Short notes on Fourier Analysis
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1 The Fourier integral transform

Let f denote a real-or complex- valued function of a real variable x such that $f(x)$ is defined over $\mathbb{R}$. The Fourier transform on f arises when considering linear integral transformations of the form

$$T_\xi f = \int_{-\infty}^{+\infty} f(x)K(x,\xi)dx \tag{1}$$

for a "nice" function K. Let's call K the kernel of the transformation. In general there are many choices for the kernel and each is useful in its own right. In our case, the goal is to choose a kernel and the domain of the parameter ξ so that T transforms derivatives f' (let's assume derivatives exist) into products

$$T_\xi f' = \xi T_\xi f \tag{2}$$

To this end, we assume henceforth that the functions K have derivatives in x and are bounded on $\mathbb{R}$ whereas the functions f live in the set C_0^m of continuously differentiable functions with m derivatives and the functions together with the derivatives decay to zero sufficiently fast so that (1) converges. Ok, these are a lot of assumptions and the natural question is: can we weaken some of these conditions? The answer is "Yes!" but one would need to consider L_p spaces and the Lebesgue integral to achieve this task.

Definition 1. *The Fourier transform of a function $f \in C_o^m$ is given by $\Phi_\xi(f(x)) = \int_{-\infty}^{+\infty} f(x)e^{-i\xi x}dx$.*

One might wonder how we picked the kernel. Well, a quick integration by parts of (1) yields $K'(x,\xi) = -\xi K(x,\xi)$ and the exponential function satisfies this and all the other above mentioned requirements. Moreover, repeated integration by parts yields the following theorem.

Theorem 1. *If $f \in C_0^m$ then $\Phi_\xi(f^{(m)}(x)) = (i\xi)^m \Phi_\xi(f(x))$.*

Now suppose we are given the Fourier transform of a function. In general, it is not obvious what the function is so the question rises of how we can retrieve our function. The following theorem answers this question.

Theorem 2. *If $f \in C_0^m$, then for all real x we have $f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Phi_\xi(f(x))e^{i\xi x}d\xi = \Phi_\xi^{-1}(\Phi_\xi(f(x))$.*

Theorem 3. *(Plancherel)*
Let $f \in C_0^m$. Then $\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |\Phi_\xi(f(x))|^2 d\xi$.

Now we list some basic properties of $\Phi_\xi(f)$. Assume that c is a real constant.

$$\Phi_\xi(f(x-c)) = e^{-i\xi c}\Phi_\xi(f(x)) \tag{3}$$

$$\Phi_\xi(f(-x)) = f(-\xi) \tag{4}$$

$$\Phi_\xi(f(cx)) = \frac{1}{|c|}\Phi_\xi(\frac{\xi}{c}), \quad c \neq 0 \tag{5}$$

$$\Phi_\xi(e^{icx}f(x)) = \Phi_\xi(\xi - c) \tag{6}$$

2 Fourier series for functions on periodic intervals

Certain functions have the property of periodicity. A function f is said to be periodic with period L if $f(x) = f(x + L)$. Perhaps the most familiar functions with this property are the sine and cosine functions. The next theorem gives us a nice way of thinking about periodic functions:

Theorem 4. *(Dirichlet's Condition)*
i) f is periodic with period $2L$ over $\mathbb{R}$
ii) f has at most, a finite number of local max and local min for $-L \leq x \leq L$
iii) f has at most a finite number of jump discontinuities over $[-L, L]$
then $f(x)$ $\frac{f(x^+)+f(x^-)}{2}$ $= \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n cos(\frac{n\pi x}{L}) + b_n sin(\frac{n\pi x}{L})]$, where $a_n = \frac{1}{L}\int_{-L}^{L} f(x)cos(\frac{n\pi x}{L})dx$ and $b_n = \frac{1}{L}\int_{-L}^{L} f(x)sin(\frac{n\pi x}{L})dx$.

By squaring the function $f(x)$ in the above theorem and its Fourier series representation and integrating both sides one can derive Parseval's theorem. Note that this just the Plancherel version for Fourier series.

Theorem 5. *With f as above we have that $\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$.*

In the more general case, suppose f is a function periodic in $[-\frac{L}{2}, \frac{L}{2}]$. If one uses the complex form for the sine and cosine function then the series looks like

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{i(\frac{2\pi nx}{L})} \tag{7}$$

where the coefficients are given by the expression

$$c_n = \frac{1}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} f(x)e^{-i(\frac{2\pi nx}{L})}dx \tag{8}$$