Jan 19/10

Higson I (K-theory from func. analysis)

M clsd mfld

D 1st order PDO of Dirac type (locally)

$D = \sum a_j \frac{\partial}{\partial x_j}$ a_j - ~~fn~~. fn val matrices. $a_i a_j^* + a_j a_i^* = \begin{cases} 2I & i=j \\ 0 & i \neq j \end{cases}$

e.g.: $D = \partial/\partial x + i\, \partial/\partial y$

Fundamental fact: dim ker D, dim coker D $< \infty$

Index(D) = dim ker D − dim coker D (D is "Fredholm")

= same for C^∞, L^2, distributions

Also: coker D = ker D^* (formal adjoint)

Idea: In index theory (à la Atiyah-Singer) it is useful to study Fredholm ops abstractly in particular via L^2 - ? (Hilbert spaces, operators, C^*-algebras etc.)

Basics of Fredholm operators:

bnded op $T: H_0 \to H_1$ + Fredholm (H_i - Hilbert)

Fredholm is "practically" invertible.

$\begin{pmatrix} T & \text{Proj} \\ \text{Proj} & 0 \end{pmatrix}$: $H_0 \oplus \ker T^* \longrightarrow H_1 \oplus \ker(T)$ is invertible!

Hence Fredholm condition is open. (in norm topo.)

The ~~ix~~ index is locally const.

Fredholm + compt = Fredholm with the same index.
(closure of fin rank op)

Atkinsons Thm: op is Fredholm iff invertible modulo cpt.

Space of Fredholm op Fred(H_0, H_1) is a "reasonable" topo sp.

Assume for the moment H_0, H_1 ∞-dim (+seperable).

Thm (Atiyah-Janich): $K(X) \cong [X; \text{Fred}]$ X is cpt topo. sp.

$\uparrow$
K-theory of vect. bun.

all ∞ dim seperable Hilb sp. are the same ~~hence Fred = Fred(H_0, H_1)~~

Suppose we have $F: X \to \text{Fred}$ thought of as family of Fred op labelled by X. Suppose $\dim(\ker F_x)$ is locally const. Then the spaces $\ker F_x \subset H_0$ constitute a vector bun, and so do $\ker F_x^*$.

We can define: $\text{Index}(F) = [\ker F] - [\ker F^*] \in K(X)$ abelian grp

This gives a map $[X, \text{Fred}] \rightsquigarrow K(X)$

Que: What if F_x = invertible?

Thm (Kuiper): If F_x is invertible $\forall x$ then F is homotopic to a constant.
(but there is no explicit constraction)

Que: What if $\dim(\ker F_x)$ is not locally const)
(Atiyah-Singer main idea) Then replace F by $FP =: F'$ where $P: H_0 \to H_0$ is a finite codim projection to fix the problem, i.e. $\dim(\ker F')$ is loc. const.

Idea: Fred is classifying space for K-theory.

Principle of Pseudo-locallity (Atiyah)

D Dirac op. (as in intro).

$D: C^\infty(M, S_0) \to C^\infty(M, S_1)$ | e.g. 2-vect bun / M and D mapping sections of one to the other.

Problem: It is not defined as an operator $\underline{L^2(M, S_0)} \to \underline{L^2(M, S_1)}$
(not bnded)

Fix: D can be "extended" to $D: H_0 \to H_1$ is such a way that: if $u_n \to u$ and $u_n \in C^\infty$, $u \in L^2$ and convergence is in L^2 and $Du_n \to v$ in L^2 then Domain$(D) \ni u$ $Du = v$. "D is closd". there is a minimal extension.

non bonded

Von Neumann: D has a "polar decomposition":

$$D = AF \qquad (\text{think } z = re^{i\theta})$$

$$A = \sqrt{DD^*}$$

$$F = \text{partial isometry}.$$

and: $\ker F = \ker D$, $\ker F^* = \ker D^*$

F is Fredholm, bnded, same index as D.

Think of D & F related by "homotopy".

⊛ Thm (Atiyah): If f is continuous fn on M ⊛

Think of it as operator defined by multi. Then

$$[F, f] = Ff - fF \quad \Big\} \text{ for example of sections.}$$

is compact ("practicaly zero").

Que: Why should we care?

Consider vect bun E on M. It can be realized as a proj-val map

$$P : M \longrightarrow M_n(\mathbb{C})$$

$$\text{s.t.} \quad P(m)\mathbb{C}^n = E_m$$

We can consider P as a projection in ~~[illegible]~~ $M_n(C(M))$ and so as a projection operator

$$P_0 : \overbrace{H_0 \oplus \cdots \oplus H_0}^{n} \longrightarrow \overbrace{H_0 \oplus \cdots \oplus H_0}^{n}$$

and same for $\quad P_1 : H_1 \oplus \cdots \oplus H_1 \longrightarrow H_1 \oplus \cdots \oplus H_1$

$$P_1 \begin{pmatrix} F & \\ & \ddots \\ & & F \end{pmatrix} P_0 : \operatorname{range}(P_0) \longrightarrow \operatorname{range}(P_1)$$

Then $P_1 \begin{pmatrix} F & \\ & \ddots \\ & & F \end{pmatrix} P_0$ is Fredholm.

Thanks to thm ⊛ we get a map:

$$\operatorname{Index}_D : K^0(M) \longrightarrow \mathbb{Z} = K(pt)$$

and more generally (via Atiyah-Janich) functorial in X.

$$\operatorname{Index}_D : K(X \times M) \longrightarrow K(X)$$

(think of X as a variable, a place-holder)

The maps Index_D (for every X) determine a class in $K_0(M)$ (K-homology of M). Denote this class by $[D]$.

Claim (Atiyah): Every class comes this way!

Que: How to define $K_0(M)$ using Fredholm $C(M)$-modules

Defn: A Fredholm $C(M)$-module is the following collection of data:

- $F: H_0 \to H_1$ Fred
- $C(M)$ actions on H_0, H_1
- $[F, f]$ compact $\forall f$.

Answer: (Kasparov) ...

but first some terminology...

Defn: A continuous field of Hilbert spaces $\{H_x\}_{x \in X}$ is

- a set of Hilb spaces
- a family of sections called "continuous sections" such that $x \longmapsto \|s(x)\|$ is continuous,
- the continuous sections are a $C(X)$-module, etc...

Defn: A bnded operator on a continuous field is... what you think... (except we require the adjoint to be continuous).

Defn: A cpt op on a cont. field parametrized by loc. cpt X is a (norm) limit of operators of the form:

$$s \longmapsto \sum_{i=1}^{n} \langle s, t_i \rangle r_i$$

where t_i and r_i are C_0-sections.

Defn: A Fred operator is one which is invertible mod compt

Ex: [redacted] $F: H_0 \to H_1$ Fred op $(X = pt)$

$\mathbb{H}_0$ = field over $[0,1]$ with fibers H_0 except 0 at 0

$\mathbb{H}_1$ = ditto.

If F is invertible (Index $F = 0$). Define $\mathbb{F}: \mathbb{H}_0 \to \mathbb{H}_1$ to be F at each fiber. This $\mathbb{F}$ is Fred in the sense of continuous fields. So $\mathbb{F}$ is a homotopy from F to $0: 0 \to 0$.

This is not true if F is not invertible! So there is s/t subtle about these definitions.

Thm (Improved Atiyah-[redacted] Janich) due to Kasparov

$K^0(X)$ = homotopy classes of Fredholm families $/X$.

Thm (Kasparov)

$K_0(X)$ = homotopy classes of Fred $C(X)$-modules