Final assignment

1. Effective *deterministic* dynamics emerging from multiscale *stochastic* systems:

   Consider the system of equations:

   $$\begin{cases} dX_t = & \left[-(Y_t + Y_t^3) + \cos(\pi t) + \sin(\sqrt{2}\pi t)\right] dt \\ dY_y = & -\epsilon^{-1}(Y_y + Y_y^3 - X_t)dt + \epsilon^{-1/2} dB_t, \end{cases} \tag{1}$$

   with initial conditions $(X_0, Y_0) = (0, 1)$.

   Integrate the system using the HMM algorithm as explained in lecture 4. Plot the mean and covariance of X_t with $\epsilon = 10^{-4}$.

2. Effective *stochastic* dynamics emerging from multiscale *stochastic* systems:

   Consider
   $$\begin{cases} dX_t = & \epsilon^{-1} Y_t \\ dY_y = & -\epsilon^{-2} Y_t^3 dt + \epsilon^{-1}(1 + X_t^2) dB_t, \end{cases} \tag{2}$$

   with initial conditions $(X_0, Y_0) = (0, 1)$. Solve the equation using the HMM algorithm.

3. Effective *stochastic* dynamics emerging from multiscale *deterministic* systems:

   Consider the system of ordinary differential equations:

   $$\begin{cases} \dot{x} = & x - x^3 + \frac{4}{90}\epsilon^{-1} y_2 \\ \dot{y}_1 = & 10\epsilon^{-2}(y_2 - y_1) \\ \dot{y}_2 = & \epsilon^{-2}(28 y_1 - y_2 - y_1 y_3) \\ \dot{y}_3 = & \epsilon^{-2}(y_1 y_2 - \frac{8}{3} y_3). \end{cases} \tag{3}$$

   The equations for (y_1, y_2, y_3) are called the Lorenz equations. In the example above, the solution is chaotic. To get a feeling what chaotic means, plot the trajectory of (y_1, y_2, y_3) with any non-zero initial condition and some small ϵ.

   Now, consider the SDE

   $$dX_t = (X_t - X_t^3)dt + \sigma dB_t. \tag{4}$$

where $\sigma = 0.126$. We wish to show that, for small ϵ, X_t approximates trajectories of $x(t)$. It is still not clear what this statement means since, for any given initial conditions, $x(t)$ is a particular deterministic solution, while X_t is a stochastic process.

Let P denote any distribution on $\mathcal{R}^3$ with a continuous density, for example, each coordinate is IID normal. Plot several sample trajectories of $x(t)$ with $\epsilon = 10^{-3}$. Initial conditions are $x(0) = 1$ and $(y_1(0), y_2(0), y_3(0))$ drawn from the distribution P. Plot the mean and variance of $x(t)$ on a macroscopic time scale independent of ϵ. Here, randomness comes from initial conditions. Compare to that of X_t.

Repeat this example with two additional distributions on $\mathcal{R}^3$ as initial conditions for (y_1, y_2, y_3). Show (through numerical examples) that after an initial relaxation time, the statistics of $x(t)$ does not depend on the initial distribution.

Try to explain how come the dynamics of x appears random. In particular, explain why we have to assume that the initial conditions for (y_1, y_2, y_3) are random although it is not too important what this distribution is.

4. Effective *deterministic* dynamics emerging from multiscale *deterministic* systems:

   Richard Tsai's class.