Wavelet-based Numerical Homogenization

Project

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Consider the model elliptic boundary value problem in one dimension,

$$-\partial_x a_\varepsilon(x)\partial_x u = f, \qquad u(0) = 0, \qquad u'(1) = 0. \tag{1}$$

Introduce the grid $\{x_k\}_{k=0}^{N-1}$ where $x_k = (k + 1/2)h$ and $h = 1/N$. Let u_k approximate $u(x_k)$ and set $U = \{u_k\}$, $F = \{f(x_k)\}$. Then use standard central differences,

$$\frac{1}{h^2}\Delta_+ A_\varepsilon \Delta_- U = F, \qquad U, F \in \mathbb{R}^N. \tag{2}$$

Here $\Delta_\pm$ are the forward/backward difference operators and A_ε is a diagonal matrix sampling $a_\varepsilon(x)$ in $x = kh$, $k = 0, \ldots, N-1$. In order to approximate the Dirichlet condition at $x = 0$, use $(\Delta_- U)_0 = 2u_0$. This leads to an overall second order method for (1).

For the numerical homogenization we will only use Haar wavelets. Also, we will not be concerned with computational costs, but rather with the approximation properties of the numerically homogenized operator. Let $N = 2^n$ for some n, sufficiently large to resolve the ε-scale in (1). In the Haar case, functions f in the scaling space V_n are piecewise constant and their scaling coefficients $s_{n,k}$ satisfy $s_{n,k} = 2^{-n/2} f(x_k)$. After appropriate rescaling we can therefore interpret (2) as a discretization in V_n,

$$L_n U_n = F_n, \qquad U_n, F_n \in V_n, \qquad L_n := 2^{2n}\Delta_+ A_\varepsilon \Delta_-,$$

where U_n and F_n contains the scaling coefficients of $u(x)$ and $f(x)$ in V_n.

a) Let $a(x, y) = 0.55 + 0.45\sin(2\pi y)b(x - 0.5)$, with $b(x) = \exp(-20x^2)$ and use $a_\varepsilon(x) = a(x, x/\varepsilon)$ for some small ε. Simulate the detailed equation and compare it with 1) the constant coefficient equation using the arithmetic mean of $a_\varepsilon(x)$, i.e. $a \equiv 0.55$, and 2) the homogenized equation. (Determine, at least numerically, the homogenized coefficient $\bar{a}(x)$ using analytical formulae.)

b) Compute the numerically homogenized operator $\bar{L}_m$, where $m < n$ is chosen so that the ε-scale averages out, i.e. $2^{-m} \geq \varepsilon$. (Make sure n is large enough though, $2^{-n} \ll \varepsilon$.) Examine the structure of $\bar{L}_m$ and verify that it its elements decay quickly away from the diagonal. Approximate $\bar{L}_m$ by crude truncation to ν diagonals. Check how many diagonals you must keep to get an acceptable solution.

c) We can write the numerically homogenized operator on the same form as the original,

$$\bar{L}_m = 2^{2m} \Delta_+ H \Delta_-, \tag{3}$$

for some matrix H, which corresponds to the "effective material coefficient" at scale m. Compute H and verify that it is strongly diagonal dominant. Approximate H by crude truncation and rebuild an approximation of $\bar{L}_m$ from the formula (3). How many diagonals do you need to keep now? Approximate H by band projection to a single diagonal, i.e. "mass lumping," $H \approx \text{band}(H, 1) = \text{diag}(H\mathbf{1})$, where $\mathbf{1}$ is the constant vector. How good is the corresponding solution? How does $\text{band}(H, 1)$ compare with the original coefficient $a_\varepsilon(x)$ and with the homogenized coefficient $\bar{a}(x)$?

d) (Optional.) Test a few other types of coefficients:

1. A three-scale system, e.g.

$$a_\varepsilon(x) = \frac{1}{2}\left(a(x - 0.1, x/\varepsilon_1) + a(x + 0.1, x/\varepsilon_2)\right), \qquad \varepsilon_1 \ll \varepsilon_2 \ll 1.$$

   Use different $m = m_1, m_2$ to capture the behavior at different scales, i.e. $2^{-m_1} \geq \varepsilon_2 \gg 2^{-m_2} \geq \varepsilon_1$.

2. A random coefficient,

$$a_\varepsilon(x) = 0.1 + b(x - 0.5)U(x),$$

   where $b(x)$ is as above, and $U(x)$ are uniformly distributed random numbers in the interval $[0, 1]$ for each x. (Use MATLAB's `rand` command.)

3. A localized coefficient,

$$a_\varepsilon(x) = \begin{cases} 0, & |x - 0.5| \geq \varepsilon, \\ \frac{1}{\varepsilon}, & |x - 0.5| < \varepsilon. \end{cases}$$