Homework: Introduction to Multiscale Modeling

1) Consider the singular perturbation problem

$$-\varepsilon\frac{d^2u}{dx^2}+\frac{du}{dx}=1,\quad 0<x<1$$
$$u(0)=0,\quad u(1)=0$$

(a) Determine the limit solution as $\varepsilon\to 0, (\varepsilon>0)$.

(b) Determine an improved approximation by matched asymptotics for $\varepsilon = 0.1$. Use the approximation from (a) for most of the interval and the solution of the homogeneous problem in an interval of length 0.2 close to the boundary with the boundary layer. Match these two approximations such that the final approximation is continuous and satisfies the boundary conditions.

(c) Solve the original singular perturbation problem numerically by a finite difference approximation. Use $\varepsilon = 0.1$ and the step size h=0.02. In the numerical approximation replace the derivatives in the original differential equation by divided differences,

$$u(x_j)\approx u_j,\ \ j=0,..,J,\ \ hJ=1$$
$$\frac{d^2u}{dx^2}\to\frac{u_{j+1}-2u_j+u_{j-1}}{h^2},\quad \frac{du}{dx}\to\frac{u_{j+1}-u_{j-1}}{2h}$$

(d) Plot the approximations from (a), (b) and (c).

2) Consider the following two-point boundary problem with oscillatory coefficient,

$$-\frac{d}{dx}\left(a(x/\varepsilon)\frac{du}{dx}\right)=1,\quad 0<x<1$$
$$u(0)=0,\quad u(1)=0$$

The oscillatory coefficient $a(x/\varepsilon)$ is a one-periodic function given by,

$$a(y)=\begin{cases}1, & 0\le y<0.5\\ 0.5, & 0.5\le y<1\end{cases}$$

(a) Determine the homogenized equation $(\varepsilon\to 0)$ of the original two-point boundary problem.

(b) Solve the homogenized equation analytically

(c) Solve the original two-point boundary problem numerically with $\varepsilon = 0.1, h = 0.02$. Use the following finite difference formula,

$$u(x_j) \approx u_j,\ j = 0,..,J,\ hJ = 1$$
$$-\frac{d}{dx}\left(a(x/\varepsilon)\frac{du}{dx}\right) \to -\frac{a(x_j + h/2)(u_{j+1} - u_j) - a(x_j - h/2)(u_j - u_{j-1})}{h^2}$$

(d) Plot the approximations from (b) and (c).

3) Consider the initial value problem for the stiff ordinary differential equation,

$$\varepsilon\frac{du}{dt} = 1 - u, \quad 0 < t < 1$$
$$u(0) = 0$$

(a) Solve the problem numerically for $\varepsilon = 0.02$ by the Euler method and step size H = 0.2 and plot the result.

$$u(t_n) \approx u_n,\ n = 0,..,N,\ HN = 1$$
$$u_{n+1} = u_n + H\varepsilon^{-1}(1 - u_n)$$

(Note that the result is clearly not satisfactory. In this simple problem we can use the analytical solution or use a, so-called, stiffly stable method but let us as an illustration apply a multiscale technique.)

(b) First resolve the transient by applying 20 microscale steps, h = 0.01 with the Euler method. Then continue with the macroscale solver as above in (a), H = 0.2, with the change that $\varepsilon^{-1}(1 - u_n)$ is replaced by an average $\left\langle \varepsilon^{-1}(1 - u_n) \right\rangle$, Compute this average by, starting from u_n take M microsteps, h = 0.01 and perform the averaging,

$$\left\langle \varepsilon^{-1}(1 - u_n) \right\rangle = \frac{1}{M}\left(\varepsilon^{-1}(1 - u_n^1) + .. + \varepsilon^{-1}(1 - u_n^M)\right), \quad (u_n^0 = u_n)$$

Use M = 5 and plot the result.