▷ Analytic Part of FMM

Improve the accuracy.
ε, fixed constant > 0.

f^B: ~~outgoing~~ far field rep.
u^A: local field rep.

(2D) $$G(x,y) = \ln|x-y| = \mathrm{Re}\left[\ln(x-y)\right]$$

We will regard $G(x,y) = \ln(x-y)$ and at the end of the computation, discard the imaginary part

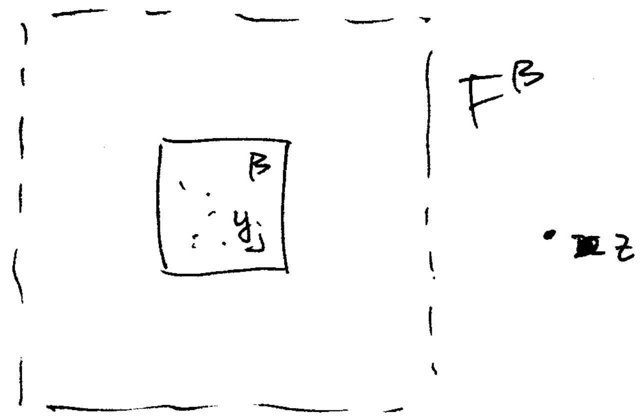
▷ Definition.

$$u^B(x) = \sum_{y_j \in B} G(x,y_j) \cdot f_j$$

f_B: is a representation that approximates $u^B(x)\big|_{x \in F^B}$

u_A: <u>local rep</u>: is a representation that approximates $u^{F^A}(x)\big|_{x\in A}$

D. Far field rep: Want to approx $u^B(x)\big|_{F^B}$.



$$G(x,y) = \ln(x-y) = \ln x + \ln\left(1-\frac{y}{x}\right) \qquad \left|\frac{y}{x}\right| \le \frac{\sqrt{2}}{3}$$

$$= \ln x + \sum_{k=1}^{\infty}\left(-\frac{1}{k}\right)\left(\frac{y}{x}\right)^k \qquad (*)$$

$$u^B(z) = \sum_{y_j\in B} G(z, y_j)\cdot f_j$$

$$= \sum_{y_j\in B}\left[\ln z + \sum_{k=1}^{\infty}\left(-\frac{1}{k}\right)\left(\frac{y_j}{z}\right)^k\right]\cdot f_j$$

$$= \left(\sum_{y_j\in B} f_j\right)\cdot \ln z + \sum_{k=1}^{\infty}\left(-\frac{1}{k}\right)\left[\sum_{y_j\in B}\left(\frac{y_j}{z}\right)^k\cdot f_j\right] \quad (*)$$

$$= \left(\sum_{y_j\in B} f_j \ln z\right) + \sum_{k=1}^{\infty}\left[\left(-\frac{1}{k}\right)\sum_{y_j\in B} y_j^k\cdot f_j\right]\cdot\frac{1}{z^k}$$

Since $\left|\frac{y_j}{z}\right| \le \frac{\sqrt{2}}{3}$, we can truncate (*) after $\underbrace{p = \log_{\frac{3}{\sqrt{2}}}\left(\frac{1}{\varepsilon}\right)}$ terms to get an ε-accuracy approximation

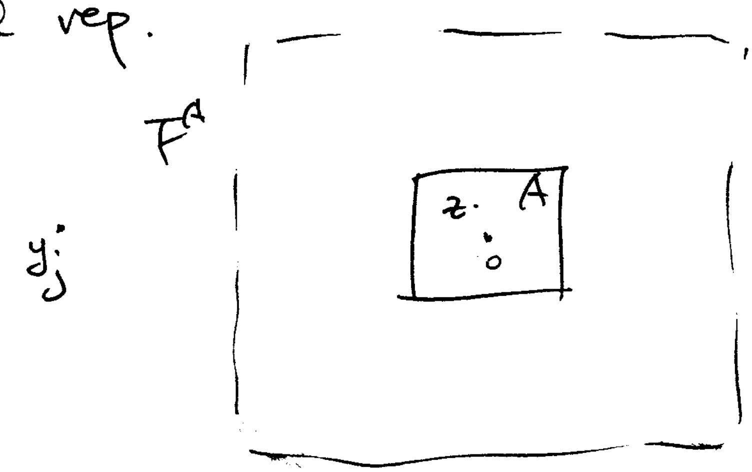
$$\Rightarrow u^B(z) \simeq \underbrace{\left(\sum_{y_j \in B} f_j\right)}_{a_0} \cdot \ln z + \sum_{k=1}^{p} \underbrace{\left(-\frac{1}{k} \sum_{y_j \in B} y_j^k f_j\right]}_{a_k} \frac{1}{z^k} + O(\varepsilon).$$

$(a_0, a_1, \dots a_p) \Rightarrow$ compact rep for $u^B(z)\big|_{F^B}$.

$\hookrightarrow$ far field rep. ("multipole representation")

▷ local rep.

$u^{F^A}(z)\big|_A$



$$\left|\frac{z}{y_j}\right| \le \frac{\sqrt{2}}{3}$$

$$\begin{aligned} G(z,y) &= \ln(z-y) \\ &= \ln(-y) + \ln\left(1-\frac{z}{y}\right) \\ &= \ln(-y) + \sum_{k=1}^{\infty}\left(-\frac{1}{k}\right)\left(\frac{z}{y}\right)^k \quad (*) . \end{aligned}$$

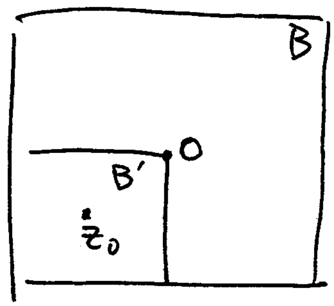
$$\begin{aligned} u^{F^A}(z) &= \sum_{y_j\in F^A} G(z, y_j) f_j \\ &= \sum_{y_j\in F^A}\left[\ln(-y_j) + \sum_{k=1}^{\infty}\left(-\frac{1}{k}\right)\left(\frac{z}{y_j}\right)^k\right]\cdot f_j \\ &= \sum_{y_j\in F^A}\left(\ln(-y_j)\cdot f_j\right) + \sum_{k=1}^{\infty}\left(-\frac{1}{k}\right)\left(\sum_{y_j\in F^A}\frac{1}{y_j^k}\cdot f_j\right)\cdot z^k \quad (*) \end{aligned}$$

As long as $p \ge \log_{\frac{3}{\sqrt{2}}}\left(\frac{1}{\varepsilon}\right)$

$$u^{F^A}(z) \approx \underbrace{\sum_{y_j\in F^A}\left(\ln(-y_j)\cdot f_j\right)}_{b_0} + \sum_{k=1}^{p}\left(-\frac{1}{k}\right)\underbrace{\left(\sum_{y_j\in F^A}\frac{1}{y_j^k} f_j\right)}_{b_k}\cdot z^k .$$

$(b_0, b_1, \cdots b_p)$. local field representation

D. Far 2 Far translation.



Given $\{a_0 \cdots a_p\}$ far field rep of B' centered at z_0, how to construct $\{b_0, b_1 \cdots b_p\}$: far field rep of the parent box B ?

$$u^{B'}(z) \simeq a_0 \ln(z - z_0) + \sum_{k=1}^{p} \frac{a_k}{(z-z_0)^k} + O(\varepsilon)$$

$$\simeq b_0 \ln z + \sum_{\ell=1}^{p} \frac{b_\ell}{z^\ell} + O(\varepsilon)$$

$$\Rightarrow \begin{cases} b_0 = a_0 \\ b_\ell = -a_0 \frac{z_0^\ell}{\ell} + \sum_{k=1}^{\ell} a_k \binom{\ell-1}{k-1} \cdot z_0^{\ell-k} \end{cases}$$

D. far-2-local translation

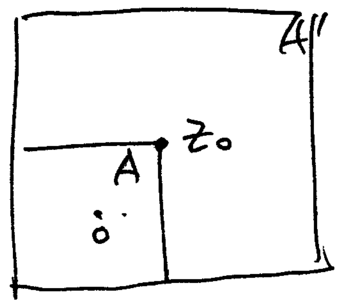
Given $\{a_0 \cdots a_p\}$ far field rep of B centered at z_0, how to construct $\{b_0 \cdots b_p\}$ local field rep at A ?

$$u^B(z) \approx a_0 \ln(z - z_0) + \sum_{k=1}^{p} \frac{a_k}{(z-z_0)^k}$$

$$\approx b_0 \cdot \ln z + \sum_{\ell=1}^{p} b_\ell \underbrace{z^\ell}_{\text{Taylor exp'n}}$$

$$\begin{cases} b_0 = a_0 \cdot \ln(-z_0) + \sum_{k=1}^{p} a_0 (-z_0)^{-k} \\ b_\ell = \left(-\frac{a_0}{\ell}\right) z_0^{-\ell} + \sum_{k=1}^{p} a_k (-z_0)^{-k} \binom{\ell+k-1}{k-1} \cdot z_0^{-\ell} \end{cases}$$

▷ local 2 local translation



Given local rep $\{a_0 \cdots a_p\}$ at A', how to const. local rep $\{b_0 \cdots b_p\}$ at A.

$$\forall z \in A, \qquad u^{F^A}(z) \simeq \sum_{k=0}^{p} a_k \cdot (z - z_0)^k$$

local rep at z_0.

$$= \sum_{\ell=0}^{p} b_k (z)^k$$

local rep at o.

$$\Rightarrow \quad \underline{b_\ell = \sum_{k=\ell}^{p} (-z_0)^{k-\ell} \binom{k}{\ell} \cdot a_k}.$$

▷ Far field rep $\qquad p+1 = O(\log(\frac{1}{\varepsilon}))$

local field rep. $\qquad p+1 = O(\log(\frac{1}{\varepsilon}))$

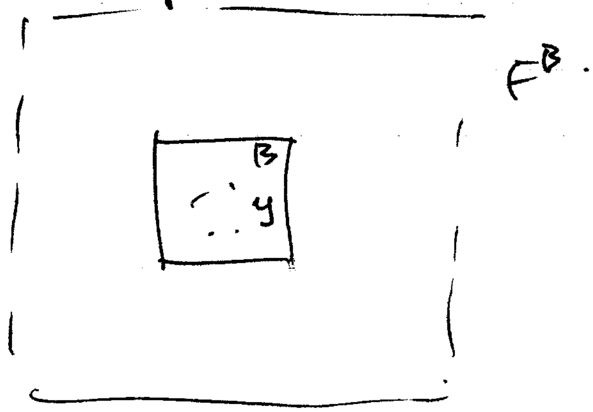
$$\left.\begin{matrix} \text{F2F} \\ \text{F2L} \\ \text{L2F} \end{matrix}\right\} \Rightarrow \text{matrix of size } (p+1) \times (p+1)$$

$$= O(p^2) = O(\log^2(\tfrac{1}{\varepsilon}))$$

$\Rightarrow$ FMM is still a linear algo with ~~complex~~ constant depending on $\underline{\log(\frac{1}{\varepsilon})}$.

▷ Other alternatives for far field representations.
local field

▷ far field rep.

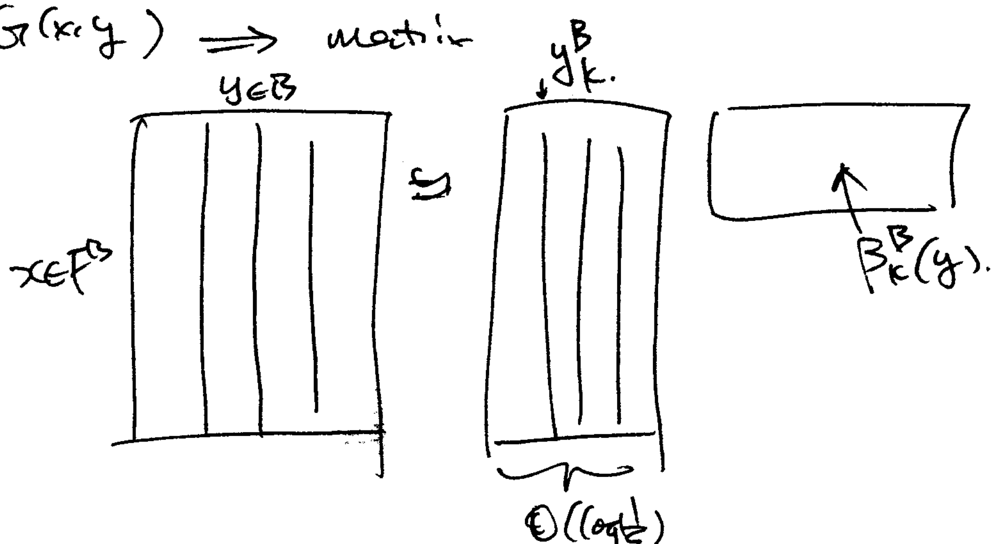


$y \in B$
$x \in F^B$.

Idea: $G(x,y)\Big|_{\substack{x\in F^B \\ y \in B}}$ is approx low rank.

rank $\approx O(\log\frac{1}{\varepsilon})$.

$G(x,y) \Longrightarrow$ matrix



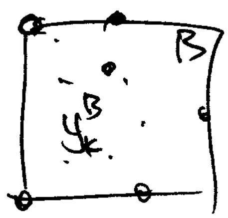
$$\underset{\substack{\uparrow \\ x\in F^B}}{G(x,}\underset{\substack{\uparrow \\ B}}{y)} \approx \sum_{k=1}^{\log\frac{1}{\varepsilon}} G(x, y_k^B) \cdot \beta_k^B(y)$$

$$u^B(z) = \sum_{y_j \in B} G(z, y_j)\cdot f_j$$

$$\approx \sum_{y_j \in B} \left[\sum_k G(z, y_k^B)\cdot \beta_k^B(y_j) \right]\cdot f_j$$

$$= \sum_k G(z, y_k^B) \cdot \left[\sum_{y_j \in B} \beta_k^B(y_j)\cdot f_j \right]$$

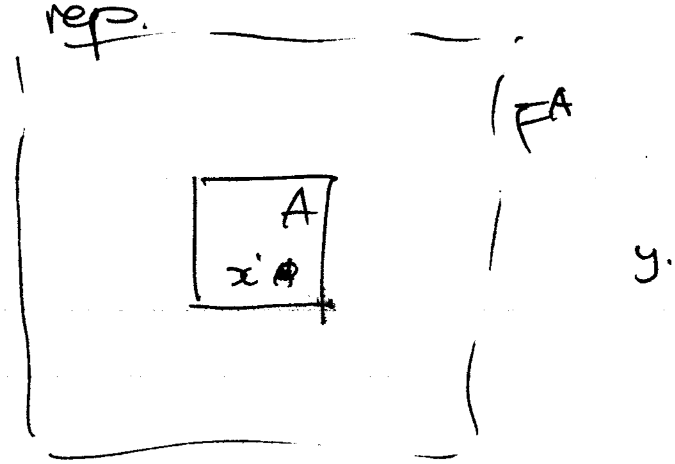
representative columns



We can reproduce $u^B(z)$ by putting mass $\left[\sum_{y_j \in B} \beta_k^B(y_j) f_j \right]$ at $\underline{\underline{\{y_k^B\}}}$.

$$\left| \{y_k\}^B \right| = \log\left(\frac{1}{\varepsilon}\right).$$

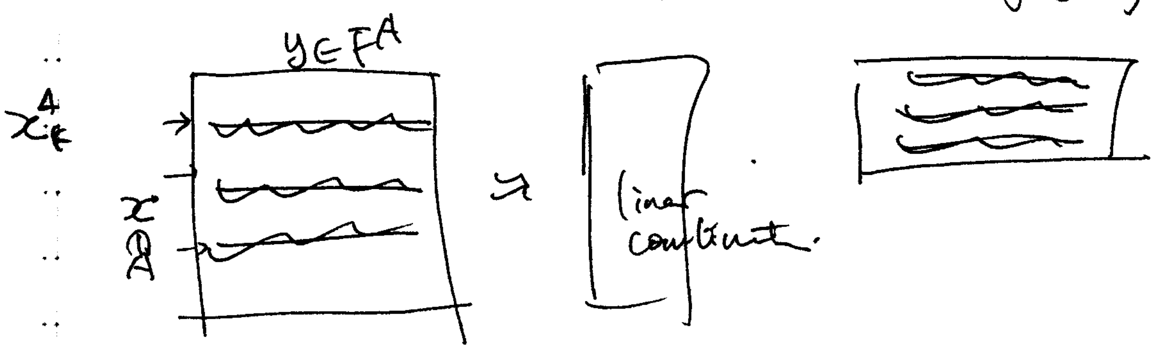
D. local rep.



$u^{F^A}(x)\big|_{x\in A}$.

$G(x,y)\big|_{\substack{x\in A\\ y\in F^A}}$ approx low rank

rank $\approx O(\log(\frac{1}{\varepsilon}))$



$$G(x,y) \approx \sum_k \underbrace{\alpha_k^A(x)}_{\substack{\text{linear}\\ \text{comb.}\\ \text{coeffs}}} \cdot \underbrace{G(x_k^A, y)}_{\text{rows}}$$

$$u^{F^A}(z) = \sum_{y_j\in F^A} G(z,y_j) f_j$$

$$\approx \sum_{y_j\in F^A} \sum_k \alpha_k^A(z)\cdot G(x_k^A, y_j)\cdot f_j$$

$$= \sum_{k} \alpha_k^A(x) \cdot \Big(\underbrace{\sum_{y_j \in F^A} G(x_k^A, y_j)\, f_j}_{\text{potential } x_k^A} \Big)$$

D. Butterfly algorithm

D. Let $x_0, x_1, \ldots, x_{N-1} \in [0, N]$
$\xi_0, \ldots, \xi_{N-1} \in [0, N]$

$f_0, \ldots, f_{N-1}$ sources.

want to compute $u_i = \sum_j G(x_i, \xi_j) \cdot f_j$

$$\begin{bmatrix} \\ u \\ \\ \end{bmatrix} \in \begin{bmatrix} & & \\ & G & \\ & & \end{bmatrix} \begin{bmatrix} \\ f \\ \\ \end{bmatrix}$$

Ex: $G(x, \xi) = e^{2\pi i \frac{x \cdot \xi}{N}}$

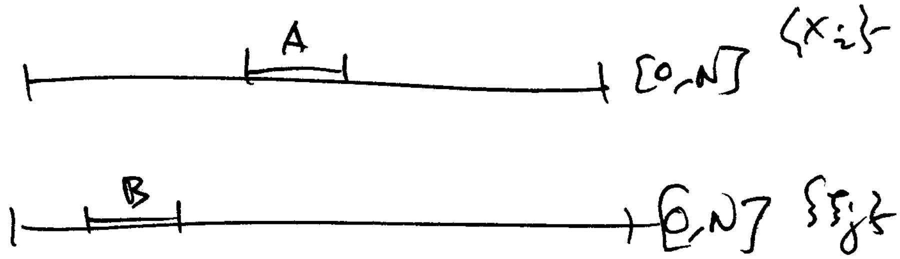
When $x_i = i, \in [0, N]$
$\xi_j = j, \in [0, N]$

$G(x_i, \xi_j) = e^{2\pi i \cdot \frac{x_i \xi_j}{N}}$ $\Rightarrow$ Fourier transform

When $x_i \to$ nonunif
$\xi_j = j,$ $\Rightarrow$ nonunif Fourier transform.

Ex: $G(x, \xi) = \begin{cases} \frac{1}{|x - \xi|}. \\ \ln|x - \xi|. \end{cases}$

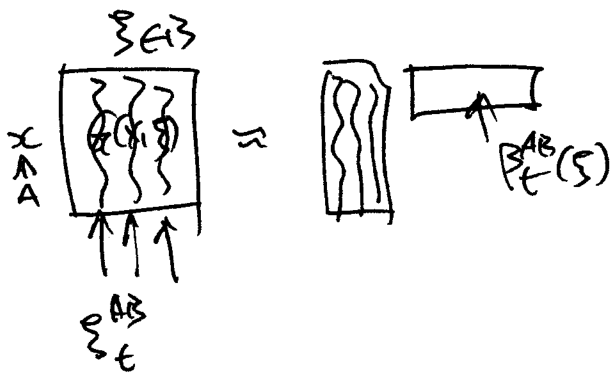
▷. Assumption.



Suppose A, B two intervals of [0,N]

with $w^A \cdot w^B \doteq N$, then

$$\left| G(x,\xi) - \sum_{t=1}^{r_\varepsilon} G(x,\xi_t^{AB})\cdot \beta_t^{AB}(\xi) \right| \le \varepsilon. \qquad \forall x \in A,\ \forall \xi \in B \qquad (*)$$



HW: $G(x,\xi) = e^{2\pi i \frac{x\cdot\xi}{N}}$ F.T.

satisfies this assumption

▷ Define. $u^B(x) = \sum_{\xi_j \in B} G(x,\xi_j) \cdot f_j$

potential generated by $\xi_j \in B$.

$u^B(x)\big|_{x\in A}$

We are interested in. $u^B(x)\big|_{x\in A}$.

Apply (*) to $\xi_j \in B$

$$\left| G(x,\xi_j) f_j - \sum_{t=1}^{r_\varepsilon} G(x, \xi_t^{AB}) \cdot \beta_t^{AB}(\xi_j) f_j \right| = O(\varepsilon) \quad \forall x\in A, \ \xi_j \in B$$

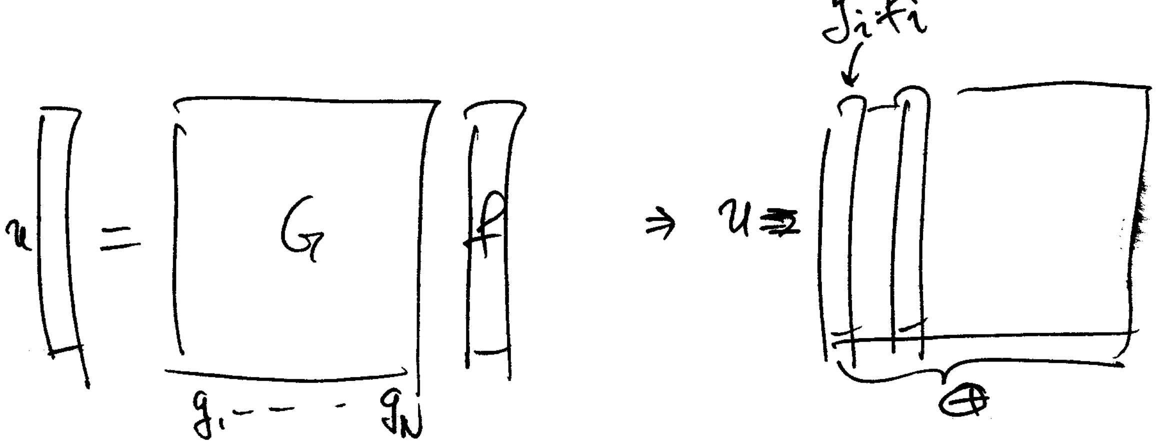
$\sum_{\xi_j \in B}$

$$\left| \underline{\underline{\sum_{\xi_j\in B} G(x,\xi_j) f_j}} - \underbrace{\sum_j \sum_{t=1}^{r_\varepsilon} G(x,\xi_t^{AB}) \cdot \beta_t^{AB}(\xi_j) f_j}_{} \right| = O(\varepsilon)$$

$$\left| u^B(x) - \sum_{t=1}^{r_\varepsilon} G(x, \xi_t^{AB}) \cdot \underbrace{\left(\sum_j \beta_t^{AB}(\xi_j) f_j \right)}_{\{f_t^{AB}\}} \right| = O(\varepsilon)$$

"Efficient rep".

Equivalent Sources.

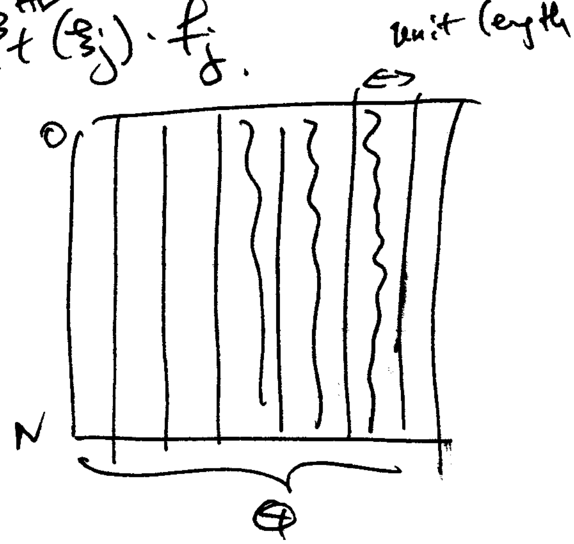


- **Step 0.** Construct $\{f_t^{AB}\}$ for

  $A = [0, N]$

  $B = [j, j+1]$.

$$f_t^{AB} = \sum_{j:\, \xi_j \in B} \beta_t^{AB}(\xi_j) \cdot f_j .$$

u = 

- **Step 1.** Construct $\{f_t^{AB}\}$ for

  $A = \frac{N}{2}[i, i+1]$ $\quad i = 0, 1$.

  $B = 2[j, j+1]$ $\quad j = 0, 1, \dots, N/2 - 1$.

P — A's parent $\quad w^P = N$

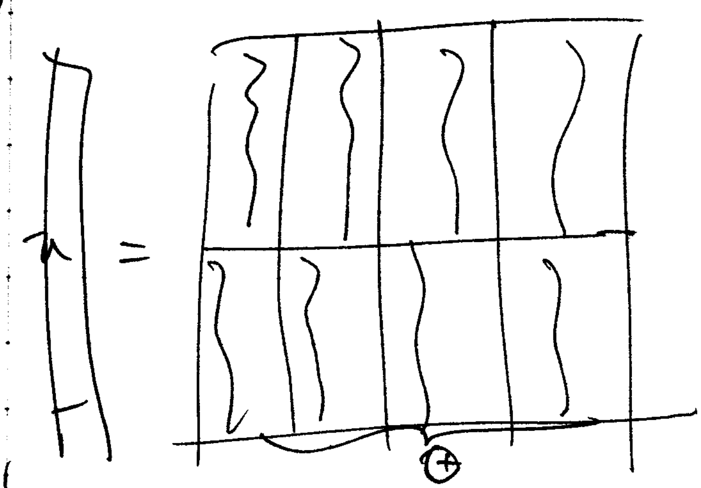
$B_1 B_2$ — B's children $\quad w^{B_1} = w^{B_2} = 1$

$$\left| u^{B_1}(x) - \sum_t G(x, \xi_t^{PB}) \cdot f_t^{PB_1} \right| = O(\varepsilon) \quad \forall x \in P$$

$$\left| u^{B_2}(x) - \sum_t G(x, \xi_t^{PB}) f_t^{PB_2} \right| = O(\varepsilon) \quad \forall x \in P.$$

+)

$$\left| u^B(x) - \left(\sum_t G(\cdot) f_t^{PB_1} + \sum G(\ ) f_t^{PB_2} \right) \right| = O(\varepsilon)_{x \in P}$$



$$\left| u^B(x) - \left[\sum_t G(x, \xi_t^{PB_1}) f_t^{PB_1} + \sum_t G(x, \xi_t^{PB_2}) f_t^{PB_2} \right] \right| = O(\varepsilon)_{x \in A}$$

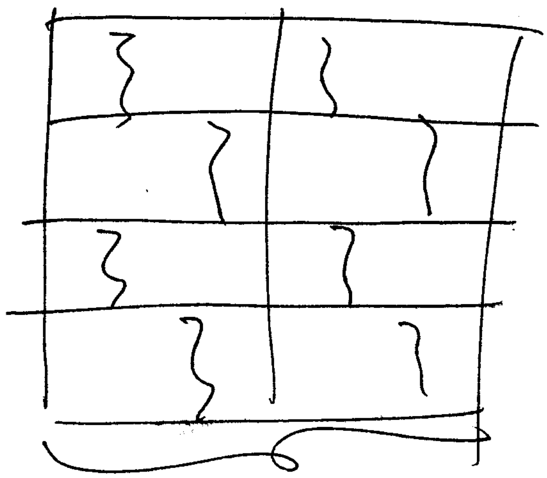
This says that $u^B(x)\big|_{x \in A}$ can be approximated by $\{f_t^{PB_1}\}$ $\{f_t^{PB_2}\}$.

Φ

"Treat those equ. sources as true sources"

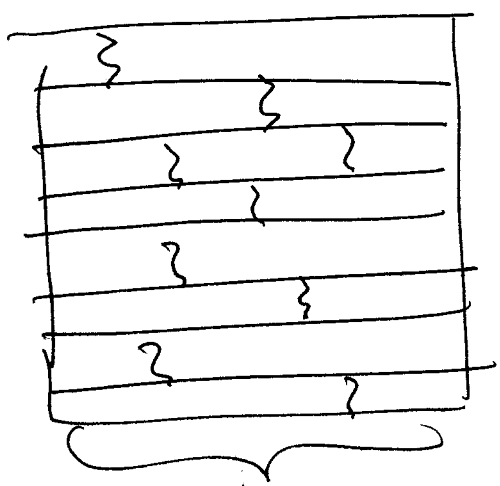
" $f_t^{AB} = \sum_s \beta_t^{AB}(\xi_s^{PB_1}) \cdot f_s^{PB_1} + \sum_s \beta_t^{AB}(\xi_s^{PB_2}) \cdot f_s^{PB_2}$ "

At step 2.

$u =$ 

Φ.

At the final step

$u =$ 

Φ

D Complexity analysis

Step 0: Construct $\{f_t^{AB}\}$ for

~~$|A| = N$~~ ~~$|B| = 1$~~

$w^A = N$ $\quad w^B = 1.$

$\#\{A\} = 1$

$\#\{B\} = N$

Cost of $\{f_t^{AB}\}$ for each pair is $O(1)$

Total complexity $= \#\{A\} \cdot \#\{B\} \cdot O(1) = 1 \cdot N \cdot 1$

$= O(N).$

Step 1: Const. $\{f_t^{AB}\}$ for

$w^A = N/2 \quad w^B = 2$

$\# \{A\} = 2$

$\# \{B\} = N/2$

Cost of $\{f_t^{AB}\}$ for each pair (A, B) is $O(1)$

Total complexity $= \#\{A\} \cdot \#\{B\} \cdot O(1) = 2 \cdot \frac{N}{2} \cdot 1$

$= O(N)$

⋮

Step $\log_2 N$

$O(N).$

Each step takes $O(N)$ operations
$\log_2 N$ steps

total complexity of the Butterfly algo
is $O(N \log N)$.