2(d+1)$ then*

$$\begin{aligned} \sum_n |t(n,E)|^s &\leq c_d(s), \\ \sum_n |j(n,E)|^s &\leq c_d(s) + 2d. \end{aligned} \tag{2.36}$$

*Moreover, $]d/(d+1), \infty[ \ni s \mapsto c_d(s)$ is a strictly decreasing $C^\infty$ function with $c_d(1) = 1$.*

**Proof.** The first bound in (2.36) follows from Lemmas 2.7 and 2.8. The second bound follows from the first, Relation (1.16), and the inequality $|a+b|^s \leq |a|^s + |b|^s$, which holds for $a, b \in \mathbf{R}$ and $0 < s \leq 1$. The regularity properties of $c_d(s)$ follow from Lemma 2.9. Finally, since the sequence $t(n, 2(d+1))$ is positive,

$$c_d(1) = \sum_n t(n, 2(d+1)) = \lambda(0, 2(d+1)) = 1.$$

□

Our next set of technical results concerns the Aizenman-Molchanov technique. The next lemma is motivated by [S].

**Lemma 2.11** *Let $r \in l^1(\mathbf{Z}^d)$ be a non-negative sequence and $R$ the corresponding convolution operator on $l^\infty(\mathbf{Z}^d)$. Assume that $\sum_n r(n) < 1$. Let $f, g \in l^\infty(\mathbf{Z}^d)$ be non-negative functions and suppose that*

$$(1-R)f \leq g.$$

*Then*

$$f \leq (1-R)^{-1}g.$$

**Proof.** Since for any $\psi \in l^\infty(\mathbf{Z}^d)$,

$$R\psi(n) = \sum_k r(n-k)\psi(k),$$

the operator $R$ is positivity preserving on $l^\infty(\mathbf{Z}^d)$ and has the norm $\sum_n r(n)$. Since

$$(1-R)^{-1} = \sum_{j=0}^{\infty} R^j,$$

the operator $(1-R)^{-1}$ is also positivity preserving on $l^\infty(\mathbf{Z}^d)$. This yields the statement. □

**Lemma 2.12** *Let $r \in l^1(\mathbf{Z}^d)$ be a non-negative sequence and $R$ the corresponding convolution operator on $l^\infty(\mathbf{Z}^d)$. Assume that*

$$r(n) \leq A\mathrm{e}^{-a|n|_+}$$

*for some $a > 0$ and that $\sum_n r(n) < 1$. Then $(1-R)^{-1}$ is the operator of convolution by the non-negative sequence $s(n)$ which satisfies*

$$s(n) \leq B\mathrm{e}^{-b|n|_+}$$

*for some $b > 0$.*

**Proof.** Let

$$\begin{aligned} \hat{r}(\phi) &:= \sum_n r(n)\mathrm{e}^{\mathrm{i}n\phi}, \\ s(n) &:= (2\pi)^{-d}\int_{\mathbf{T}^d}(1-\hat{r}(\phi))^{-1}\mathrm{e}^{-\mathrm{i}n\phi}\mathrm{d}\phi. \end{aligned}$$

Since $\hat{r}(\phi)$ is an analytic function on $\mathbf{T}^d$ and $1 > \max |\hat{r}(\phi)|$, the function $(1-\hat{r}(\phi))^{-1}$ is also analytic on $\mathbf{T}^d$. Thus, the sequence $s(n)$ decays exponentially and $(1-R)^{-1}$ is the operator of convolution by $s(n)$. Finally, since $(1-R)^{-1}$ is positivity preserving we derive that $s(n)$ is a non-negative sequence. $\Box$

The final result we will need is the following well-known rank-one perturbation formula. Let $\tilde{V}$ and $m \in \mathbf{Z}^d$ be given. Set

$$V = \tilde{V} + \alpha(\delta_m|\cdot)\delta_m,$$

$\tilde{h}(E) = h_0(E) + \tilde{V}$, $h(E) = H_0 + V$. Then the resolvent identity yields (see e.g. [S])

**Lemma 2.13** *For any $n$ and $z$,*

$$(\delta_n|(h(E)-z)^{-1}\delta_m) = \frac{(\delta_n|(\tilde{h}(E)-z)^{-1}\delta_m)}{1+\alpha(\delta_m|(\tilde{h}(E)-z)^{-1}\delta_m)}.$$

# 3 The strong coupling regime

In this section we prove Theorem 1.1.

We fix $s \in ]d/(d+1), 1[$ such that $k_s > 0$. Let $\delta \in ]0, k_s[$. Since $\liminf k_s(n) = k_s$, there exists an $l$ such that for all $n$ with $|n|_+ > l$,

$$k_s(n) < k_s - \delta =: k_{s,\delta}. \tag{3.37}$$

By changing the distributions $\mu_n$ within the box $|n|_+ \leq l$ we may assume that (3.37) holds for all $n$. By Lemma 2.2, such a change does not affect Theorem 2.1.

Let $m \in \mathbf{Z}^d$ and $E \notin \sigma(H_0)$ be given. For $\varepsilon > 0$ we set

$$G(n) \equiv G(m,n;E+\mathrm{i}\varepsilon) := (\delta_m|(h(E)-E-\mathrm{i}\varepsilon)^{-1}\delta_n), \tag{3.38}$$

and write $z = E + \mathrm{i}\varepsilon$. The function $G$ satisfies the equation

$$\sum_k j(n-k,E)G(k) + (\lambda V(n) + U_0(n) - z)G(n) = \delta_{mn}.$$

Then,

$$\mathbf{E}\left(|\lambda V(n) + U_0(n) - z|^s |G(n)|^s\right) \leq \delta_{mn} + \sum_k |j(n-k,E)|^s \mathbf{E}(|G(k)|^s). \tag{3.39}$$

($\mathbf{E}$ stands for the expectation). It follows from Lemma 2.13 that

$$|G(n)|^s = \frac{|a|^s}{|\lambda V(n) + b|^s},$$

where $a$ and $b$ are functions of $\{V(l)\}_{l\neq n}$. Let $\alpha = U_0(n) - z$. Averaging only over $V(n)$ we get

$$\begin{aligned}\int |a|^s \frac{|\lambda V+\alpha|^s}{|\lambda V+b|^s}\mathrm{d}\mu_n(V) &= |a|^s|\lambda|^s|\lambda|^{-s}\int \frac{|V+\lambda^{-1}\alpha|^s}{|V+\lambda^{-1}b|^s}\mathrm{d}\mu_n(V)\\ &\geq k_{s,\delta}|a|^s|\lambda|^s|\lambda|^{-s}\int \frac{1}{|V+\lambda^{-1}b|^s}\mathrm{d}\mu_n(V)\\ &= k_{s,\delta}|\lambda|^s\int \frac{|a|^s}{|\lambda V+b|^s}\mathrm{d}\mu_n(V),\end{aligned} \tag{3.40}$$

where we used the relations (1.4) and (3.37). Averaging over $\{V(l)\}_{l\neq n}$ we get

$$\mathbf{E}\left(|\lambda V(n)+U_0(n)-z)|^s|G(n)|^s\right) \geq k_{s,\delta}|\lambda|^s\mathbf{E}(|G(n)|^s).$$

Let

$$\mathfrak{g}(n) := \mathbf{E}(|G(n)|^s).$$

Note that $\mathfrak{g}\in l^\infty(\mathbf{Z}^d)$ ($\mathfrak{g}(n)\leq 1/\varepsilon^s$). Relations (3.39) and (3.40) yield that

$$(1-k_{s,\delta}^{-1}|\lambda|^{-s}R)\mathfrak{g} \leq k_{s,\delta}^{-1}|\lambda|^{-s}\delta_m,$$

where $R$ is the operator of convolution by $|j(n,E)|^s$. By the choice of $s$ (recall Lemma 2.10)

$$\sum |j(n,E)|^s \leq c_d(s)+2d.$$

If $\lambda$ is such that

$$k_{s,\delta}|\lambda|^s > c_d(s)+2d, \tag{3.41}$$

then it follows from Lemma 2.11 that

$$\mathfrak{g} \leq k_{s,\delta}^{-1}|\lambda|^{-s}(1-k_{s,\delta}^{-1}|\lambda|^{-s}R)^{-1}\delta_m.$$

Lemma 2.12 and the estimate (1.19) yield that there exist constants $C_E$ and $a(E)>0$ such that

$$\mathfrak{g}(n) \leq C_E\mathrm{e}^{-a(E)|n-m|_+}.$$

Therefore, for all $E\notin\sigma(H_0)$,

$$\sup_{\varepsilon>0}\mathbf{E}\left(|G(m,n;E+\mathrm{i}\varepsilon)|^s\right) \leq C_E\mathrm{e}^{-a(E)|n-m|_+}.$$

Since $\delta$ in (3.37) is arbitrary, Theorem 1.1 follows from Lemma 2.5.

# 4 The weak coupling regime

In this section we prove Theorem 1.2.

We fix $s \in ]d/(d+1), 1[$ such that $K_s < \infty$. Let $\delta > 0$. Since $\limsup K_s(n) = K_s$, there exists an $l$ such that for all $n$ with $|n|_+ > l$,

$$K_s(n) < K_s + \delta =: K_{s,\delta}. \tag{4.42}$$

By changing the distributions $\mu_n$ within the box $|n|_+ \leq l$ we may assume that (4.42) holds for all $n$.

Let $m \in \mathbf{Z}^d$ and $E \notin \sigma(H_0)$ be given. The resolvent identity yields that

$$\begin{aligned}(\delta_m|(h(E) - E - \mathrm{i}\varepsilon)\delta_n) &= (\delta_m|(h_0(E) - E)^{-1}\delta_n) \\ &\quad - \textstyle\sum_k (\lambda V(k) - \mathrm{i}\varepsilon)(\delta_m|(h(E) - E - \mathrm{i}\varepsilon)^{-1}\delta_k)(\delta_k|(h_0(E) - E)^{-1}\delta_n).\end{aligned} \tag{4.43}$$

Using the relation (1.15) and shorthand (3.38) we rewrite (4.43) as

$$G(n) = -t(n-m, E) + \sum_k (\lambda V(k) - \mathrm{i}\varepsilon) t(n-k, E) G(k).$$

Then,

$$\mathbf{E}(|G(n)|^s) \leq |t(n-m,E)|^s + \sum_k |t(n-k,E)|^s \mathbf{E}\left((|\lambda|^s |V(k)|^s + |\varepsilon|^s)|G(k)|^s\right). \tag{4.44}$$

Averaging first over $V(k)$ and then over $\{V(l)\}_{l \neq k}$, we derive from (1.4), (4.42) and Lemma 2.13 that

$$\mathbf{E}\left(|V(k)|^s |G(k)|^s\right) \leq K_{s,\delta} \mathbf{E}(|G(k)|^s). \tag{4.45}$$

Let

$$\begin{aligned}\mathfrak{g}(n) &:= \mathbf{E}(|G(n)|^s), \\ f(n) &:= |t(n-m,E)|^s.\end{aligned}$$

Clearly, $\mathfrak{g}, f \in l^\infty(\mathbf{Z}^d)$ and we derive from (4.44) and (4.45) that

$$(1 - (|\varepsilon|^s + |\lambda|^s K_{s,\delta})R)\mathfrak{g} \leq f,$$

where $R$ is the operator of convolution by $|t(n,E)|^s$. By the choice of $s$ (recall Lemma 2.10)

$$\sum_k |t(n,E)|^s \leq c_d(s).$$

We choose $\lambda$ such that

$$|\lambda|^s K_{s,\delta} < c_d(s)^{-1},$$

and $\varepsilon_0 > 0$ such that

$$|\varepsilon_0|^s + |\lambda|^s K_{s,\delta} < c_d(s)^{-1}.$$

In the sequel we assume that $0 < \varepsilon < \varepsilon_0$. Lemma 2.11 yields that

$$\mathfrak{g} \le (1 - (|\varepsilon|^s + |\lambda|^s K_{s,\delta})R)^{-1} f,$$

and it follows from Lemma 2.12 and the estimate (1.19) that for some constants $C_E$ and $a(E) > 0$

$$\mathfrak{g}(n) \le C_E \mathrm{e}^{-a(E)|n-m|_+}.$$

Therefore, for all $E \notin \sigma(H_0)$,

$$\sup_{0<\varepsilon<\varepsilon_0} \mathbf{E}(|G(m,n;E+\mathrm{i}\varepsilon)|^s) \le C_E \mathrm{e}^{-a(E)|n-m|_+}.$$

Since $\delta$ in (4.42) is arbitrary, Theorem 1.2 follows from Lemma 2.5.

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