1. Let u be a smooth solution to

$$\begin{aligned} u_{tt} - \triangle u &= F(t,x) && \text{in} \quad \mathbb{R}_+ \times \mathbb{R}^n \\ u &= f(x) && \text{for} \quad t = 0 \\ u_t &= g(x) && \text{for} \quad t = 0 \ , \end{aligned}$$

with F, f, and g compactly supported. Assuming finite speed of propagation, show that for $t \in [0,T]$,

$$\mathcal{E}(t) \equiv \left[\int_{R^n} \left(u_t^2 + |\nabla u|^2 \right) dx \right]^{\frac{1}{2}} \leq \mathcal{E}(0) + T \sup_t \left[\int_{\mathbb{R}^n} |F|^2 dx \right]^{\frac{1}{2}} .$$

If you're having trouble showing the above, assume that F is 0 and show that $\mathcal{E}$ is constant in time.

2. Describe what it would take for

$$v(t,x) \quad = \quad A(t,x) e^{ik\phi(t,x)}$$

to be an approximate solution to

$$\begin{aligned} u_{tt} - \triangle u &= 0 && \text{in} \quad \mathbb{R}_+ \times \mathbb{R}^n \\ u &= f(x) && \text{for} \quad t = 0 \\ u_t &= g(x) && \text{for} \quad t = 0 \ . \end{aligned}$$

In your explanation, consider that v satisfies

$$\begin{aligned} v_{tt} - \triangle v &= \tilde{F}(t,x) && \text{in} \quad \mathbb{R}_+ \times \mathbb{R}^n \\ v &= \tilde{f}(x) && \text{for} \quad t = 0 \\ v_t &= \tilde{g}(x) && \text{for} \quad t = 0 \end{aligned}$$

exactly.

3. Implement a finite difference solver for

$$\begin{aligned} u_{tt} &= u_{xx} \\ u|_{t=0} &= e^{-10x^2} e^{ik(x^3+x)} \\ u_t|_{t=0} &= \left[-ik(3x^2+1) \right] e^{-10x^2} e^{ik(x^3+x)} \ , \end{aligned}$$

for a given constant k and $t \in [0,1]$. To avoid the need for boundary conditions in your simulation, enlarge your computational domain so that the boundaries don't affect the solution for $(t,x) \in [0,1] \times [-1,1]$.

(a) What happens if the CFL condition is violated?

(b) How big can you make k before the computations become infeasible?

(c) Based on you previous answer, how big could you make k in 2 and 3 spacial dimensions?

4. Consider asymptotic solutions of the form

$$u = A(t,x,y)e^{ik\phi(t,x,y)}$$

to the variable speed wave equation

$$u_{tt} - c(x,y)\,[u_{xx} + u_{yy}] \quad = \quad 0\ .$$

(a) Derive the Eikonal and amplitude equations.

(b) Derive the system of ODEs that the characteristics satisfy.