Higson III. Applications of K-homology

Recall: K-homo. & asymptotics of t.D as t → 0.

Atiyah's cycle for K₀(X):

- F: H₀ → H₁, a Fred. op. bounded
  - H₁ are C(X)-repr
  - [F,f] should be cpt

Kapranov's equiv. relation: homotopy using cts yields.
This gives

\[ K^0(X \times Y) \to K^0(Y) \quad (\text{A-H K-theory}) \]

- Functorial & multiplicative in Y
  \[ K^0(X \times Y) = \text{[vect. bun on } X \times Y]\]
  \[ = \text{[families of vector } X \text{ param. by } Y]\]

From F we get

- [family of Freds param. by Y].

Pseudo-locality follows from

- \( (tD ± i)^{-1} \) cpt v. t

- \( \lim_{t \to 0} [(tD ± i)^{-1}, f] \to 0 \)

\[ [(tD ± i)^{-1}, f] = (tD ± i)^{-1} [f, (tD ± i)] (tD ± i)^{-1} \]

\[ = (-t \cdot \text{)} [f, tD] (-t \cdot \text{)} \]

\[ = t (-\text{)} [f, D] (-\text{)} \]

\[ \text{all bndd} \]

D Dirac op. F = phase(D) i.e. D = F|D|

\[ (0 \circ F \circ 0)^{\text{th}} = h(D) + \text{opt for } h(x) \}

For example \( h(x) = \frac{2}{\pi} \arctan(x) \)

\[ \text{Diagram: } h(x) \to x \]
\[ \int \frac{1}{1 + t^2} dt = \int \left[ \frac{1}{1 + t^2} \right] dt \]
\[ = \int \left[ (1 + t^2)^{-1} \right] dt \]
\[ D(1 + t^2)^{-1} = t^2 \left[ (tD + i) - (tD - i) \right] \]

Hence the integrand is continuous to opt-op valued and uniformly bounded. Hence we can integrate \( \int \)
for small enough \( \epsilon \).
\[ \int h(D) \, dt = \text{opt up to } \epsilon \]

**Geo. K-homo of Baum:** \( K^{\text{geo}}(X) \)

\[ K_0^{\text{geo}}(X) = \frac{\text{geometric cycles}}{\text{equiv. relation}} \]

Geo. cycle is a triple: \( (M, spinc, E, \text{on } X) \)

Equivalence relation: bordism, direct sum, disjoint union

Modification to produce spinc sphere bundles.

There is also \( K_2^{\text{geo}}(X) \). Geo. K-homo is \( \mathbb{Z}/2\mathbb{Z} \)-graded.

There is nat. trans: \( K_0^{\text{geo}}(X) \to K_0(X) \)

The index theorem is:

\[ \begin{array}{ccc}
K_0^{\text{geo}}(X) & \xrightarrow{\text{index}} & K_0(X) \\
\downarrow & & \downarrow \\
K_0^{\text{geo}}(pt) & \to & K_0(pt)
\end{array} \]

Application: \( \pi \text{ grp. } X \to B\pi. \quad \sigma_1, \sigma_2 : \pi \to U(N) \)

A relative eta invariant for operators on \( M \)

\[ \text{(no of pos. eval)} - \text{(no of neg. eval)} \]
Fact: We get \( p(r_1, r_2) : K_f(B\Pi) \rightarrow \mathbb{R}/\mathbb{Z} \).

The relative eta inv. \( p \) is a homotopy inv., mod \( \mathbb{Z} \),
when applied to the signature operator.

\[ \sim x \sim \]

The world's simplest index thm: \( M^3 \), \( H \leq TM \) plane bun.
Suppose \( TM/H \) trivial. Suppose \( [H, H] \) generates \( TM/H \).
Choose \( Z \) which trivializes \( TM/H \). We get a symp. form
\[ H_p \times H_p \rightarrow \mathbb{R}/H_p \cong \mathbb{R} \]
hence an orientation.

Choose \( 2C\epsilon H(H) \) Poincare dual to Euler class.
Fix metric on \( H \). Define \( \Delta_H \) Laplacian in \( H \)-direction.
\( \Delta_H \) not elliptic. Consider \( \Delta_H + i\alpha Z \), \( \alpha : \mathbb{R} \rightarrow \mathbb{R} \).

Thm: Suppose \( \text{Im}(\alpha) \cap \{ \text{odd integers} \} = \emptyset \).
Then \( \Delta_H + i\alpha Z \) is Fred
(dame stability as for Dirac \( -\bar{\partial}\partial, L^2, \text{etc all the same} \).

Thm: Index \( (\Delta_H + i\alpha Z) = \sum_k (k-1) \text{winding } \sum_{k \text{odd}}^\infty (\frac{z-K}{k+K}) \)

This is proved by interesting deformations.

The pf uses \( T_H M \) the "tang bunt" of Heisenberg groups
\[ G_p = H_p \oplus T_p/H_p \ (\text{non abelian}) \]

1) the dual \( T_H^* M \) is a bundle of grp \( C^* \)-algebras.
2) deformation of \( T_H M \) back to \( TM \) (going from non-abelian
to abelian).

\[ \sim x \sim \]

The \( [G, R] = 0 \) Thm:
\( i.e.: \text{quantization commutes with reduction thm} \).

Set up: \( M \) Kahler mfd
\( (K\text{-homo can provide an environment for thm}) \)
\( J: TM \otimes J^2 = -1 \)

\( h(x, y) = g(x, y) - i\omega(x, y) \)

We're interested in \( \omega \)'s that satisfy a prequantum condition. For a Hermitian line bundle \( L \), a connection \( \nabla \)

Suppose we have a group \( G \) acting: \( G \times M \rightarrow M \)

We can differentiate sections of \( L \) over \( \mathbb{R} \).

We have:

\[ \nabla_x = D_x - i\mu_x \]  

\( \mu_x \) is a real scalar.

We get

\[ \omega(x, y) + y(\mu_x) = 0 \]

or

\[ J \text{ grad } \mu_x = X \]

\( \mu_x \) is called a moment map: \( \mu: M \rightarrow \mathbb{R} \).

Assume \( 0 \in \mathbb{R} \) is a regular value.

\( G \) acts (locally) freely on \( \mu^{-1}(0) \)

\[ \frac{M}{G} = \mu^{-1}(0)/G \]

We can reduce \( L \otimes \mathbb{R} \), \( L/\mathbb{R} \)

**Thm:** \( \text{Index(\)Dalbeault\) } M_{/G} \rightarrow L_{/G} \)

\[ \oplus \text{ multiplicity of trivial repn } \left( \text{Index(\)Dalbeault\) } M_{/G} \right) \]

We can define \( K^*_G(X) \) (\( G \)-Hilbert sp., etc.)

\( K^*_G(X) \) (\( S^1 \) and \( G \)-action, etc.)

We have: \( R: K^*_G(X) \rightarrow K^*_G(X/G) \)

\[ [F: H_0 \rightarrow H_1] \mapsto [F|_{G_0}: H_0^G \rightarrow H_1^G] \]

This reduction compatible with \( H_{/G} \).

RHS of \( \otimes \) is 0.

Problem: define \( R \) for geo \( K \)-hom. and show \([\mu, R] = 0\)