Exercises

(1) Let $u = (x, y, z)$ and

$$f_\epsilon(x,y,z) = \begin{pmatrix} a & \frac{1}{\epsilon} & 0 \\ -\frac{1}{\epsilon} & b & 0 \\ 0 & 0 & -\frac{1}{10} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ x^2 + cy^2 \end{pmatrix}. \tag{5.1}$$

The equation for u is

$$u' = f_\epsilon(u), u(0) = (1, 0, 1).$$

Take $\epsilon = 10^{-4}$,$a = b = 0$ and $c = 1$. Find approximations for $z(t)$ in $0 < t \leq 1$ using the following schemes and compare with the analytical solution. Plot the trajectories of your approximations of $x(t)$ and $y(t)$ on the xy plane, and the graph $z(t)$ as a function of time. Explain what you observe in each case.

(a) Forward Euler using $\triangle t = \epsilon/50$.

(b) Backward Euler for x and y and Forward Euler for z, using $\triangle t = 0.1$.

(c) Verlet method or Midpoint rule for x and y, and Forward Euler for z, using $\triangle t = \epsilon/50$.

(d) Solve this problem by the HMM-FE-fe method, with $\mathcal{Q} = \mathcal{R} = I$. $h = \epsilon/50$, $H = 0.1$, and $hM = 2 \cdot 10^{-3}$.

(e) Derive linear stability criteria on H for HMM-FE-fe, assuming that $h = c_0\epsilon$.

(f) Let $a = b = 1$ in the system defined above. Solve it by the same HMM-FE-fe scheme with the same parameters as in (d). Does this scheme correctly approximate the behavior of z in the time interval $0 < t \leq 1$? Explain.

Algorithm. *HMM-FE-fe scheme for* $u' = f_\epsilon(u)$

Macroscale with Forward Euler: $U^{n+1} = U^n + HF^n$, $U^0 = \mathcal{Q}(u_0)$

Microscale with Forward Euler:

$$u^n_{k+1} = u^n_k + hf_\epsilon(u^n_k), k = 0, \pm 1, \cdots, \pm M$$

$$u^n_0 = \mathcal{R}(U^n)$$

Averaging:

$$F^n := \frac{1}{2M} \sum_{k=-M}^{M} K^{\cos}(\frac{k}{2M}) f_\epsilon(u^n_k),$$

$$K^{\cos}(t) = \frac{1}{2}\chi_{[-1,1]}(t)\left(1 + \cos(\pi t)\right),$$

$$\chi_{[-1,1]}(x) = \begin{cases} 1, & -1 \leq x \leq 1 \\ 0, & \textit{otherwise} \end{cases}$$

(2) Following the previous problem. Define the slow variable

$$\xi(x,y) = x^2 + y^2 \text{ and } \xi(t) := x^2(t) + y^2(t),$$

where $x(t)$ and $y(t)$ are defined in (5.1).

(a) Show that $d\xi/dt$ can be approximated by averaging:

$$\left| \frac{d\xi}{dt}(t_n) - \int_{-\infty}^{\infty} -\frac{d}{dt} K^{\cos}(\frac{t_n - t}{2Mh}) \left(x^2(t) + y^2(t)\right) dt \right| \leq C\eta^p.$$

Find p.

(b) Modify your previous HMM-FE-fe code as follows and determine if the dynamics of z is accurately approximated by this new scheme. Plot your approximations as in the previous problem. Explain your findings.

(c) Do the same thing as in the previous problem, but with $c = 0$. Does your multiscale algorithm work? Why?

Algorithm. *Constrained HMM-FE-rk4 scheme for* $u' = f_\epsilon(u)$

Macroscale with Forward Euler: $U^{n+1} = U^n + HF^n$, $U^0 = \mathcal{Q}(u_0)$

Microscale with Runge-Kutta-4:

$$u^n_{k+1} = rk4(u^n_k, h), k = 0, \pm 1, \cdots, \pm M$$

$$u^n_0 = \mathcal{R}(U^n).$$

rk4 is a explicit Runge-Kutta 4 routine using step size h.

$$rk4(y, h) = y + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

$$k_1 = hf_\epsilon(y), k_2 = hf_\epsilon(y + \frac{1}{2}k_1), k_3 = hf_\epsilon(y + \frac{1}{2}k_2), k_4 = hf_\epsilon(y + k_3).$$

Averaging:

$$dz^n := \frac{1}{2M} \sum_{k=-M}^{M} K^{\cos}(\frac{k}{2M})(x^n_k \cdot x^n_k + cy^n_k \cdot y^n_k - \frac{z^n_k}{10}).$$

$$d\xi^n := \frac{1}{2M} \sum_{k=-M}^{M} G(\frac{k}{2M})\left(x^n_k \cdot x^n_k + y^n_k \cdot y^n_k\right),$$

where $G(\frac{k}{2M}) := \frac{-1}{2Mh}\frac{d}{dt}K^{\cos}(\frac{t}{2Mh})$.

Evaluate effective force: *Find a unit vector* dX^n *such that*

$$d\xi^n = \nabla_{x,y}\xi|_{x^n_k, y^n_k} \cdot dX^n.$$

$$F^n := \begin{pmatrix} dX^n \\ dz^n \end{pmatrix}.$$