Introduction to Wavelet Based Numerical Homogenization

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Wavelet based numerical homogenization

[Beylkin,Brewster,Engquist,Dorobantu,Levy,Gilbert,O.R.,...]

Suppose

$$L_{j+1} u = f, \qquad L_{j+1} \in \mathcal{L}(V_{j+1}, V_{j+1}) \qquad u, f \in V_{j+1},$$

is a discretization (e.g. FD, FEM) of a differential equation on scale-level $j+1$ where L_{j+1} contains small scales.

Want to find an *effective* discrete operator $\bar{L}_{j'}$, with $j' \ll j$ that computes the coarse part of u.

C.f. classical homogenization.

Wavelet based numerical homogenization

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Example (Elliptic eq, Haar)

$$\partial_x r(x/\varepsilon)\partial_x u_\varepsilon = f, \qquad \Rightarrow \qquad L_{j+1} = \frac{1}{h^2}\Delta_+ R^\varepsilon \Delta_-.$$

where R^ε is diagonal matrix sampling $r(x/\varepsilon)$, and $2^j \sim 1/\varepsilon$. Here one could use

$$\bar{L}_{j'} = \frac{1}{h^2}\Delta_+ \bar{R}\Delta_-.$$

Wavelet transforms

Simple to extract the coarse and fine part of $u = \{u_k\}$:

$$\mathcal{W}u = \begin{pmatrix} U_f \\ U_c \end{pmatrix}, \qquad u \in V_{j+1} \quad U_f \in W_j, \quad U_c \in V_j.$$

For compactly supported wavelets, $\mathcal{W}$ is *sparse*. It is also orthonormal, $\mathcal{W}^T\mathcal{W} = I$.

In Haar basis on $[0, 1]$,

$$\mathcal{W} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & \cdots & & \\ 0 & 0 & 1 & -1 & 0 & \cdots \\ \vdots & \vdots & & \ddots & \ddots & \\ 0 & 0 & \cdots & 0 & 1 & -1 \\ 1 & 1 & 0 & \cdots & & \\ 0 & 0 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & & \ddots & \ddots & \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix} \in \mathbb{R}^{2^{j+1} \times 2^{j+1}}.$$

Start from equation

$$L_{j+1}u = f, \qquad L_{j+1} \in \mathcal{L}(V_{j+1}, V_{j+1}) \qquad u, f \in V_{j+1}.$$

Decompose equation in coarse and fine part (use $\mathcal{W}^T\mathcal{W} = I$)

$$\mathcal{W}L_{j+1}\mathcal{W}^T\mathcal{W}u = \mathcal{W}f \qquad \Rightarrow$$

$$\begin{pmatrix} A_j & B_j \\ C_j & L_j \end{pmatrix} \begin{pmatrix} U^f \\ U^c \end{pmatrix} = \begin{pmatrix} F^f \\ F^c \end{pmatrix}.$$

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Eliminate U^f,

$$(L_j - C_j A_j^{-1} B_j)U^c = F^c - C_j A_j^{-1} F^f.$$

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Supposing f smooth so $F^f = 0$ and $F^c = f$.

$$(L_j - C_j A_j^{-1} B_j) U^c = f.$$

We call the matrix

$$\bar{L}_j = L_j - C_j A_j^{-1} B_j, \qquad \bar{L}_j \in \mathcal{L}(V_j, V_j),$$

the (numerically) homogenized operator. Since

- Half the size of original L_{j+1}.
- Given $\bar{L}_j, f$ we can solve for coarse part of solution, U^c.
- Takes influence of fine scales into account.

Compare with classical homogenization:

$$L = \nabla R(x/\varepsilon)\nabla \quad \Rightarrow \quad \bar{L} = \nabla \int R(x) dx \nabla - \nabla \int R(x) \frac{\partial \chi}{\partial x} dx \nabla$$

where χ solves the (elliptic) cell problem.

Wavelet based numerical homogenization

Reduction can be repeated,

$$\bar{L}_j \to \bar{L}_{j-1} \to \bar{L}_{j-2} \to \dots, \qquad \bar{L}_j \in \mathcal{L}(V_j, V_j),$$

to discard suitably many small scales / to get a suitably coarse grid. Also, condition number improves

$$\kappa(\bar{L}_j) < \kappa(L_{j+1}).$$

Wavelet based numerical homogenization

Problem: L sparse (banded) $\not\to$ $\bar{L}$ sparse (banded). (Must invert A_j.)

However: Approximation properties of wavelets imply elements of A_j^{-1} decay rapidly away from diagonal.

Therefore: $\bar{L}$ diagonally dominant in many important cases and can be well approximated by a banded matrix. (Cf. a (local) differential operator.)

Different Approximation Strategies

1. "Crude" truncation to ν diagonals,
2. Band projection to ν diagonals, defined by

$$M\boldsymbol{x} = \text{band}(M, \nu)\boldsymbol{x}, \qquad \forall \boldsymbol{x} \in \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_\nu\}.$$

$$\boldsymbol{v}_j = \{1^{j-1}, 2^{j-1}, \dots, N^{j-1}\}^T, \qquad j = 1, \dots, \nu.$$

   C.f. "probing", [Chan, Mathew], [Axelsson, Pohlman, Wittum].
3. The above methods used on the matrix H instead, where e.g.

$$L_{j+1} = \frac{1}{h^2}\Delta_+ R \Delta_- \qquad \Rightarrow \qquad \bar{L}_j = \frac{1}{(2h)^2}\Delta_+ H \Delta_-.$$

   H can be seen as the effective material coefficient.
4. The above methods used on the matrix A_j^{-1} instead, where $\bar{L}_j = L_j - C_j A_j^{-1} B_j$. [Levy, Chertock]

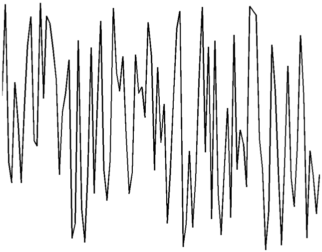
Elliptic 1D case

Consider the elliptic one-dimensional problem

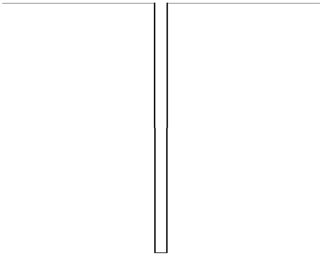
$$\partial_x a^\varepsilon(x)\partial_x u = 1, \qquad u(0) = u'(1) = 0,$$

with standard second order discretization.
Try two cases:

$a^\varepsilon(x) = $ "*noise*"

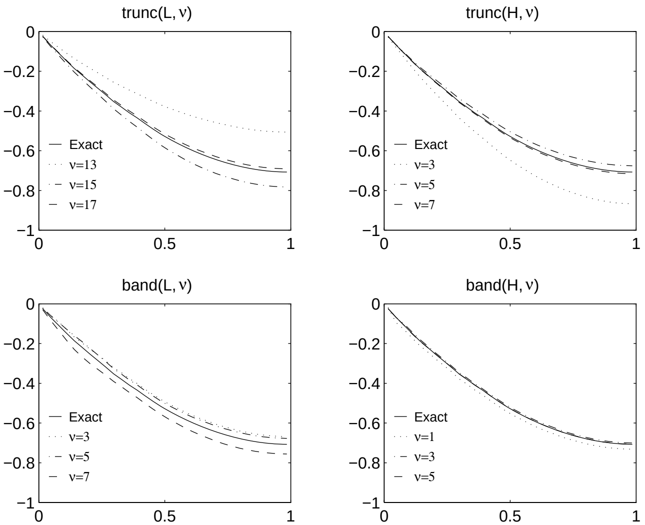


$a^\varepsilon(x) = $ "*narrow slit*"



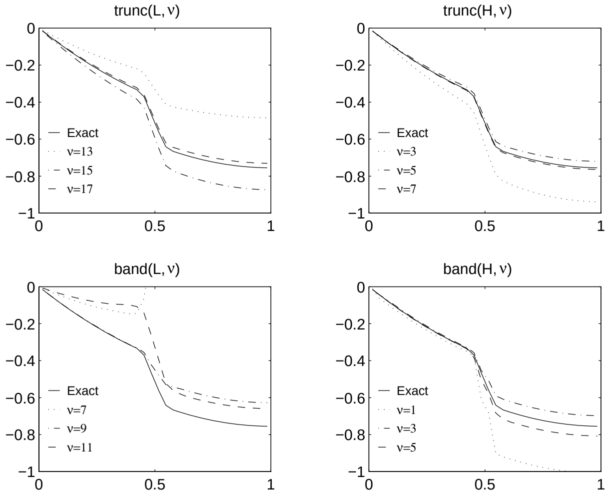
Elliptic 1D case – noise

Different approximation strategies



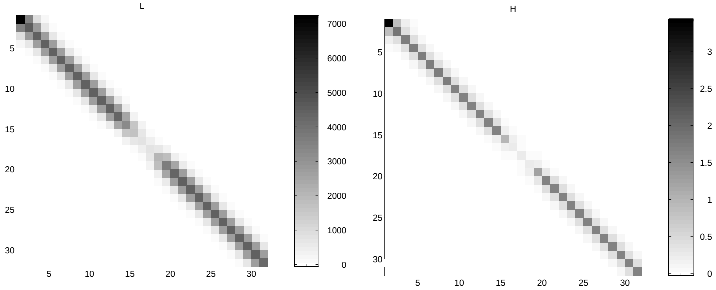
Elliptic 1D case – narrow slit

Different approximation strategies



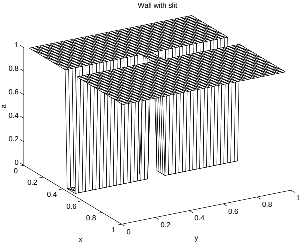
Elliptic 1D case – narrow slit

Matrix element size

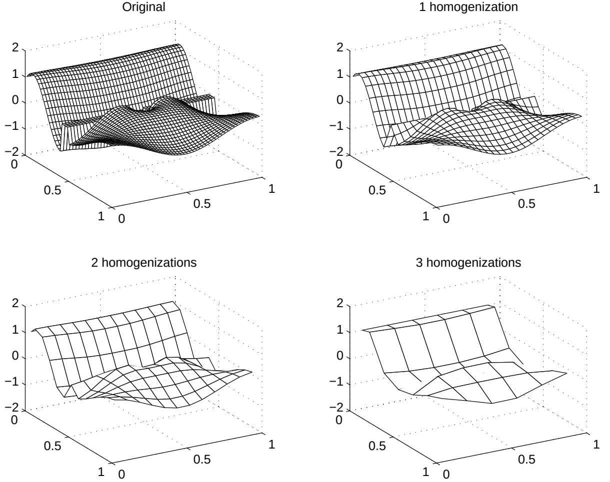


Simulate a wave hitting a wall with a small opening modeld by Helmholtz

$$\nabla a(x, y)\nabla u + \omega^2 u = 0,$$



Examples
Helmholtz 2D case



Examples

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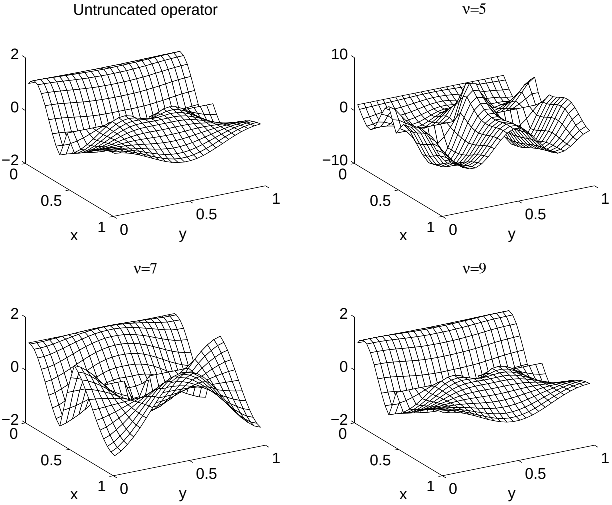

Helmholtz 2D case

Matrix element size