The Penetration Function and its Application to Microscale Problems

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ABSTRACT

The presentation will adress the recovery of the microscale features on a unit ball $\Xi \subset \Omega \subset R^2$ from the macroscale solution U by the global-local approach. Consider the microscale equation

$$\begin{aligned} \text{div } A(x) \text{ grad } u_n = 0 \quad &\text{on} \quad \Xi, \\ u_n = g_n \quad &\text{on} \quad \partial\Xi \end{aligned}$$

Here $A(x)$ is a symmetric measurable matrix which characterizes the microstructure, $u_n \subset H^1(\Xi)$ is the weak solution of the problem and $g_n \in S_n(\Xi) \subset H^{\frac{1}{2}}(\partial\Xi)$ is the trace of the macrosolution U on $\partial\Xi$. If $g_n = g$, where g is the trace of the microsolution u on Ω, then obviously $u_n = u$.

We will consider a class Υ of the microstructures A. Further we will assume that S_n is a $2n$ dimensional space of the trigonometric polynomials of degree n on $\partial\Xi$. This is the "deal" approximation space because it is directly related to the Kolmogorov n-width theory.

We define the *penetration function*,

$$\Phi(\Upsilon, R, n) = \sup \frac{\|u-u_n\|_{E(\Xi_R)}}{\|u\|_{E(\Xi)}},$$

where g_n is the $H^{\frac{1}{2}}(\partial\Xi)$ projection of the trace g on $\partial\Xi$ of the microscale solution u on S_n, Ξ_R is the ball of radius $0 < R < 1$, $\| u \|_E$ is the energy norm and the supremum is taken over all $A \in \Upsilon$ and $g \in H^{\frac{1}{2}}(\partial\Xi)$. Note that for general microstructure $g \in H^{\frac{1}{2}}(\partial\Xi)$ only and is not smoother.The penetration function characterizes the best possible accuracy in the microscale feature on the ball Ξ_R which could be obtained by the global-local approach if only is known that $A \in \Upsilon$.

We prove that if Υ is the class of measurable matrices $A(x)$ with bounds $0 < \lambda_1$, and $\lambda_2 < \infty$ of the minimal and maximal eigenvalues then

$$\Phi(\Upsilon, R, n) \geq C(R) n^{-\frac{1}{2}} lg^4 n.$$

This is only a lower estimate and it is possible that $\Phi \geq C n^{-\alpha}$ with α much smaller than $\frac{1}{2}$.

We also prove the upper estimate

$$\Phi(\Upsilon, R, n) \leq C(R) n^{-\alpha},$$

where α depends only on the contrast $\kappa = \frac{\lambda_1}{\lambda_2}$ with $\alpha = \frac{\kappa}{2+\kappa}$. Nevertheless this estimate is likely very pessimistic.

To see the accuracy of the estimates we will analyze numerically the penetration function for the following class Υ_0 of matrices $A(x)$.

$$\Upsilon_0 = \{A(x) = \begin{bmatrix} a(x) & 0 \\ 0 & a(x) \end{bmatrix}, a(x) \text{ has only values } \lambda_1 \text{ or } \lambda_2\}$$

and present the numerical results. They indicate that both estimates are inaccurate. We will also show that for $a(x)$ analytic on the closed Ξ, then the penetration function Φ decreases exponentially with n.

The showed results indicate that the reconstruction of the microstructure from the macro solution is very inaccurate in contrast to the usual folklore.