RTG Mini-Course
Perspectives in Geometry Series*

Jacob Lurie†

Lecture IV: Applications and Examples (1/29/2009)

Let Σ be a Riemann surface of genus g, then we can consider BDiff(Σ), the classifying space of the orientation preserving diffeomorphism group of Σ. This can be considered as the classifying space for connected surfaces of genus g. In particular, we can form a bundle E with fiber Σ as:

$$\begin{array}{c} (\mathrm{EDiff}(\Sigma) \times \Sigma) / \mathrm{Diff}(\Sigma) \\ \downarrow \\ \mathrm{BDiff}(\Sigma) \end{array}$$

On E, there is a canonical 2-dimensional vector bundle V with fiber $T_x\Sigma$. We can then consider its Euler class $e(V) \in H^2(E; \mathbb{Q})$, and its powers $e^{n+1} \in H^{2n+2}(E; \mathbb{Q})$. Since E is a surface bundle with compact fiber, we can integrate these classes to get classes $\kappa_n \in H^{2n}(\mathrm{BDiff}(\Sigma); \mathbb{Q})$, and these classes give a map

$$\mathbb{Q}[\kappa_1, \kappa_2, ...] \to H^*(\mathrm{BDiff}(\Sigma); \mathbb{Q}).$$

This is in fact a map of graded rings where κ_i is degree $2i$.

Mumford Conjecture: *This map is an isomorphism in a range of degrees that goes $\to \infty$ as $g \to \infty$*

How is the Mumford conjecture related to TQFT's? Recall we have been studying an $(\infty, 2)$-category Bord_2; we have an inclusion:

$$\mathrm{BDiff}(\Sigma) \subseteq \text{Classifying space for all closed surfaces} = \mathrm{Hom}_{\mathrm{Hom}_{\mathrm{Bord}_2}(\emptyset,\emptyset)}(\emptyset, \emptyset)$$

*These notes are by no means a substitute for the lectures, found at http://www.ma.utexas.edu/users/plowrey/dev/rtg/perspectives.html

†Scribes: Braxton Collier, Parker Lowrey, Michael B. Williams

In general, if $\mathcal{C}$ is an (∞, n)-category, you can extract from $\mathcal{C}$ a topological space $|\mathcal{C}|$, called the "classifying space" of $\mathcal{C}$. This is done by inverting all k-morphisms in $\mathcal{C}$, which yields an ∞-groupoid; we have seen that this can be identified with a topological space.

We have a natural map of $\mathrm{Hom}_{\mathrm{Hom}_{\mathrm{Bord}_2}(\emptyset,\emptyset)}(\emptyset, \emptyset)$ into $\Omega^2|\mathrm{Bord}_2|$ since the former are a space of 2-morphisms in a simplicial category. Let Y_g be the component of $\Omega^2|\mathrm{Bord}_2|$ containing the image of $\mathrm{BDiff}(\Sigma)$ under

$$\mathrm{BDiff}(\Sigma) \subseteq \mathrm{Hom}_{\mathrm{Hom}(\phi,\phi)_{\mathrm{Bord}_2}}(\phi, \phi) \to \Omega^2|\mathrm{Bord}_2|$$

To prove the Mumford conjecture, use 3 steps:

1. Prove that the map $H^*(Y_g, \mathbb{Q}) \to H^*(\mathrm{BDiff}(\Sigma))$ is an isomorphism in a range of degrees $\to \infty$ as $g \to \infty$.

2. "Understand $|\mathrm{Bord}_2|$": construct a homotopy equivalence of $|\mathrm{Bord}_2| \cong X$ for some understood topological space X.

3. Compute $H^*(\Omega^2 X, \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, ...] \oplus ...$

The first step is the hardest and involves Haar stability, we won't be talking about this. The second step is related to the cobordism hypothesis: there is a statement for all Bord_n that transforms the classifying spaces to something more recognizable. Once these two steps are done, the last step is just a calculation. So our goal is to understand step 2 using the cobordism hypothesis. Recall:

Cobordism Hypothesis: *If $\mathcal{C}$ is a symmetric monodial (∞, n) category that has duals (see notes from lecture 3 for definition)*, $\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n, \mathcal{C}) \cong \{\text{objects of } \mathcal{C}\}^{hSO(n)}$

where the latter object is the fixed points of the $SO(n)$ action on the classifying space of objects of $\mathcal{C}$.

Specialize to the case where $\mathcal{C}$ is an ∞-groupoid, $\mathcal{C} \cong \pi_{\leq\infty} X$, and consider the following relationships between the algebraic structure of $\mathcal{C}$ and the topological structure of X:

- Symmetric monoidal structure on $\mathcal{C} \Leftrightarrow X$ is an E_∞-space (i.e. has a commutative multiplication).

- $\mathcal{C}$ has duals $\Leftrightarrow \pi_0 X$ is an abelian group $\Rightarrow X$ is an ∞-loop space, i.e. there exist pointed spaces $X(n)$ such that $X \cong \Omega^n(X(n))$.

If $\mathcal{C}$ is an ∞-groupoid, then

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n, \mathcal{C}) \cong \mathrm{Fun}^{\otimes}(|\mathrm{Bord}_n|, \mathcal{C}) = \mathrm{Map}_{\infty-\text{loop spaces}}(|\mathrm{Bord}_n|, X).$$

We have a similar picture if we look at the framed bordism theory. By the cobordism hypothesis, $\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{fr}, \mathcal{C}) = \mathrm{Map}_{I.L.}(|\mathrm{Bord}_n^{fr}|, X) \cong X$, i.e. $|\mathrm{Bord}_n^{fr}|$ is free on one point, here the $I.L$ stands for maps of infinite loop spaces; it is thus homotopy equivalent to $QS^0 := \lim_n \Omega^n S^n$.

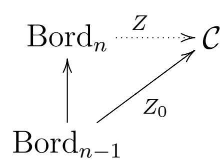
What about our original classifying space of interest? The cobordism hypothesis tells us $\text{Map}_{I.L.}(|\text{Bord}_n|, X)^{hSO(n)} = (X)^{h\text{SO}(n)}$. This tells us that $|\text{Bord}_n| \cong |\text{Bord}_n^{fr}|/SO(n)$, where the right hand side is a quotient in the world of infinite loop spaces. Therefore, this is isomorphic to the coinvariants of a homotopy action of SO(n) on QS^0. In turns out this action is given by the J homomorphism. This object has a name: $\Omega^{\infty-n}\text{MTSO}(n)$

Example: For $n = 2$, we have $\Omega^2|\text{Bord}_n| \cong \Omega^\infty\overline{\text{MTSO}(2)}$

The above has now shown
Theorem (G-M-T-W): $|\text{Bord}_n| \cong \Omega^{\infty-n}\text{MTSO}(n)$.

This theorem is equivalent to the special case of the cobordism hypothesis where $\mathcal{C}$ is an ∞-groupoid. We are now done discussing the Mumford conjecture. In the remaining minutes we will concentrate on Bord_2, but first some remarks.

Remark: The proof of the cobordism hypothesis uses induction on n. How do we understand the case of n based on $n-1$? We have the diagram:



From induction, Z_0 is equivalent to giving an object of $\mathcal{C}$ which is an SO($n-1$) fixed point for some action of SO($n-1$) on $\mathcal{C}$.

Question: What do we need to get from Z_0 to Z?

Answer: Need to supply one piece of information, namely, what does Z do on an n-dimensional disc (which isn't in Bord($n-1$)) where D^n is regarded as an n-morphism, and the source/target are in Bord($n-1$). In other words: we need to supply an n-morphism in $\mathcal{C}$

$$Z(D^n) : Z_0(S^{n-1}) \to 1$$

in an $SO(n)$-equivariant way, satisfying a non-degeneracy condition (which we won't make explicit).

Example: $n = 2$.
$Z : \text{Bord}_2 \to \mathcal{C}$ is equivalent to giving:

1. $\text{Bord}_1 \cong \text{Bord}^{fr} \to \mathcal{C}$, which is just an object $X \in \mathcal{C}$
2. $Z(D^2) = \eta \in \text{Hom}(Z_0(S^1), 1)^{S^1}$ such that η is "non-degenerate".

Example: The B-model.
Let $\mathcal{C}$ be a symmetric monoidal $(\infty, 2)$-category defined as follows:

1. Objects of $\mathcal{C}$ are algebraic varieties over $\mathbb{C}$.

2. Morphisms $X \to Y$ in $\mathcal{C}$ are chain complexes of quasi-coherent sheaves on $X \times Y$.
3. Composition: the convolution product (i.e. pulling back, tensoring and pushing forward) defined using the diagram:

$$\begin{array}{ccccc} & & X \times Y \times Z & & \\ & \swarrow & \downarrow & \searrow & \\ X \times Y & & Y \times Z & & X \times Z \end{array}$$

   Note: Some care must be taken, we may need to take a free resolution in the tensor product.
4. 2-morphisms are maps of chain complexes (again, in the possibly derived sense)
5. higher morphisms: Chain homotopies, chain homotopies between chain homotopies, etc.
6. The monodial structure is coming from products of varieties. Clearly this is symmetric and a point is the identity.

We want to find some field theories. By the cobordism hypothesis we just need the following:

Goal: Lets find a dualizable object of $\mathcal{C}$. This is easy, because every object in $\mathcal{C}$ is self-dual: first we need a candidate for evaluation and coevaluation:

$$* \xrightarrow{coev} X \times X \xrightarrow{ev} *.$$

We will let both the coev and ev be given by the structure sheaf of the diagonal in $X \times X$. One can check that this gives the necessary conditions for duality. This is already enough for a 1-dimensional field theory. To a point it assigns a variety X. Once given X, the only other interesting thing we can ask is what it assigns a circle.

If we let $Z_0 : \mathrm{Bord}_1 \to \mathcal{C}$ be our functor, then $Z_0(S^1) =$"dim$X$":

$$* \xrightarrow{coev} X \times X \xrightarrow{ev} * \qquad \left(* \xrightarrow{compose} *\right)$$

Calculating this out explicitly, we see that $ev \circ coev$ corresponds to $R\Gamma(\mathcal{O}_X \otimes^L_{\mathcal{O}_{X\times X}} \mathcal{O}_X)$, where we treat $\mathcal{O}_X$ as a bimodule (one can easily compute that $\mathcal{O}_X$ is supported on the diagonal when treated as a bimodule). However, this is the definition of Hochschild homology, so $Z_0(S^1)$ is the Hochshild homology of X.

To get a 2-dimensional field theory, we need

$$\eta \in \mathrm{Hom}_{ch.cmplx}(Z_0(S^1), 1)^{S^1} = \mathrm{Hom}_{ch.cmplx}(\mathrm{Hoch}_*(X), \mathbb{C})^{S^1}.$$

So, we are looking for a trace of Hochschild homology that is "circle" invariant. This implies that rather then Hochschild homology, we should be using cyclic cohomology. This is something we only expect for a Calabi-Yau. If X is Calabi-Yau variety of dimension n, then

$$\mathrm{Hoch}_{-n}(X) \cong H^n(X; \mathcal{O}_x) \cong \ H^n(X; \omega) \xrightarrow{tr} \mathbb{C}$$

Disclaimer: The above trace map doesn't fit in with $\mathrm{Hom}_{ch.cmplx}(\mathrm{Hoch}_*(X), \mathbb{C})^{S^1}$, unless we introduce appropriate twistings. Ignoring this aspect, we use the trace map to give us η, which is automatically circle invariant due to degree considerations.

So, we get η above if X is Calabi-Yau. To get a two dimensional field theory we need some non-degeneracy. In this case, non-degeneracy of $\eta \Leftrightarrow$ Serre duality. The upshot is we get a two dimensional field theory associated to X; it is the "B-model".