ODE Tutorial RTG summer program 7/21/08

Start w/ example

$$y' = 3y \qquad (1)$$
$$y(0) = C \qquad (2)$$

solution?

$$y(t) = Ce^{3t}$$

family of integral curves (picture*)

In general

$$y' = f(t,y)$$
$$y(t_0) = y_0$$

$f : [a,b] \times E \to \mathbb{R}$, $t_0 \in [a,b]$ $y_0 \in E$ $\qquad E \subseteq \mathbb{R}^n$ open?

Looking for $y : [t_0, T] \to E$.

Equivalent to (by fund thm calc)

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds$$

ordinary $\Leftrightarrow$ derivative/dependence one variable
first order $\Leftrightarrow$ first derivative

3 important questions

(1) local existence: Does our equation (or system) have a solution $y(t)$ defined near t_0?

(2) existence in the large (e.g. global existence) On what t-ranges do we have a solution? (this may depend on y_0)

(3) uniqueness of solutions.

Examples

$$y' = y^2 \qquad y(0) = c \ (>0)$$

$$y(t) = \frac{c}{1-ct} \qquad \text{solution ? } \checkmark$$

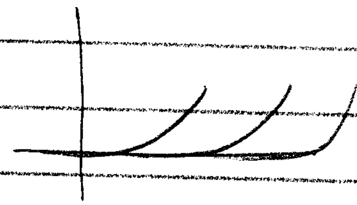
only exists for $-\infty < t < \frac{1}{c}$. Domain of sol. can depend on i.c.

(2) $y' = |y|^{1/2}$ $\quad y(0) = 0$

$y(t) \equiv 0$

AND For any $c > 0$

$$y(t) = \begin{cases} 0 & \text{if } t \le c \\ \frac{(t-c)^2}{4} & \text{if } t \ge c \end{cases}$$



check it. nonuniqueness. graph.

Picard's Thm

Let $f : [t_0, t_0 + a] \times E \to \mathbb{R}$ cont. $E = \{y : |y - y_0| < b\}$ and uniformly Lipschitz cont WRT y. Let M be a bound for $|f(t,y)|$ on $\mathbb{R}$, $\alpha = \min\left(a, \frac{b}{M}\right)$.
Then

$$y' = f(t,y) \qquad y(t_0) = y_0$$

has a unique sol $y = y(t)$ on $[t_0, t_0 + \alpha]$

Def We say f is lipschitz if $\exists K$ s.t.

$$|f(x) - f(y)| \le K|x - y| \quad \text{for all } x, y \in \operatorname{dom} f.$$

(Sketch of pf)

$$y_0(t) := y_0$$

$$y_1(t) := y_0 + \int_{t_0}^{t} f(s, y_0)\,ds$$

$$y_{n+1}(t) := y_0 + \int_{t_0}^{t} f(s, y_n(s))\,ds.$$

Assume $\lim_{n\to\infty} y_n(t)$ exists. $y(t) := \lim_{n\to\infty} y_n(t)$

$$\underbrace{\lim_{n\to\infty} y_{n+1}(t)}_{=\, y(t)} = y_0 + \underbrace{\lim_{n\to\infty} \int_{t_0}^{t} f(s, y_n(s))\,ds}_{=\, \int_{t_0}^{t} f(s, y(s))\,ds}.$$

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds. \qquad f \text{ cont. bounded} \ldots$$

How do we know $\lim_{n\to\infty} y_n(t)$ exists? ANALYSIS!
requires Lipschitz assumption.

Note what do we mean by
$\lim_{n\to\infty} y_n(t) = y(t)$?

fix t^*. $y_n(t^*) = y(t^*)$ "pt wise convergence"

In some function space norm

$$\|y\|_{L^\infty} = \sup_{t\in[a,b]} |y(t)|$$

$$\|y\|_{L^1} = \int_a^b |y(t)|\,dt$$

$$\|y\|_{L^2} = \left(\int_a^b |y(t)|^2 dt\right)^{1/2}$$

$$\lim_{n\to\infty} y_n(t) = y(t) \quad \text{in } \|\cdot\|_* \quad \text{"strong convergence"}$$

$$\lim_{n\to\infty} \|y_n - y\|_* = 0 \quad \text{in real \#s sense.}$$

<u>examples</u>

$$y_n(t) = t^n \quad \text{on } [0,1]$$

ptwise limit $f(t) = \begin{cases} 0 & \text{if } t<1 \\ 1 & \text{if } t=1 \end{cases}$

strong limit $\|\cdot\|_{L^1}$ is $g(t) \equiv 0$

$$\|y_n - g\|_{L^1} = \int_0^1 |y_n(t) - g(t)|\,dt$$

$$= \int_0^1 t^n - 0\,dt$$

$$= \left(\frac{1}{n+1}t^{n+1} - 0\right]_0^1 = \frac{1}{n+1} - 0 \to 0$$

strong limit $\|\cdot\|_{L^\infty}$ DNE

e.g. $\|y_n - f\|_{L^\infty} = \sup_{t\in[0,1]} |y_n(t) - f(t)| =$

$$= \sup_{t\in[0,1)} |t^n| = 1$$

OR

$$\|y_n - g\|_{L^\infty} = \sup_{t\in[0,1]} |t^n - 1| = 1.$$

different norms work differently. $\longrightarrow$ Little o and Big O.

In Picard we are talking strong limit in L-∞ norm.

Uniqueness? Suppose $z(t)$ exists s.t.

$$z(t) = y_0 + \int_{t_0}^{t} f(s, z(s))\, ds.$$

we show $y_n(t) \to z(t)$ uniformly

don't worry about it.

$\square$

Systems of Equations

$$(\star)\ \begin{cases} y_1' = F_1(t, y_1, \dots y_n) & y_1(0) = y_1^0 \\ y_n' = F_n(t, y_1, \dots, y_n) & y_n(0) = y_n^0 \end{cases}$$

If $F_1, \dots F_n$ and partials $\frac{\partial F_i}{\partial y_j}$ cont. in a a region R of $\mathbb{R} \times \mathbb{R}^n$ and $(t_0, y_1^0, \dots y_n^0) \in R$ Then there is an interval $(t_0 - h, t_0 + h)$ in which there exists a unique sol. to the system $(\star)$.

Note on superposition

$$y' = f(t, y) \quad \text{where } f \text{ linear in } y.$$

$$z' = f(t, z)$$

$$(\alpha y + \beta z)' = \alpha y' + \beta z' = \alpha f(t, y) + \beta f(t, z)$$
$$= f(t, \alpha y + \beta z)$$

linear combo's of sol's to a linear eqn's are also sol's to the eqn.

example

$$y' - p(t) y = 0$$

sols of form $y(t) = C \exp\left\{ \int_0^t p(s)\, ds \right\}$

Gronwall's (if there is time)

Suppose we know that for some cont. function $\beta(t)$

$$u'(t) \le \beta(t)\, u(t) \qquad t > 0$$

then

$$u(t) \le u(0) \exp\left\{ \int_0^t \beta(s)\, ds \right\} \quad \text{for any } t > 0.$$

pf

define

$$v(t) := \exp\left\{ \int_0^t \beta(s)\, ds \right\}$$

then

$$v'(t) = \beta(t)\, v(t)$$

$$\frac{d}{dt}\left(\frac{u(t)}{v(t)}\right) = \frac{u'v - v'u}{v^2} \le \frac{\beta uv - \beta vu}{v^2} = 0$$

$$\frac{u(t)}{v(t)} \le \frac{u(0)}{v(0)} = u(0)$$

$$u(t) \le u(0)\, v(t) = u(0) \exp\left\{ \int_0^t \beta(s)\, ds \right\}.$$

$\square$

Add a bit on Euler scheme + you're done!

(+ pictures)

1^{st} and 2^{nd} order linear ODEs constant coefficients

$$ay'' + by' + cy = f(x)$$

characteristic eqn. (solve homogeneous eqn 1^{st})

$$ar^2 + br + c = 0$$

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

solutions $y_1(t) = e^{r_1 t}$, $y_2(t) = e^{r_2 t}$

$$ay_1''(t) + by_1'(t) + cy = 0$$

$$= ar_1^2 e^{r_1 t} + br_1 e^{r_1 t} + ce^{r_1} =$$

$$\underbrace{(ar_1^2 + br_1 + c)}_{\checkmark} e^{r_1 t} = 0$$

complex roots.

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double roots

complex roots $\rightsquigarrow$ oscillations!

$$y_1 = e^{(\lambda - i\mu)t} \qquad y_2 = e^{(\lambda + i\mu)t}$$

$$e^{(\lambda - i\mu)t} + e^{(\lambda + i\mu)t} = e^{\lambda t}(\cos\mu t + i\sin\mu t) + e^{\lambda t}(\cos\mu t - i\sin\mu t)$$

$$= 2e^{\lambda t}\cos\mu t$$

$$e^{(\lambda + i\mu)t} - e^{(\lambda - i\mu)t} = 2ie^{\lambda t}\sin\mu t \quad \text{similarly.}$$

Summary of 2^{nd} order linear ODE

$$y'' + p(t)y' + q(t)y = g(t) \quad \text{inhomogeneous}$$
$$y'' + p(t)y' + q(t)y = 0 \quad \text{homogeneous}$$

- homogeneous, p, q constants ⇒ we can solve it.
- solutions to 2^{nd} order ODE 2 linearly independent sols to homogeneous eqn + particular sol to inhomogeneous problem discribes the sol space.

AKA fundamental solutions

homogeneous problem,

→ If we have one fundamental sol, we can find another (linearly independent) one. Called Reduction of Order

→ Obtaining particular solutions to inhomogeneous problem.

- Method of Undetermined Coefficients

  clever guessing for constant coefficient case.

- Variation of Parameters

  - must know fund sols., to find a particular solution.

1^{st} order linear ODE

$$y' + p(t)\,y = g(t)$$

choose integrating factor $\mu(t)$ s.t.

$$\mu(t)\,p(t)\,y = \mu'(t)\,y \qquad \text{WHY?}$$

$$\cancel{\mu(t)y' + \mu(t)p(t)y = \mu t}$$

$$\mu(t)\left[y' + p(t)y\right] = \mu(t)g(t)$$

$$\mu(t)y' + \mu'(t)\,y = \mu(t)\,g(t)$$

$$\left(\mu(t)\,y\right)' = \mu(t)\,g(t)$$

$$\mu(t)\,y = \int_0^t \mu(s)\,g(s)\,ds + C$$

$$y = \frac{1}{\mu(t)} \int_0^t \mu(s)\,g(s)\,ds + C$$

What's $\mu(t)$

$$\mu(t)\,p(t) = \mu'(t)$$

$$\frac{\mu'(t)}{\mu(t)} = p(t)$$

$$\ln(\mu(t)) = \int^t p(s)\,ds$$

$$\mu(t) = \exp\left\{\int^t p(s)\,ds\right\}.$$