Higson I (K-theory from func. analysis)

\[ D = \sum_{ij} a_{ij} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \text{ is fundamental matrices.} \quad a_{ij}^* + a_{ij} = \sum_{i,j} \begin{pmatrix} i & 0 \\ 0 & j \end{pmatrix} \]

\[ \text{e.g.: } D = \gamma x + i \gamma y \]

Fundamental fact: \( \dim \ker D, \dim \coker D < \infty \)

\[ \text{Index}(D) = \dim \ker D - \dim \coker D \quad (D \text{ is } \text{Fredholm}) \]

\[ = \text{same for } C^0, L^2, \text{ distributions} \]

Also: \( \coker D = \ker D^* \) (formal adjoint)

Idea: In index theory (à la Atiyah-Singer) it is useful to study Fredholm ops abstractly in particular via \( L^2 \) ops. (Hilbert spaces, operators, \( C^* \)-algebras etc.)

Basics of Fredholm operators:

\[ \text{bounded op } T: H_0 \rightarrow H_1 + \text{Fredholm} \quad (H_i, \text{ Hilbert}) \]

Fredholm is "practically" invertible.

\[ H_0 \otimes \ker T^* \rightarrow H_1 \otimes \ker(T) \text{ is invertible!} \]

Hence Fredholm condition is open. (in norm topo.)

The \( \text{index} \) is locally const.

Fredholm + compact \( \Rightarrow \) Fredholm with the same index.

(closure of the rank op)

Atkinho: Thm: op is Fredholm iff invertible modulo cpt.

Space of Fredholm op Fred(\( H_0, H_1 \)) is a "reasonable" topo sp.

Assume for the moment \( H_0, H_1 \) 0-dim (+separable).

\[ \text{Thm (Atiyah-Janich): } K(X, \mathbb{Z}) \cong [X, \text{Fred}] \quad X \text{ is cpt } \]

\[ \text{all 0-dim separable } \quad \uparrow \]

Hilb sp are the same hence Fred = Fred(\( H_0, H_1 \)) K-theory of vect. bun.
Suppose we have \( F : X \rightarrow \text{Fred} \) thought of as family of Fred op labelled by \( X \). Suppose \( \dim(\ker F_x) \) is locally constant. Then the spaces \( \ker F_x \) \( \subset \) to constitute a vector bun, and do do \( \ker F_x^+ \).

We can define: Index \( (F) = [\ker F] - [\ker F^+] \in K(X) \)

This gives a map \( [X, \text{Fred}] \rightarrow K(X) \) abelian group.

Qn: What if \( F = \text{invertible} \)?

Thm (Kuiper): If \( F \) is invertible \( \forall x \) then \( F \) is homotopic to a constant.

(but there is no explicit construction)

Qn: What if \( \dim(\ker F_x) \) is not locally constant?

(Atiyah-Singer main idea) Then replace \( F \) by \( FP_\varepsilon : F' \)
where \( P : H_0 \rightarrow H_0 \) is a finite codim projection to fix the problem, i.e. \( \dim(\ker F') \) is loc. const.

Idea: Fred is classifying space for \( K \)-theory.

Principle of Pseudo-locality (Atiyah)

\( D \) Dirac op. (as in intro).

\[ D : C^0(H, S_0) \rightarrow C^0(H, S_1) \]

\( e.g. \) L2-vec bun \( \rightarrow H0 \) and \( D \) mapping sections of one to the other.

Problem: It is not defined as an operator \( L^2(H, S_0) \rightarrow L^2(H, S_1) \)

(Not bounded)

Fix: \( D \) can be "extended" to \( D : H_0 \rightarrow H_1 \) is such a way that if \( u_n \rightarrow u \) and \( u_n \in C^0 \), \( u \in L^2 \) and convergence is in \( L^2 \) and \( Du_n \rightarrow v \) in \( L^2 \) then Domain \( D \) \( \exists u \)

\( Du = v \). "D is closed." There is a minimal extension.
Von Neumann: D has a "polar decomposition":

\[ D = AF \quad (\text{think } z = re^{i\theta}) \]

\[ A = \sqrt{DD^*} \]

F = Partial isometry.

and: \( \ker F = \ker D \), \( \ker F^* = \ker D^* \)

F is Fredholm, bounded, same index as D.

Think of D & F related by "homotopy".

\( \text{(Thm (Atiyah): If } \text{F is continuous } \Phi_n \text{ on } M \text{)} \)

Think of it as operator defined by multiplying.

\[ DF + I = FF - FF \]

is compact ("practically zero").

For example of sections.

Why would we care?

Consider vector bundle \( E \) on M. It can be realized as a projection map

\[ P : M \to M_n(C) \]

\[ \text{st. } P(m)C^n = E_m \]

We can consider P as a projection in \( \text{Mat}(C(M)) \)

and so as a projection operator

\[ P_0 : H_0 \oplus \cdots \oplus H_0 \to H_0 \oplus \cdots \oplus H_0 \]

and same for \( P_1 : H_1 \oplus \cdots \oplus H_1 \to H_1 \oplus \cdots \oplus H_1 \)

\[ P_1 \left( F, P \right) P_0 : \text{range}(P_0) \to \text{range}(P_1) \]

Then \( P_1(F, P)P_0 \) is Fredholm.

Thanks to Thm & we get a map:

\[ \text{Index}_D : K^0(M) \to \mathbb{Z} = K^0(\text{pt}) \]

and more generally (via Atiyah - Janceich) functorial

\[ \text{Index}_D : K(X \times M) \to K(X) \text{ in } X. \]
(think of X as a variable, a placeholder)

The maps Index (for every X) determine a class in $K_0(M)$ (K-homology of M). Denote this class by $T$.  

Claim (Atiyah): Every class comes this way.  

Question: How to define $K_0(M)$ using Fredholm $C(M)$-modules?  

Definition: A Fredholm $C(M)$-module is the following collection of data:  
- $F$: $H_0 \rightarrow H_1$ Fred  
- $C(M)$ acts on $H_0, H_1$  
- $H_0, H_1$ compact if f.  

Answer: (Kasparov) --  
but, first some terminology...  

Definition: A continuous field of Hilbert spaces $\{H_x\}_{x \in X}$ is  
- a set of Hilb spaces  
- a family of sections called "continuous sections" such that $x \rightarrow \|s(x)\|$ is continuous,  
- the continuous sections are a $C(X)$-module, etc...  

Definition: A bounded operator on a continuous field is...  
what you think... (except we require the adjoint to be continuous).  

Definition: A ctp op on a cont. field parameterized by loc. ctp $X$ is a (norm) limit of operators of the form:  
$$s \mapsto \sum_{i} <s, t_i>r_i$$  
where $t_i$ and $r_i$ are $C_0$-sections.  

Definition: A Fred operator is one which is invertible mod ctp.
Ex: $F: H_0 \to H_1, \text{ Fred op (} X = \text{pt) }$

$H_0 = \text{ field over } [0,1] \text{ with fibers } H_0 \text{ except } 0 \text{ at } 0$

$H_1 = \text{ ditto.}$

If $F$ is invertible (Index $F = 0$), define $F: H_0 \to H_1$ to be $F$ at each fiber. This $F$ is Fred in the sense of continuous fields. So $F$ is a homotopy from $F$ to $0: 0 \to 0$.

This is not true if $F$ is not invertible! So there is a subtle about these definitions.

Thus (Improved Atiyah-Samash due to Kasparov)

$K^0(X) = \text{ homotopy classes of Fredholm families of } X.$

Thus (Kasparov)

$K_0(X) = \text{ homotopy classes of Fred } C(X)-\text{modules}$