Wavelets

- Tool for describing functions on different scales (or level of detail) via a multiresolution decomposition.
- Basis for $L^2(\mathbb{R})$ with special properties.

Example

Decomposition into different scales

$$f_3(x) = f_2(x) + d_2(x) = f_1(x) + d_1(x) + d_2(x) = \cdots =$$

$$= \underbrace{f_0(x)}_{\text{Average}} + \underbrace{d_0(x)}_{\text{detail } 0} + \underbrace{d_1(x)}_{\text{detail } 1} + \underbrace{d_2(x)}_{\text{detail } 2}$$

Why?

- Explicit description of f viewed on different scales: $f_1, f_2, f_3, \ldots$
  and of the parts of f belonging to different scales: $d_1, d_2, d_3 \ldots$
  Useful for understanding/analyzing phenomena & in design of numerical algorithms.
- Size of $d_j(x) \to 0$ rapidly with j when f is piecewise smooth –
  $\Rightarrow$ Can approximate f by just a few d_j
- Difference, $d_j(x)$ indicates where f is non-smooth
  $\Rightarrow$ useful tool ~~for~~ e.g. edge detection & in mesh refinement

Basis & inner product for L

$L^2(\Omega)$ = space of square integrable functions on Ω

$$\int_{\Omega} f^2(x)\,dx < \infty \iff f \in L^2(\Omega)$$

Some generalizations of finite-dimensional lin alg:
(vectors → functions = "infinite length vectors")

- Two functions f, g are orthogonal if
$$\langle f, g \rangle := \int_{\Omega} f(x) g(x)\,dx = 0$$
- Norm : $\|f\|_{L_2} = \sqrt{\langle f, f \rangle} = \int f^2\,dx.$
- Orthonormal basis (ONB)
  $\{\phi_k(x)\}$ is an (ONB) if
  - $f \in L_2 \Rightarrow \exists f_k : f(x) = \sum f_k \phi_k(x)$
  - $\langle \phi_k, \phi_\ell \rangle = \delta_{k-\ell}$

  $\Rightarrow f_k = \langle f, \phi_k \rangle$

(
- Parseval : $\|f\|^2 = \sum f_k^2$
- Cauchy-Schwarz : $|\langle f, g \rangle| \leq \|f\|_{L_2} \cdot \|g\|_{L_2}$

)

__Example__ : $L^2([0, 2\pi])$ has basis

__Fourier__ $\left\{\frac{\sin nx}{\sqrt{\pi}}\right\} \cup \left\{\frac{\cos nx}{\sqrt{\pi}}\right\} \cup \left\{\frac{1}{\sqrt{2\pi}}\right\}$

__Polynomials__ $1, x, x^2, \ldots$ (not orthogonal or normalized though)

Wavelet & scaling spaces

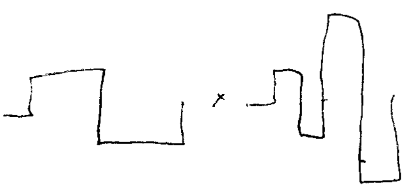
Example continued

Introduce function spaces $V_j, W_j \subset L^2(\mathbb{R})$ for the different scales:

V_j – "scaling spaces", spaces of functions viewed on scale j, "f_j"
Here: piecewise constant functions, piecewidth $= 2^{-j}$

W_j – "wavelet spaces", spaces of differences (detail) between level j & $j+1$, "d_j".

Every $f \in V_{j+1}$ can be decomposed uniquely as
$f = f_j + d_j$ with $f_j \in V_j$ & $d_j \in W_j$
$\Rightarrow V_{j+1} = V_j \oplus W_j$, direct sum

Moreover, here
$\langle f_j, d_j \rangle = 0$ $\int$  dx

$f_j \perp d_j \; \forall f \Rightarrow V_j \perp W_j$

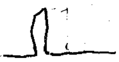
W_j is "orthogonal complement of V_j in V_{j+1}"

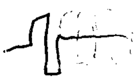
In fact:
$f_j = P_{V_j} f$
$d_j = P_{W_j} f$
projections. (defined as $\langle f - f_j, g \rangle = 0 \; \forall g \in V_j$)

Remark: $V_{j+1} = W_j \oplus W_{j-1} \oplus \dots \oplus W_{j-k} \oplus \underline{V_{j-k}} \dots$

Bases for V_j & W_j

Example continued

 $\{\phi_{j,k}\}$ orthonormal basis for V_j

 $\{\psi_{j,k}\}$ orthonormal basis for W_j = the wavelets

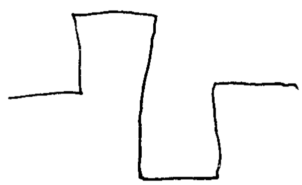
Observation: These are all dilated and translated versions of two functions!

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$$

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$$

where

$\phi(x) =$  → "scaling (shape) function"

$\psi(x) =$  "mother wavelet"

Moreover:
$$\int \phi(x)\phi(x-k)\,dx = \delta_k$$
$$\int \phi(x)\psi(x-k)\,dx = 0$$
$$\int \psi(x)\psi(x-k)\,dx = \delta_k$$

Since $V_{j+1} = V_j \oplus W_j$, $V_j \perp W_j$

the set $\{\phi_{j,k}\}_{k\in\mathbb{Z}} \cup \{\psi_{j,k}\}_{k\in\mathbb{Z}}$ is orthonormal basis for V_{j+1}.

$V_{j+1} = W_j \oplus W_{j-1} \oplus V_{j-1}$

$\{\psi_{j,k}\} \cup \{\psi_{j-1,k}\} \cup \{\phi_{j-1,k}\}$ ONB for V_{j+1}

$j \to \infty$, as before ...

$\{\psi_{j,k}\}$? ON basis for $L^2(\mathbb{R})$?

Structure of wavelet system, summary

- Scaling & wavelet spaces V_j & W_j describing different scales such that $\underbrace{V_{j+1}}_{\text{finer scale}} = \underbrace{V_j}_{\text{average/coarse part}} \oplus \underbrace{W_j}_{\text{difference/detail/fine part.}}$

- Bases for V_j & W_j

$$V_j = \text{span}\{\phi_{j,k}\} \quad \text{scaling functions}$$
$$W_j = \text{span}\{\psi_{j,k}\} \quad \text{wavelets}$$

generated by scaling function & mother wavelet

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$$
$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$$

$\Rightarrow$ $\{\psi_{j,k}\}$ basis for $L^2(\mathbb{R})$

- $\int \psi(x)\,dx = 0$ $\quad$ ($\Rightarrow$ basic approximation property)

System described above called "Haar" system. Simplest one possible, first proposed early 20^{th} century.

Advantages:
- ϕ, ψ compactly supported (good for numerics)
- all bases orthogonal & $V_j \perp W_j$

Disadvantages:
- ϕ, ψ discontinuous
- $\int x^p \psi(x)\,dx \neq 0 \quad p > 0$

} relatively bad numerical & approximation properties

Generalize Haar and find better wavelet systems!

$\boxed{\text{Examples}}$

Wavelets - generalizations

Want to create "better wavelet systems than Haar, ~~with~~ so

Eg:
- Smoother wavelets
- Wavelets with many vanishing moments:

$$\int \psi(x)\, x^p\, dx = 0 \quad p = 0, \dots, M-1$$

cf. Haar: $M = 1$

$\Rightarrow$ give good approximation prop.

Other "good" properties:
- Compactly supported ϕ & ψ.
- Symmetric ϕ, ψ; $\hat{\phi}, \hat{\psi}$.
- etc. for $\hat{\phi}, \hat{\psi}$
- Orthonormal bases.

Multiresolution analysis

One can show that we just need to find spaces V_j & a scaling function $\phi(x)$ that forms a multiresolution analysis (MRA):

i.e. V_j & ϕ satisfy:

1. $\dots \subset V_j \subset V_{j+1} \subset \dots \subset L^2$

   nested spaces

2. $f(t) \in V_j \Longleftrightarrow f(2t) \in V_{j+1}$

   spaces ~~have~~ describe different scales

3. $\overline{\bigcup V_j} = L^2 \qquad \bigcap V_j = \{0\}$ (technical)

4. $\{\phi(x-k)\}$ is an ON-basis for V_0

From these requirements it then follows that a full wavelet system can be created:

- $\phi_{j,k}(x) := 2^{j/2}\phi(2^j x - k)$ is an ON basis for V_j (Exercise)
- W_j spaces defined as orthogonal complement of V_j in V_{j+1} ($V_{j+1} = V_j \oplus W_j$, $V_j \perp W_j$)
- ~~There~~ exists a mother wavelet $\psi(x)$ such that
$$\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k)$$
is an ON basis for W_j (and for $L^2(\mathbb{R})$)

- There are many examples of V_j & ϕ satisfying 1) - 4).
- Of particular interest ~~for~~ numerical computations are those with compact support for ϕ & ψ. (Discovered in 88' by Daubechies.)
- The orthogonality condition can be relaxed, and non-orthogonal wavelet systems can also be built.

Filter coefficients

Since $\phi \in V_0 \subset V_1$ and V_1 is spanned by $\phi_{1,k}(x) = \sqrt{2}\,\phi(2x-k)$ we can find $\{h_k\}$ such that

$$(*) \quad \phi(x) = \sum_k \sqrt{2}\, h_k\, \phi(2x-k)$$

This is called a <u>refinement equation</u>

The $\{h_k\}$ are the <u>(lowpass) filter coefficients</u> for the wavelet system. Characterizes it.

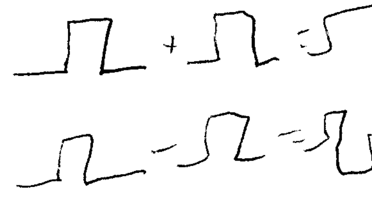
Similarly, since $\psi \in W_0 \subset V_1$,

$$\psi(x) = \sum_k \sqrt{2}\, g_k\, \phi(2x-k)$$

for some (highpass) filter coefficients $\{g_k\}$,

<u>Example</u>: Haar: $h_k = \left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$

$g_k = \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}$



<u>Note</u>: Compact support of $\phi, \psi \Leftrightarrow$ $\{h_k\}\{g_k\}$ finite length (FIR) filter or (impulse response)

= finite # non-zero entries.

$\circledast$ We can build $\perp$ wavelet systems directly from filter coefficients because of the following

<u>If</u>

- $\sum_k h_k h_{k+2\ell} = \delta_\ell = \begin{cases} 1 & \ell = 0 \\ 0 & \text{else.} \end{cases}$

(**) - $\sum_k h_k = \sqrt{2}$

plus some technical assumptions

(e.g. $\sum |h_k| |k|^\varepsilon < \infty \quad \varepsilon > 0$), $\left(\inf_{|\xi| \le \pi/2} |\hat{h}(\xi)| \geq \delta > 0 \right)$

$$\left\{ \begin{array}{l} \left(\hat{h}(\xi) := \frac{1}{\sqrt{2}} \sum_k h_k e^{ik\xi} = \frac{1}{2}(1 + e^{i\xi})^N \hat{f}(\xi) \right. \\ \sum |f_k| |k|^\varepsilon < \infty \quad \varepsilon > 0 \quad) \\ |\hat{f}(\xi)| < 2^{N-1} \end{array} \right\} \Rightarrow \text{gives regular } \phi$$

<u>Then</u>

- (*) defines a scaling function
- $g_k = (-1)^k h_{1-k}$ defines a mother wavelet via (*')
- which together defines an MRA.
-

Finding compactly supported orthogonal wavelets

$\Leftrightarrow$

Finding finite length $\{h_k\}$ satisfying (**)

<u>Exercise</u>: Show that (**) are <u>necessary</u> conditions if (*) holds & $\{\phi(x-k)\}$ are orthogonal.

Approximation properties

Basic approximation mechanism of wavelets: vanishing moments. (and space localization)

Definition: Mother wavelet $\psi(x)$ has M vanishing moments if

$$\int x^p \psi(x)\,dx = 0, \quad \text{for } p = 0, 1, \dots, M-1$$

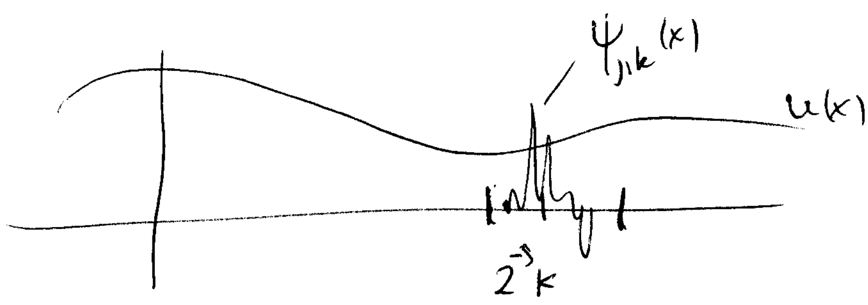
- Implies that $\int p(x)\psi(x)\,dx = 0$ when $p(x)$ polynomial of degree $< M$
- Same holds for $\psi_{j,k}(x)$ (Exercise.)
- There exist compactly supported wavelets with arbitrary many vanishing moments. (Daubechies, CPAM '88')

Suppose $u(x) = \sum u_{j,k}\psi_{j,k}(x)$ is smooth $\in C^M(\mathbb{R})$

Then coefficients given by

$$u_{j,k} = \langle u, \psi_{j,k} \rangle = \int_\Omega u(x)\psi_{j,k}(x)\,dx$$

where Ω = support of $\psi_{j,k}(x)$



Taylor expand $u(x)$ around $x_0 \in \Omega$

$$u(x) = u(x_0) + (x-x_0)u'(x_0) + \cdots + \frac{(x-x_0)^{M-1}}{(M-1)!}u^{(M-1)}(x_0) + \frac{(x-x_0)^M}{M!}R(x)$$

$$\Rightarrow \quad u_{j,k} = \underbrace{\int_\Omega \sum_{n=0}^{M-1} u^{(n)}(x_0)\frac{(x-x_0)^n}{n!}\psi_{j,k}(x)\,dx}_{=0} + \int_\Omega \frac{(x-x_0)^M}{M!}R(x)\psi_{j,k}(x)\,dx$$

since ψ has M vanishing moments!

$$\Rightarrow \quad |u_{j,k}| \le \int_\Omega \frac{|x-x_0|^M}{M!}|R(x)|\cdot|\psi_{j,k}(x)|\,dx$$

Note: $\quad |x-x_0| \le |\Omega|$

$\quad\quad\quad |R(x)| \le \sup_{x\in\Omega}|u^{(M)}(x)|$

$$\Rightarrow \quad |u_{j,k}| \le \frac{|\Omega|^M}{M!}\cdot\sup_{x\in\Omega}|u^{(M)}(x)|\cdot\int_\Omega|\psi_{j,k}(x)|\,dx$$

Cauchy Schwarz: $\displaystyle\int_\Omega fg\,dx \le \left(\int_\Omega f^2dx\cdot\int_\Omega g^2dx\right)^{1/2}$

$$\Rightarrow \quad \int_\Omega|\psi_{j,k}(x)|\,dx \le \Big(\underbrace{\int_\Omega 1^2dx}_{=|\Omega|}\cdot\underbrace{\int\psi_{j,k}^2\,dx}_{=1}\Big)^{1/2} \le |\Omega|^{1/2}$$

$$\therefore \quad |u_{j,k}| \le \frac{|\Omega|^{M+1/2}}{M!}\cdot\sup_{x\in\Omega}|u^{(M)}(x)| \le C2^{-j(M+1/2)}\sup_{x\in\Omega}|u^{(M)}(x)|$$

- $u_{j,k}$ (details) decay rapidly with j locally around $\psi_{j,k}$ when $u(x)$ smooth & M large.

- Most $u_{j,k}$ can be neglected. Just need to keep those where $u(x)$ has abrupt changes / discontinuities.

- Wavelet approximates also piecewise smooth

Time-frequency analysis

Localization in time & frequency
a special property of wavelet bases.

Consider three ways to represent a signal. $\boxed{\text{Examples}}$

1. Sample values

   Basis: $\delta(x - x_j)$ Fourier transform: $e^{-i\omega x_j}$

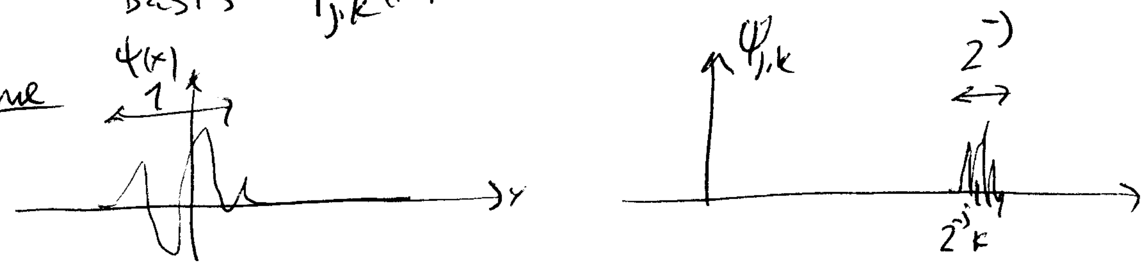
2. ~~Frequen~~ Fourier decomposition

   Basis: $e^{i\omega_j x}$ Fourier transform $\delta(\omega - \omega_j)$

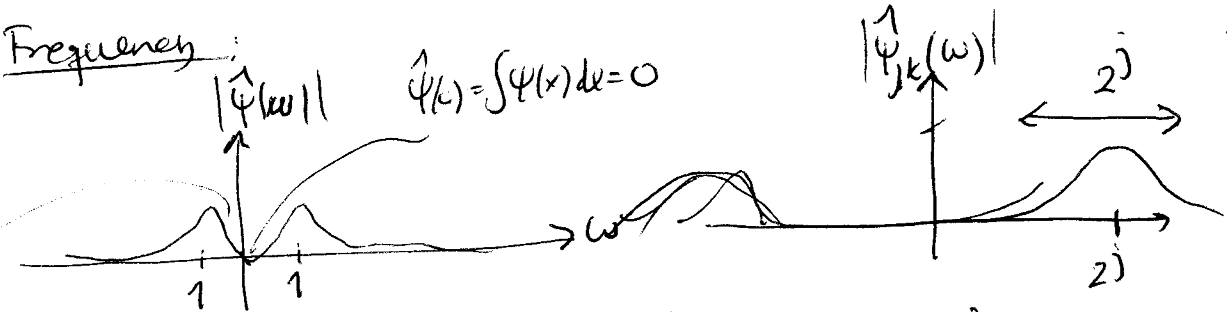
3. Wavelets:

   Basis $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$

Time



Frequency:



$$\hat\psi_{j,k}(\omega) = \int 2^{j/2}\psi(2^j x - k)\, e^{-i\omega x}\, dx = \{y = 2^j x - k\}$$

$$= 2^{-j/2}\int \psi(y)\, e^{-i\omega 2^{-j}(y+k)}\, dy =$$

$$= 2^{-j/2} e^{i\omega 2^{-j}k}\, \hat\psi(\omega 2^{-j})$$

Note: $\hat\psi^{(p)}(\omega) = \mathcal{F}\left((ix)^p\psi(x)\right) \Rightarrow \int x^p \psi(x)\,dx = \frac{1}{i^p}\hat\psi^{(p)}(0) = 0$

Finding filtercoefficients $\{h_n\}$ (sketch) 10

1) Define $\hat{h}(\xi) = \frac{1}{\sqrt{2}} \sum h_n e^{in\xi}$ ~ Fourier transform of $\{h_n\}$ / Generating function

2) $\sum_k h_k = \sqrt{2} \iff \hat{h}(0) = 1$

$\sum_k h_k h_{k+2\ell} = \delta_\ell \iff |\hat{h}(\xi)|^2 + |\hat{h}(\xi+\pi)|^2 = 1$

Proof: $|\hat{h}(\xi)|^2 + |\hat{h}(\xi+\pi)|^2 =$

$$= \sum_{k,\ell} h_k e^{ik\xi} \cdot h_\ell e^{-i\ell\xi} + h_k e^{ik(\xi+\pi)} \cdot h_\ell e^{-i\ell(\xi+\pi)}$$

$$= \sum_{k,\ell} h_k h_\ell e^{i(k-\ell)\xi} \left(1 + e^{i(k-\ell)\pi}\right) = \{\ell \to \ell + k\}$$

$$= \sum_{k,\ell} h_k h_{k+\ell} e^{-i\ell\xi} \Big(1 + \underbrace{e^{-i\ell\pi}}_{=(-1)^\ell}\Big) =$$

$$= \sum_{k,\ell} h_k h_{k+2\ell} e^{-i\ell\xi} = 1 \text{ iff } \sum h_k h_{k+2\ell} = \delta_\ell$$

3) Compactly supported $\iff \{h_n\}$ finite length
$\iff \hat{h}(\xi) = P(e^{i\xi})$, P some polynomial with real coeff.

4) M vanishing moments $\Rightarrow$

$$\hat{h}(\xi) = \left(\frac{1+e^{i\xi}}{2}\right)^M \cdot Q(e^{i\xi})$$

Q some polynomial, real coeff.

(Follows from:
$\hat{\psi}(2\xi) = e^{-i(\xi+\pi)} \hat{h}(\xi+\pi) \cdot \hat{\phi}(\xi)$ and

the fact that $\partial_\xi^p \hat{\psi}(0) = 0$

$0 \le p < M$

5) We get:

$$Q(1) = 1$$

$$\left|\frac{1+e^{i\xi}}{2}\right|^{2M} |Q(e^{i\xi})|^2 + \left|\frac{1-e^{i\xi}}{2}\right|^{2M} |Q(-e^{i\xi})|^2 = 1$$

Make ansatz for $Q = c_0 + c_1 z + c_2 z^2 + \dots$

Solve for $c_0, c_1, c_2 \dots$ etc.

Ex: $M = 1 \Rightarrow Q = 1$ since $|1+z|^2 + |1-z|^2 = 1 + |z|^2$

$\Rightarrow \hat{h}(\xi) = \frac{1+e^{i\xi}}{2} \Rightarrow h_n = \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ i.e. Haar

Ex: $M = 2 \Rightarrow Q = a + b\,z$ with

$a = \frac{1+\sqrt{3}}{2}$ $b = \frac{1-\sqrt{3}}{2}$

$$\hat{h}(\xi) = \left(\frac{1+e^{i\xi}}{2}\right)^2 (a + b e^{i\xi})$$

$$= \underbrace{\frac{a}{4}}_{h_0/\sqrt{2}} + \underbrace{\frac{2a+b}{4}}_{h_1/\sqrt{2}} e^{i\xi} + \underbrace{\frac{2b+a}{4}}_{h_2/\sqrt{2}} e^{i2\xi} + \underbrace{\frac{b}{4}}_{h_3/\sqrt{2}} e^{i3\xi}$$

$\Rightarrow$ Daubechies 4.

Wavelet transforms

Suppose we have a function $u(x) \in V_{j+1}$ given.

$$u(x) = \sum_k u_k \phi_{j+1,k}(x)$$

In Numerical algorithms we only need to keep track of the $\{u_k\}$ coefficients. Suppose they are given.

Note: If $\{\phi_{j+1}\}$ is Haar $\rightarrow$ $u_k \sim$ sample values of $u(x)$ at $x = 2^{-j}k$ (actually $2^{-j/2} u_k(x)$)

In general $u_k \sim$ local average of $u(x)$ at $x = 2^{-j}k$.

$\not\approx$ sample value.

We want to decompose $u(x)$ into a wavelet representation:

$$u(x) = \sum_{j'=0}^{j} d_{j'}(x) + u_0(x) = \sum_{j',k} u^f_{j',k} \psi_{j',k}(x) + \sum_k u_0 \phi_{0k}(x) \quad (*)$$

We start with one step:

$$u(x) = \underset{\in V_j}{u_j(x)} + \underset{\in W_j}{d_j(x)} =: \sum u^c_{j,k} \phi_{j,k}(x) + \sum u^f_{j,k} \psi_{j,k}(x)$$

Then:

$$\left.\begin{aligned} u^c_{j,k} &= \langle u, \phi_{j,k} \rangle = \sum_\ell u_\ell h_{\ell-2k} \\ u^f_{j,k} &= \langle u, \psi_{j,k} \rangle = \sum_\ell u_\ell g_{\ell-2k} \end{aligned}\right\} \text{Exercise}$$

$\therefore$ • The transform $\{u_k\} \longrightarrow \{u^c_{j,k}\} \cup \{u^f_{j,k}\}$ can be done only using filter coefficients $\{h_k\}$ & $\{g_k\}$

- For compactly supported wavelets, computing $u_{j,k}^c$ & $u_{j,k}^f$ can be done in $O(1)$ time.

Suppose now $u(x)$ is of finite length / compact support say $\subset [0,1]$.

$\Rightarrow$ # coefficients $\{u_k\}$ on level $j+1$ is 2^{j+1}. $= N$

Let $\bar{u} = \{u_k\}$, vector $\in \mathbb{R}^{2^{j+1}}$.

$U_j^f = \{u_{j,k}^f\}$ $\quad U_j^c = \{u_{j,k}^c\}$ $\quad \in \mathbb{R}^{2^j}$

Transform $\bar{u} \to u_j^f \cup u_j^c$ cost $O(2^{j+1})$

To do the full transform (*) we continue this hierarchically:

$$
\begin{tikzcd}
\bar{u} \arrow[r] \arrow[dr] & U_j^f & & & \\
& U_j^c \arrow[r] \arrow[dr] & u_{j-1}^f & & \\
& & u_{j-1}^c \arrow[r] \arrow[dr, no head] & u_{j-2}^f & \\
& & & u_{j-2}^c & \cdots \quad u_0^f \\
& & & & u_0^c
\end{tikzcd}
$$

Cost $\quad O(2^{j+1}) + O(2^j) + O(2^{j-1}) + \cdots$

Total cost $\displaystyle\sum_{k=0}^{j+1} 2^k = 2^{j+2} - 1 = O(2^{j+1}) = O(N)$

where N is the total number of unknowns we started from.

$\therefore$ Wavelet transform cost $O(N)$ $\qquad$ Inverse transform done

cf. FFT $= O(N \log N)$

In matrix form we can write

$$U = \begin{pmatrix} U_j^f \\ U_j^c \end{pmatrix} = \underbrace{\begin{pmatrix} g_0 \; g_1 \; g_2 \; \cdots & & \\ & g_0 \; g_1 \; \cdots & \\ \hline h_0 \; h_1 \; h_2 \; \cdots & & \\ & h_0 \; h_1 \; h_2 \; \cdots & \end{pmatrix}}_{W_j} \begin{pmatrix} \bar{u} \end{pmatrix}$$

Then:

$$W_j^T W_j = I \quad \text{(orthonormal)}$$

$$W_j \quad \text{sparse} \quad \text{(when } h, g \text{ FIR)}.$$

Numerical homogenization

Example

$$\partial_x r(x/\varepsilon)\partial_x u^\varepsilon = f \qquad u(0)=u(1)=0$$

Numerically we can solve this by discretizing:

$$x_j = j\Delta x \qquad u_j \approx u(x_j)$$

$$f(x_j) = \partial_x r^\varepsilon \partial_x u(x_j) \approx \left(r\!\left(\tfrac{x_{j+1}}{\varepsilon}\right) \frac{u_{j+1}-u_j}{\Delta x} - r\!\left(\tfrac{x_j}{\varepsilon}\right) \frac{u_j - u_{j-1}}{\Delta x} \right) / \Delta x$$

$$= \frac{u_{j+1} - R_j^\varepsilon u_j + u_{j-1}}{\Delta x^2} \qquad R_j^\varepsilon := r\!\left(\tfrac{x_{j+1}}{\varepsilon}\right) + r\!\left(\tfrac{x_j}{\varepsilon}\right)$$

$$\Rightarrow \underbrace{\begin{pmatrix} -R_1^\varepsilon & 1 & & \\ 1 & -R_2^\varepsilon & 1 & \\ & & \ddots & \\ & & 1 & -R_N^\varepsilon \end{pmatrix}}_{L^\varepsilon \in \mathbb{R}^{N\times N}} \underbrace{\begin{pmatrix} u_1^\varepsilon \\ \vdots \\ \vdots \\ u_N^\varepsilon \end{pmatrix}}_{=\vec{U}^\varepsilon} = \Delta x^2 \underbrace{\begin{pmatrix} f_1 \\ \vdots \\ \vdots \\ f_N \end{pmatrix}}_{\vec{f}} \qquad f_j := f(x_j)$$

$$N = 1/\Delta x$$

Need $\Delta x \sim \varepsilon$ to get accurate result. $\Rightarrow N = O(1/\varepsilon)$

Solving system cost $O(N) = O(1/\varepsilon)$ (normally $O(N^3)$ but here sparse can be solved faster)

We know from homogenization theory that

$u^\varepsilon - \bar{u} = O(\varepsilon)$ where $\bar{u}$ solves homogenized eq:

$$(*) \quad \partial_x \bar{r} \partial_x \bar{u} = f \qquad \bar{u}(0)=\bar{u}(1)=0 .$$

We can decompose $L^\varepsilon = \Delta_+ R \Delta_-$ where Cf

Discretizing in the same way:

$$\bar{L}\,\vec{u} = \Delta x^2 \vec{f}$$

where $\bar{L} \in \mathbb{R}^{N\times N}$ but $N = O(1)$ indep. of ε.

Cost $= O(1)$ $\Rightarrow$ Much cheaper and almost as accurate when $\varepsilon \ll 1$.

Numerical homogenization:

Compute $\bar{L}$ (a matrix) directly from original eq. or from L^{ε} (another, bigger matrix) (without having to know explictly about (*)).

Remark: Both L^{ε} & $\bar{L}$ are sparse, banded, matrices. This is natural since they approximate local differential operators. We would like the numerically homogenized operator to have these properties too.

Taking $N = 2^{j+1}$ we can think of the vectors $\bar{u}, \bar{f}$ as coefficient vectors for some functions in V_{j+1} and L^{ε} as ~~an~~ linear operator $V_{j+1} \to V_{j+1}$