Jan 26th 10

Higson III Applications of K-homology

Recall: K-homo. & asymptotics of $t\cdot D$ as $t \to 0$.

Atiyah's cycle for $K_0(X)$:

$F: H_0 \to H_1$ a Fred. op. bounded

H_i are $C(X)$-reps

$[F,f]$ should be cpt

Kapranov's equiv. relation: homotopy using cts fields.

This gives

$$K^0(X\times Y) \longrightarrow K^0(Y) \quad \begin{pmatrix} \text{A-H} \\ \text{K-theory} \end{pmatrix}.$$

functorial & multiplicative in Y

$K^0(X\times Y) = [\text{vect bun on } X\times Y]$

$= [\text{families of vect bun on } X \text{ param. by } Y]$

From F we get

$[\text{family of freds param. by } Y]$.

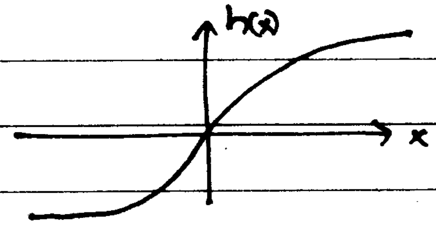
Pseudolocality follows from

- $(tD\pm i)^{-1}$ cpt $\forall t$
- $\|[(tD\pm i)^{-1}, f]\| \xrightarrow[t\to 0]{} 0$

$$[(tD\pm i)^{-1}, f] = (tD\pm i)^{-1}[f,(tD\pm i)](tD\pm i)^{-1}$$
$$= (\text{-"-})[f, tD](\text{-"-})$$
$$= t\underbrace{(\text{-"-})[f,D](\text{-"-})}_{\text{all bnded}}$$

D Dirac op. $F = \text{phase}(D)$ i.e. $D = F|D|$

$\begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix}$ ~~is $h(D)$ + cpt for~~



for example $h(x) = \frac{2}{\pi}\arctan(x)$

$$[\mathrm{actan}(D), f] = \int_0^1 [\tfrac{d}{dt} \arctan(tD), f]\, dt$$

$$= \int_0^1 [D(1+t^2D^2)^{-1}, f]\, dt$$

$$D(1+t^2D^2)^{-1} = t^{-1}\left[(tD+i)^{-1} - (tD-i)^{-1} \right].$$

hence the integrand is continuous $t>0$ cpt-op-valued and uniformaly bnded. hence we can integrate $\int_\epsilon^1$ for small enough ϵ.

$$[h(D), f] = \text{cpt up to } \epsilon$$

Geo. K-homo of Baum: $K^{geo}(X)$

$$K_0^{geo}(X) = \{\text{geometric cycles on } X\} \big/ \text{equiv. relation}$$

geo. cycle is a triple: $(M^{spin^c, 2k}, E^{\text{cx vbun}}_{\text{on } M}, f: M \to X)$

equivalence relation: bordism, directsum/ ~~dis~~ disjoint union modification to produce spin^c sphere bundles.

there is also $K_1^{geo}(X)$. Geo. K-homo is $\mathbb{Z}/2\mathbb{Z}$ - graded.

There is nat. trans: $K^{geo}(X) \xrightarrow{\mu(X)} K_0(X)$

the index theorem is the comm. of this diag

$$\begin{array}{ccc} K_0^{geo}(X) & \xrightarrow{\mu(X)} & K_0(X) \\ \downarrow & & \downarrow \\ K_0^{geo}(pt) & \xrightarrow{\mu(pt)} & K_0(pt) \end{array}$$

Application: π grp. $X \longrightarrow B\pi$. $\sigma_1, \sigma_2 : \pi \longrightarrow U(N)$

A relative eta invariant for operators on M^{odd}

(# of pos. eval) − (# of neg. eval).

<u>Fact</u>: We get $\rho(r_1, r_2) : K_1(B\pi) \to \mathbb{R}/\mathbb{Z}$.
the relative eta inv. ρ is a homotopy inv., mod $\mathbb{Q}$, when applied to the signature operator.

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The World's simplest index thm: M^3. $H \subseteq TM$ plane bun. suppose TM/H trivial. suppose $[H,H]$ generates TM/H. Choose Z which trivializes TM/H. We get a symp. form

$$H_p \times H_p \xrightarrow[{[\,,\,]}]{} T_p/H_p \simeq \mathbb{R}$$

hence an orientation.
Choose $2C \in H_1(M)$ Poincare dual to Euler class.
Fix metric on H. Define Δ_H Laplacian in H-direction.
Δ_H not elliptic. Consider $\Delta_H + i\alpha Z$, $\alpha : M \to \mathbb{C}$.
<u>Thm</u>: Suppose $\mathrm{Im}(\alpha) \cap \{\text{odd integers}\} = \phi$. Then $\Delta_H + i\alpha$ is <u>Fred</u> (same stability as for Dirac ... C^∞, L^2, etc all the same).
<u>Thm</u>: $\mathrm{Index}(\Delta_H + i\alpha Z) = \sum_{k \text{ odd integer}} (k-1) \text{ winding } \#_C\left(\frac{\alpha - k}{\alpha + k}\right)$

this is proved by interesting deformations.
the pf uses 1) $T_H M$ the "tang bun" of Heisenberg groups

$$G_p = H_p \oplus T_p/H_p \quad (\text{non abelian})$$

2) the dual $T_H M^\vee$ is a bundle of grp C^*-algebras.
3) deformation of $T_H M$ back to TM (going from non-abelian to abelian).

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<u>The $[Q,R] = 0$ Thm</u>:
i.e.: Quantization commutes with reduction thm.
Set up: M Kahler mfld
(K-homo can provide ~~the~~ environment for thm)

$J: TM \circlearrowleft$, $J^2 = -1$

$h(X,Y) = g(X,Y) - i\omega(X,Y)$ ← symplectic form.

we're interested in ω's that satisfy a prequantum condition $\exists L$ hermitian line bun $/M$, ∇ connection s.t. $\nabla^2 = i\omega$

Suppose we have a cpt grp G acting: $G \times M \to M$

We can differentiate section of L wrt $X \in \mathfrak{g}$.

We have:

$$\nabla_X = D_X - i\mu_X$$

← real scalar fn.

we get

$$\omega(X,Y) + Y(\mu_X) = 0$$

or

$$J \operatorname{grad} \mu_X = X$$

μ_X is called a <u>moment map</u>: $\mu: M \to \mathfrak{g}^*$.

Assume $0 \in \mathfrak{g}^*$ is a regular value.

G acts (locally) freely on $\mu^{-1}(0)$

$$M/\!/G := \mu^{-1}(0)/G \quad \underline{\text{reduction of } M}$$

We can reduce $\omega/\!/G$, $L/\!/G$

<u>Thm</u>: $\operatorname{Index}(\text{Dalbeault}_{M/\!/G,\, L/\!/G}) =$

$$(*) \quad = \underset{\text{of triv. repn}}{\text{multiplicity}} \left(\operatorname{Index}(\text{Dalbeault}_{M,L}) \right).$$

in the language of geo K-homo we have (M, L, id) cycle for $K_0^G(M)$, then $(M/\!/G, L/G, f = \text{inclusion})$ cycle for $K_0(M/G)$

We can define $K_*^G(X)$ (G-Hilbert sp., etc.)

$K_*^{geo, G}(X)$ (M^{spin} w/ G-action, etc)

reduction ↓

We have: $R: K_*^G(X) \longrightarrow K_*(X/G)$

$$[F: H_0 \to H_1] \longmapsto [F|_{H_0^G} : H_0^G \to H_1^G]$$

this reduction compatible with RHS of $(*)$

Problem: define R for geo K-homo. & show $[\mu, R] = 0$ ← Q