Homework — Instructor: Lexing Ying

Problem 1. In the discussion of the fast Fourier transform, we assumed that the size of the vector n is an integer power of 2. Derive a similar fast Fourier transform when $n = 3^k$ for $k \in \mathbb{Z}$.

Problem 2. Assume that a matrix M is of form

$$M = \begin{bmatrix} y_0 & y_{-1} & \cdots & y_{-(n-1)} \\ y_1 & y_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & y_{-1} \\ y_{n-1} & \cdots & y_1 & y_0 \end{bmatrix}.$$

Find an algorithm that computes $y = Mx$ in $O(n \log n)$ steps and implement it in Matlab.

Hint: Extend M to a $2n \times 2n$ cyclic matrix. In Matlab, the functions `fft(...)` and `ifft(...)` implement the forward and inverse Fourier transforms.

Problem 3. Assume that M is an $n \times n$ matrix of form $M = (\alpha^{jk})_{0 \le j,k \le n-1}$ where α is a fixed constant. Find an algorithm that computes $y = Mx$ in $O(n \log n)$ time and implement it in Matlab.

Hint: Multiply M with two diagonal matrices, $A = diag(\alpha^{-j^2/2})$ on the left and $B = diag(\alpha^{-k^2/2})$ on the right. What is the resulting matrix? Can we reuse the result of Problem 2 now?

Problem 4. In the discussion of the butterfly algorithm, we assumed that, for any two intervals A, B in $[0, N]$ with widths $w^A w^B = N$ and for any $\varepsilon > 0$, there exists a number r_ε that depends only on ε and functions $\{\alpha_t^{AB}(x)\}_{1 \le t \le r_\varepsilon}$ and $\{\beta_t^{AB}(\xi)\}_{1 \le t \le r_\varepsilon}$ such that

$$\left| G(x,\xi) - \sum_{t=1}^{r_\varepsilon} \alpha_t^{AB}(x) \beta_t^{AB}(\xi) \right| \le \varepsilon \quad \forall x \in A, \forall \xi \in B$$

Prove this for the case of $G(x, \xi) = e^{2\pi i x \xi / N}$.

Hint: Renormalize A and B to the interval $[0, 1]$ and use Taylor expansion to the kernel $G(x, \xi)$. You also need to use the Stirling formula at a certain point.