Introduction to Wavelets

Exercises

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We suppose that V_j and $\phi(x)$ are the scaling spaces and scaling function for an orthogonal multiresolution analysis (MRA),

1. $\cdots \subset V_j \subset V_{j+1} \subset \cdots \subset L^2(\mathbb{R})$
2. $f(t) \in V_j \Leftrightarrow f(2t) \in V_{j+1}$
3. $\overline{\cup V_j} = L^2(\mathbb{R})$ and $\cap V_j = \{0\}$
4. $\{\phi(x-k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_0.

Exercise 1

Let $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$. Show that $\{\phi_{j,k}(x)\}$ with $k \in \mathbb{Z}$ is an orthonormal basis for V_j.

Exercise 2

Suppose $\{h_k\}$ are the filter coefficients for the wavelet system,

$$\phi(x) = \sum_k h_k \phi_{1,k}(x) = \sum_k \sqrt{2} h_k \phi(2x-k). \tag{1}$$

Assume that $\int \phi(x)dx \neq 0$ and that only a finite number of $\{h_k\}$ are non-zero. Show that

a) $\sum_k h_k = \sqrt{2}$,

b) $\sum_k h_k h_{k-2\ell} = \delta_\ell = \begin{cases} 1, & \ell = 0, \\ 0, & \text{otherwise.} \end{cases}$
(Hint: Use the orthogonality of $\{\phi(x-k)\}$.)

Exercise 3

Show that if the mother wavelet $\psi(x)$ has M vanishing moments,

$$\int x^p \psi(x) dx = 0, \qquad p = 0, \ldots, M-1,$$

then the same is true for every wavelet basis function $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$.

Exercise 4

Suppose $u \in V_{j+1}$ is given by

$$u(x) = \sum_k u_k \phi_{j+1,k}(x).$$

It can then be decomposed into $u(x) = U^c(x) + U^f(x)$ with the coarse part $U^c(x) \in V_j$ and fine part $U^f(x) \in W_j$

$$U^c(x) = \sum_k U_k^c \phi_{j,k}(x), \qquad U^f(x) = \sum_k U_k^f \psi_{j,k}(x).$$

Show that

$$U_k^c = \sum_\ell u_\ell h_{\ell-2k}.$$

(Hint: Use (1) and the fact that $U_k^c = \langle u, \phi_{j,k} \rangle$.)

Extra: In the same way you can show that if

$$\psi(x) = \sum_k \sqrt{2} g_k \phi(2x-k),$$

then

$$U_k^f = \sum_\ell u_\ell g_{\ell-2k}.$$