Consider the wave equation in 2 spacial dimensions, $x = (x_1, x_2)$

$$u_{tt} - u_{x_1x_1} - u_{x_2x_2} = 0 ,$$

with initial data

$$\begin{aligned} u|_{t=0} &= e^{-100|x|^2} e^{ik(-x_1+x_2^2)} \\ u_t|_{t=0} &= \left[ik\sqrt{1+4x_2^2}\right] e^{-100|x|^2} e^{ik(-x_1+x_2^2)} , \end{aligned}$$

so that the waves propagate in the positive x_1 direction.

1. Derive the Eikonal and transport equations for solutions of the form,

$$u = A(t,x)e^{ik\phi(t,x)} .$$

2. Write a function in some programming language (Matlab would be the easiest using `ode45`) to solve the ODEs that define the bicharacteristics for the Eikonal equation. In the notation from class, the bicharacteristics are

$$(T(s), X(s), \tau(s), \xi(s)) ,$$

where $X = (X_1, X_2)$ and $\xi = (\xi_1, \xi_2)$.

Your code should take as inputs, t, $X(0) = (y_1, y_2)$ and $\xi(0) = (\eta_1, \eta_2)$ and return $(T(s_0), X(s_0), \tau(s_0), \xi(s_0))$ for s_0 such that $T(s_0) = t$. You can use the analytic solution to find this s_0.

Note: You do not need to numerically integrate the τ equation, since $\tau^2 = |\xi|^2$ for all s. Choose the root for τ which gives you propagation in the positive x_1 direction.

3. Enlarge your ODE system to also compute ϕ and its second derivatives on the bicharacteristic originating from $(0, y_1, y_2, \eta_1, \eta_2)$. You will need initial conditions for ϕ and all second derivatives involving x_1 and x_2. Remember that you can get derivatives involving t using derivatives of the Eikonal equation directly. Also, compute the amplitude on this bicharacteristic (it will need an initial value as well).

4. Consider a representation of the initial data as follows,

$$A(x)e^{ik\phi(x)} \approx \frac{k}{2\pi}\int_\Omega A(y)e^{ik(T_2^y[\phi](x)+i|x-y|^2/2)}dy ,$$

where,

$$\begin{aligned} A(x) &= e^{-100|x|^2} \\ \phi(x) &= -x_1 + x_2^2 , \end{aligned}$$

and $T_2^y[\phi](x)$ is the second order Taylor polynomial of ϕ about the point y as a function of x. The domain Ω is the square $[-0.2, 0.2]^2$.

Now, looking at $T_2^y[\phi](x) + i|x-y|^2/2)$ as the entire initial phase and $A(y)$ as the amplitude, decide what you need to send as input to your code from the previous part, so that you can calculate the phase, its derivatives, and the amplitude at a given time t for the characteristics originating from (y_1, y_2) at $t = 0$.

5. For a fixed y, fixed t, and the appropriate initial conditions from the previous part, calculate ϕ, its first and second derivatives and the amplitude at s_0 (this is the same s_0 as before). Then form

$$\begin{aligned} \psi(t,x;y) &= \phi(s_0) + \nabla_x\phi(s_0)\cdot(x-y) + \frac{1}{2}(x-y)\cdot H_x\phi(s_0)(x-y) \\ A(t,x;y) &= A(s_0)\ , \end{aligned}$$

where $\nabla_x\phi = (\phi_{x_1}, \phi_{x_2})$ and $H_x\phi$ is the 2×2 Hessian matrix of ϕ containing its x derivatives.

Finally, compute the wave field for one Gaussian beam for $k = 10^4$,

$$v(t,x;y) = \frac{k}{2\pi}A(t,x;y)e^{ik\psi(t,x;y)}\ .$$

Computationally, you will need to evaluate $v(t,x;y)$ on some grid: fix a value for t and y (say something like $t = .25$, $y = (0,0)$, but you should be able to vary these values later) and create a mesh for (x_1, x_2) (say on the rectangle $[-0.2, 1.2]\times[-0.2, 0.2]$), then evaluate $v(t,x;y)$ on this grid.

6. Finally, loop over your code in the last part to compute

$$u(t,x) = \int_\Omega v(t,x;y)dy\ .$$

As with x, you will need a grid on Ω to calculate this integral. Test your code by computing $u(t,x)$ at $t = 0$ and comparing the result to the initial condition for u. Make several plots showing u (its real, and absolute values) at $t = 0, 0.25, 0.50, 0.75, 1.00$. You may find it useful to look at the article "Superpositions and higher order Gaussian beams" available at

`http://www.intlpress.com/CMS/2008/issue6-2/`

and more specifically, sections 3.1 and 3.4.