PERSPECTIVES IN GEOMETRY LECTURES

These are notes, taken by Samuel Isaacson, on Tony Pantev's 2010 "Perspectives in Geometry" series at the University of Texas in Austin.

1. Basics in non-commutative Hodge theory

First, we need to clear up some terminology:

- *nc (non-commutative) Hodge theory:* this is Hodge theory on the cohomology groups of "non-commutative manifolds" (or varieties).
- *Non-abelian Hodge theory:* This is Hodge theory on cohomology of manifolds with non-abelian coefficients.

We need to combine these eventually.

Non-commutative Hodge structures arise in mirror symmetry but can be applied to symplectic topology.

Fact 1.1. The cohomology of a compact symplectic manifold is equipped with a nc Hodge structure.

1.1. **Hodge theory of compact Kähler manifolds.** If X is compact Kähler, then $\mathrm{H}^\bullet(X;\mathbf{C})$ carries extra structure called a Hodge structure. This structure comes from the fact that we can compare different cohomology theories:

$$\mathrm{H}^\bullet_{\mathrm{dR}}(X;\mathbf{C}) = \frac{\ker\left(A^\bullet_{\mathbf{C}}(X) \xrightarrow{d} A^{\bullet+1}_{\mathbf{C}}(X)\right)}{\operatorname{im}\left(A^{\bullet-1}_{\mathbf{C}}(X) \xrightarrow{d} A^\bullet_{\mathbf{C}}(X)\right)}$$

We also have

$$\mathrm{H}^\bullet_{\mathrm{B}}(X;\mathbf{C}) = \text{the singular cohomology of } X$$

and

$$\mathrm{H}^\bullet_{\mathrm{Dol}}(X;\mathbf{C}) = \frac{\ker\left(A^\bullet_{\mathbf{C}}(X) \xrightarrow{\overline{\partial}} A^{\bullet+1}_{\mathbf{C}}(X)\right)}{\operatorname{im}\left(A^{\bullet-1}_{\mathbf{C}}(X) \xrightarrow{\overline{\partial}} A^\bullet_{\mathbf{C}}(X)\right)}$$

Then

$$\mathrm{H}_{\mathrm{dR}} \cong \mathrm{H}_{\mathrm{B}} \quad \text{(de Rham's theorem)}$$

and

$$\mathrm{H}_{\mathrm{Dol}} \cong \mathrm{H}_{\mathrm{dR}} \quad \text{(follows from Hodge's theorem and Kähler identities } 2\Delta_{\overline{\partial}} = \Delta_d\text{).}$$

This construction uses a Kähler metric; however, the isomorphism does not depend on this choice. Note that this isomorphism, as specified, is not unique; but we can refine this statement.

Using these isomorphisms,we have

$$\begin{array}{ccccc} \mathrm{H}^w_{\mathrm{B}}(X;\mathbf{C}) & \xrightarrow{\cong} & \mathrm{H}^w_{\mathrm{dR}}(X;\mathbf{C}) & \xleftarrow{\cong} & \mathrm{H}^w_{\mathrm{Dol}}(X;\mathbf{C}) \\ \uparrow & & & & \| \\ \mathrm{H}^w_{\mathrm{B}}(X;\mathbf{Z}) & & & & \bigoplus_{p+q=w} \mathrm{H}^{p,q}(X) \end{array}$$

(note $\mathrm{H}^{p,q}(X) \cong \mathrm{H}^q(X;\Omega_X^p)$).

Provisional definition 1.2. A *pure Hodge structure of weight* w is a triple

$$(V, V_{\mathbf{Z}}, \bigoplus_{p+q=w} V^{p,q} \cong V)$$

where

1. V is a complex vector space.
2. $V_{\mathbf{Z}}$ is a finitely generated abelian group mapping to a full lattice in V (embedding on the free part).

Let's ignore $\mathbf{Z}$ and work with $\mathbf{Q}$-*structures* instead:

$$(V, V_{\mathbf{Q}}, V = \bigoplus V^{p,q})$$

The condition we require is that for all p and q, $\overline{V^{p,q}} = V^{q,p}$.

Remark 1.3. This is a great deal of information. If X is a compact Riemann surface, then

$$\left(\mathrm{H}^1_{\mathrm{dR}}(X;\mathbf{C}), \mathrm{H}^1(X;\mathbf{Z}), \mathrm{H}^1(X;C) = \mathrm{H}^{1,0} \oplus \mathrm{H}^{0,1}\right)$$

reconstructs X (this is called Torelli's theorem).

We will slightly modify the above definitions.

Definition 1.4. A *pure Hodge structure of weight* w is a triple

$$(V, V_{\mathbf{Q}}, F^\bullet V)$$

where $F^\bullet V$ is a decreasing filtration of length w:

$$V = F^0V \supseteq F^1V \supseteq \cdots \supseteq F^wV \supseteq 0.$$

We demand that $\mathrm{gr}^p_F \mathrm{gr}^q_{\overline{F}} V = 0$ unless $p + q = w$. Another way to state this condition is that F and $\overline{F}$ are w-opposed.

Here, in the geometric setting, we want

$$F^pV = \bigoplus_{a \geq p} \mathrm{H}^a(X;\Omega_X^{w-a}) \cong \bigoplus_{a\geq p} \mathrm{H}^{w-a,a}.$$

Note that $\mathrm{H}^{p,q} = F^p \cap \overline{F^q}$. The reason this is better (a filtration as opposed to a direct sum decomposition) is that if we have a family of compact Kähler manifolds $f : \mathscr{X} \to S$ (here f is a holomorphic map) then we get a bundle of cohomologies

$$\mathscr{H}^w(\mathscr{X}/S) \to S$$

with fiber

$$\mathscr{H}^w(\mathscr{X}/S)_s \cong \mathrm{H}^w_{\mathrm{dR}}(\mathscr{X}_s;\mathbf{C})$$

which is a holomorphic vector bundle. We also get complex subbundles $\mathscr{H}^{p,q}(\mathscr{X}/S)$ and $F^p\mathscr{H}(\mathscr{X}/S)$. However, the former are not necessarily holomorphic as subbundles, while the latter are holomorphic.

Definition 1.5. This filtration is called the *Hodge filtration.* If

$$(V, V_{\mathbf{Q}}, F^\bullet V) \quad \text{and} \quad (V', V'_{\mathbf{Q}}, F^\bullet V')$$

are two pure Hodge structures, then a map between them is a linear map $f : V \to V'$ so that $f(V_{\mathbf{Q}}) \subseteq V'_{\mathbf{Q}}$ and $f(F^pV) = F^pV' \cap f(V)$ (note this condition is stronger than you might expect).

We get a category $\mathbf{HS_Q}$ whose objects are direct sums of Hodge structures of various weights and whose maps are maps of Hodge structures.

Fact 1.6. $\mathbf{HS_Q}$ is an abelian category.

To generalize the notion of a Hodge structure, we need to reformulate the data of a filtration.

1.2. **First reformulation of the notion of a Hodge structure.** Our first reformulation is the following: we think about a filtration on a vector space in terms of algebraic geometry. We have

$$\begin{aligned}\{\text{graded f.d. vector space}\} &\leftrightarrow \{\text{representation of } S^1\}\\ &\leftrightarrow \{\text{representation of } \mathbf{C}^\times\}\end{aligned}$$

and

$$\begin{aligned}&\{\text{filtered f.d. vector space}\}\\ &\quad\leftrightarrow \{\text{finite rank vector bundle on } \mathbf{A}^1_{\mathbf{C}} \text{ equivariant for the action of } \mathbf{C}^\times\}\end{aligned}$$

The second correspondence is given by the Rees module construction: $(V, F^\bullet V)$ produces a trivial bundle $V \times \mathbf{A}^1_{\mathbf{C}}$ coming with an action of $\mathbf{C}^\times$. As an equivariant bundle, it corresponds to the module

$$\xi(V, F^\bullet V) = \sum u^{-i} F^i V \subseteq V \otimes \mathbf{C}[u, u^{-1}]$$

of $\mathbf{C}[u]$. This construction is called the *Rees bundle.* Here we have the following identification of the fibers at 0 and 1:

$$\begin{aligned}\xi(V, F^\bullet V)_1 &= V\\ \xi(V, F^\bullet V)_0 &= \operatorname{gr}_F V\end{aligned}$$

This leads to the Hodge structure on non-abelian cohomology or on homotopy types. If X is a compact Kähler manifold and T a k-truncated homotopy type (i.e., $\pi_i(T, x) = 0$ if $x \in T$ and $i > k$), then $\operatorname{Hom}(X, T)$ has a Hodge filtration in the sense of $\mathbf{C}^\times$-equivariance over $\mathbf{A}^1$.

1.3. **Second reformulation of the notion of a Hodge structure.** Our second reformulation involves recasting the data of a filtration into differential geometric data. The idea here is to promote V to a holomorphic vector bundle $\mathscr{V}$ on $\mathbf{A}^1$ (namely, the trivial bundle) and promote $F^\bullet V$ to a holomorphic connection ∇ on $\mathscr{V}|_{\mathbf{A}^1\setminus\{0\}}$ whose flat sections give back $F^\bullet V$. This will be the same as what we had before if the pole of ∇ is *logarithmic*, i.e, if $\nabla = d + A\frac{du}{u}$.

2. Meromorphic connections and Stokes phenomena

Recall that yesterday, we spoke about pure Hodge structures and recast the notion of filtration as geometric data. Today, we'd like to generalize this (in particular, we'd like to drop the logarithmic requirement). Our setup is the following: we work on $\mathbf{A}^1_{\mathbf{C}}$ with coordinate u. We want to understand pairs (H, ∇) where H is an algebraic vector bundle over $\mathbf{A}^1$ and ∇ is a connection

$$H \to H \otimes \Omega^1_{\mathbf{A}^1}(\infty \cdot \{0\})$$

(so ∇ is a *meromorphic connection*). We want to classify such data up to gauge equivalence. This really only depends on the germ near 0. So we make a replacement

$$(H, \nabla) \leftrightarrow (\mathscr{H}, \nabla)$$

where $\mathscr{H}$ is a rank n free module over $\mathbf{C}\{u\}$ and

$$\nabla_{d/du} : \mathscr{H} \to \mathscr{H} \otimes \mathbf{C}\{u\}[u^{-1}]$$

satisfies the Leibniz rule. Here $\mathbf{C}\{u\}$ means "convergent power series in u."

Definition 2.1. $(\mathscr{H}, \nabla)$ has a *regular singularity at* 0 if we can find a basis so that

$$\nabla_{d/du} = \frac{d}{du} + u^{-1} A_0$$

where $A_0 \in \mathrm{Mat}_{n\times n}(\mathbf{C})$.

In general, if we choose a basis for $\mathscr{H}$, then

$$\nabla_{d/du} = \frac{d}{du} + u^{-N} \sum_{i\geq 0} A_i u^i$$

where the latter term is denoted $A(u)$. If $G \in \mathrm{GL}_n(\mathbf{C}\{u\}[u^{-1}])$, then

$$A_{G\nabla} = G^{-1}AG + G^{-1}dG$$

The *Riemann-Hilbert correspondence* is the following:

$$\{(\mathscr{H}, \nabla) \text{ with regular singularities}\} \leftrightarrow \{\text{conjugacy classes of } \exp(2\pi i A_0)\}.$$

What if $(\mathscr{H}, \nabla)$ does not have regular singularities? Here, the picture is complicated.

2.1. **Classification of irregular** $(\mathscr{H}, \nabla)$**.** We'll follow the plan below:

(1) First, there is a formal classification (no convergence condition). This is a theorem of Fuchs and Levelt-Turrittin.
(2) This is the actual classification; it is more recent and due to Malgrange-Deligne (see also Wazow, Hukuhara).

2.1.1. *Step 1.* We classify $(\mathscr{H}, \nabla)$ where

$$\nabla : \mathscr{H} \to \mathscr{H} \otimes \mathbf{C}(\!(u)\!)$$
$$G \in \mathrm{GL}_n\left(\mathbf{C}(\!(u)\!)\right)$$

Here $\mathbf{C}(\!(u)\!)$ denotes formal Laurent series in u.

Definition 2.2. A *Puiseux tail* is an expression of the form

$$\sum_{\substack{\lambda\in\mathbf{Q}\\ \lambda\leq -1}} c_\lambda u^\lambda$$

with only finitely many $c_\lambda \neq 0$ and u^λ is some branch of this power. These are taken modulo equivalence: we say

$$\sum_\lambda c_\lambda u^\lambda \sim \sum_\lambda c_\lambda e^{2\pi i m\lambda} u^\lambda$$

when $m \in \mathbf{Z}$. Note that this is a well-defined polynomial on the positive real axis and the expression is an analytic continuation. Every equivalence class under $\sim$ has finitely many representatives.

If $\underline{g}$ is a Puiseux tail, then $\underline{g}$ gives rise to an $\mathscr{E}^{\underline{g}} = (\mathscr{H}^{\underline{g}}, \nabla)$—a connection with an irregular singularity. The flat sections of this connection are the functions $\exp(g)$ for all $g \in \underline{g}$. More invariantly, we have a cover $\rho_N : \mathbf{C} \to \mathbf{C}$ sending t to $t^N = u$. Each g is a well-defined polynomial of t^{-1}. Then there is a connection

$$\big(\mathbf{C}\{t\}, d - dg\big).$$

Now $\mathscr{E}^{\underline{g}}$ is the pushforward of this connection.

Theorem 2.3 (Levelt-Turrittin)**.**

$$\big\{\textit{category of differential modules over } \mathbf{C}(\!(u)\!)\big\}$$
$$\bigoplus_{\underline{g}} \mathscr{E}^{\underline{g}} \otimes \big\{\textit{category of modules with regular singularities}\big\}$$

So for every $(\mathscr{H}, \nabla)$*, we can find a formal isomorphism*

$$(\mathscr{H}, \nabla) \cong \bigoplus_{\underline{g}_i} \mathscr{E}^{\underline{g}_i} \otimes (\mathscr{R}_i, \nabla_i)$$

where each $\mathscr{R}_i$ *is a module with regular singularities.*

2.1.2. *Step 2.* We need some extra data to reconstitute $\mathscr{E}^{\underline{g}_i} \otimes (\mathscr{R}_i, \nabla_i)$ into a convergent irregular connection. The extra data is the so-called *Stokes filtration* on $\mathscr{S} = (\mathscr{H}, \nabla)|_{S^1}$ labelled by some finite sets moving with a point on the circle. More precisely, labeling is given by a local system **Del** of vector spaces on S^1. For every $U \subseteq S^1$,

$$\Gamma(U, \mathbf{Del}) = \left\{ \omega \;\middle|\; \omega = \Bigg(\sum_{\substack{\lambda < -1 \\ \lambda \in \mathbf{Q}}} c_\lambda u^\lambda \Bigg) du \right\}$$

Here we choose branches of u^λ in the sector of $\mathbf{C}$ given by U. The germs of sections of **Del** are ordered: suppose $\omega', \omega'' \in \mathbf{Del}(U)$ and $\varphi \in U$. Then

$$\omega' - \omega'' = c^a u^a + \{\text{higher order terms}\}.$$

We say $\omega' <_\varphi \omega''$ if and only if

$$\Re \left(\frac{c_a e^{i\varphi(a+1)}}{a+1} \right) < 0.$$

For every $\varphi \in S^1$ and $\omega \in \mathbf{Del}_\varphi$, define $(\mathscr{S}_{\le \omega})_\varphi \subseteq S_\varphi$ where

$$(\mathscr{S}_{\le \omega})_\varphi = \big\{ s \in \Gamma(re^{i\varphi}, \mathscr{H})^\nabla \;\big|\; se^{-\int \omega} \text{ has moderate growth when } r \to 0 \big\}.$$

This condition means

$$\|e^{-\int \omega} s\|_{re^{i\varphi}} \sim \underline{O}(r^N)$$

where $N > 0$ is an integer.

Theorem 2.4 (Malgrange-Deligne)**.**

$$\{(\mathscr{H}, \nabla) \textit{ connections with poles}\} \leftrightarrow \{(\mathscr{S}, \{\mathscr{S}_{\le \omega}\}_{\omega \in \mathbf{Del}})\}$$

is an equivalence of categories.

There are special types of connections:

$$\begin{aligned}&\{\text{all meromorphic connections}\}\\ &\supseteq \{\text{mero. connections without ramification}\}\\ &\supseteq \{\text{mero. connections of exponential type}\}\\ &\supseteq \{\text{mero. connections with regular singularities}\}\end{aligned}$$

This corresponds to the chain of inclusions

$$\mathscr{E}^{\underline{g}} \otimes (\mathscr{R}, \nabla) \supseteq \mathscr{E}^{g} \otimes (\mathscr{R}, \nabla) \supseteq \mathscr{E}^{u} \otimes (\mathscr{R}, \nabla) \supseteq (\mathscr{R}, \nabla).$$

(in the second term, $g \in \mathbf{C}[u^{-1}]$).

3. Non-commutative Hodge structures

Yesterday, we classified all meromorphic finite rank connections in one variable. This classification consisted of two parts:

(1) The formal classification: we have

$$(\mathscr{H}, \nabla) = \bigoplus \mathscr{E}^{\underline{g}_i} \otimes (\mathscr{R}_i, \nabla_i).$$

Here, $\mathscr{E}^{\underline{g}_i}$ is a connection of the form $d - dg$, $g = \sum c_a u^a$, $a \in \mathbf{Q}_{\leq 1}$, and $(\mathscr{R}_i, \nabla_i)$ is a connection with regular singularities.

(2) The actual classification:

$$(\mathscr{H}, \nabla) \leftrightarrow (\mathscr{S}, (\mathscr{S}_{\leq \omega})_{\omega \in \mathbf{Del}})$$

Here $\mathscr{S}$ is the restriction of $(\mathscr{H}, \nabla)$ to S^1 and $\mathscr{S}_\omega$ is the Stokes filtration.

Definition 3.1. A *pure nc Hodge structure* is a triple $(H, \mathscr{E}_B, \mathbf{iso})$ where

(1) $H \to \mathbf{A}^1$ is an algebraic $\mathbf{Z}/2$-graded vector bundle.
(2) $\mathscr{E}_B$ is a $\mathbf{Z}/2$-graded local system of $\mathbf{Q}$-vector spaces on $\mathbf{A}^1 \setminus \{0\}$.
(3) $\mathbf{iso}$ is an isomorphism $\mathscr{E} \otimes \mathscr{O} \to H|_{\mathbf{A}^1 \setminus \{0\}}$.

Note that $\mathbf{iso}$ gives a flat holomorphic connection

$$\nabla : H|_{\mathbf{A}^1 \setminus \{0\}} \to H|_{\mathbf{A}^1 \setminus \{0\}} \otimes \Omega^1.$$

We require the following conditions:

(1) *Filtration axiom:* There is a frame for H near $0 \in \mathbf{A}^1$ such that ∇ has at most a second order pole in this frame, i.e.,

$$\nabla = d + A(u)u^{-2}du$$

(2) ∇ has at most a regular singularity at ∞.
(3) $\mathbf{Q}$-*structure axiom:* The Stokes filtrations are compatible with $\mathscr{E}_B$, i.e.,

$$(\mathscr{E}_B \cap \mathscr{S}_{\leq \omega}) \otimes \mathbf{C} = \mathscr{S}_{\leq \omega}.$$

(4) *Opposedness axiom:* The rational structure $\mathscr{S}_B = \mathscr{E}_B \cap \mathscr{S}$ gives a real structure on $\mathscr{S}$. This gives a complex conjugation map $\tau : \mathscr{S} \to \mathscr{S}$. Let $\gamma : \mathbf{P}^1 \to \mathbf{P}^1$ be the map $u \mapsto 1/u$. Then $H|_{\{u||u| \leq 1\}}$ and $\gamma^* \overline{H}|_{\{u||u| \geq 1\}}$ glue along S^1 via τ. This gives a holomorphic vector bundle $\hat{H}$ on $\mathbf{P}^1$. We require that $\hat{H}$ is trivial.

Remark 3.2. Suppose we have a pure rational Hodge structure of weight w, i.e.,

$$(V, V_{\mathbf{Q}}, F^{\bullet}V)$$

Then it gives rise to a natural pure nc Hodge structure as follows: define

$$\mathscr{T}_{w/2} = \left(\mathscr{O}, d - \frac{w}{2}\frac{du}{u}\right)$$
$$H = \xi(V, F^{\bullet}V) \to \mathbf{A}^1.$$

This is called the *Tate Hodge structure.* The Tate commutative Hodge structure is

$$\mathbf{Q}(1) = (\mathbf{C}, 2\pi i\mathbf{Q}, \mathbf{C}^{1,1})$$
$$\mathbf{Q}(n) = \mathbf{Q}(1)^{\otimes n} = (\mathbf{C}, \mathbf{C}^{n,n}, (2\pi i)^n)$$

Then we take

$$\left(\mathscr{T}_{w/2} \otimes H, \left(d - \frac{w}{2}\frac{du}{u}\right) \otimes \mathrm{id}_H + \mathrm{id}_{\mathscr{T}_{w/2}} \otimes d\right)$$

Denote the connection by ∇. It has a regular singularity with monodromy

$$\begin{cases} \mathrm{id} & \text{if } w/2 \text{ is even} \\ -\mathrm{id} & \text{if } w/2 \text{ is odd} \end{cases}$$

Moreover, ∇ preserves any rational structure on Hs, i.e., $(\mathscr{T}_{w/2} \otimes H)_{\mathbf{Q}}$.

This is a nc Hodge structure and it gives a functor

$$\begin{array}{ccc} \mathbf{HS_Q} & \longrightarrow & \mathbf{ncHS_Q} \\ \downarrow & & \uparrow \subseteq \\ (\mathbf{HS_Q})/\otimes\mathbf{Q}(1) & \longrightarrow & (\mathbf{ncHS_Q})^{\mathrm{reg}} \end{array}$$

This realizes commutative Hodge structures modulo Tate twists as a full subcategory of $\mathbf{ncHS_Q}$.

Fact 3.3. The categories

$$\mathbf{ncHS_Q} \supseteq (\mathbf{ncHS_Q})^{\mathrm{exp}} \supseteq (\mathbf{ncHS_Q})^{\mathrm{reg}}$$

are abelian categories.

3.1. **Historical remarks.**

(1) This structure was first written down about 20 years ago by Cecotti and Vafa (charges of superconformal field theories have nc Hodge structures)
(2) Kioji and Saito also studied this sort of data around the same time in ("exponential systems in singularity theory")
(3) See also Hertling ("TER structure") and Sabbah ("pure twistor $\mathscr{D}$-module")

3.2. **Examples.**

(1) Let (X, ω) be a compact symplectic manifold. Suppose that $\dim_{\mathbf{R}} X = 2d$. Any time we have such a manifold, we get a "large volume limit family" $(X, \log q\omega)$ indexed by $q \in \mathbf{C}$. The 3-point genus 0 Gromov-Witten invariants of (X, ω) give a deformation of the cup product on $\mathrm{H}^{\bullet}(X; \mathbf{C})$:

$$\star_q : \mathrm{H}^{\bullet}(X; \mathbf{C})^{\otimes 2} \to \mathrm{H}^{\bullet}(X; \mathbf{C}) \otimes \mathbf{C}_q$$

where

$$\mathbf{C}_q = \left\{ \sum_{i=1}^{\infty} c_i q^{E_i} \;\middle|\; c_i \in \mathbf{C},\, E_i \in \mathbf{R}, \text{ and } E_i \to \infty \right\}.$$

We assume that $\star_q$ is convergent near 0. (This is true if X is Kähler.) Now set

$$\begin{aligned} \mathscr{H} &= \mathrm{H}^{\bullet}(X;\mathbf{C}) \otimes \mathbf{C}\{u,q\} \\ \mathscr{H}^0 &= \Big(\bigoplus_{k \equiv d \pmod 2} \mathrm{H}^k \Big) \otimes \mathbf{C}\{u,q\} \\ \mathscr{H}^1 &= \Big(\bigoplus_{k \equiv d+1 \pmod 2} \mathrm{H}^k \Big) \otimes \mathbf{C}\{u,q\}. \end{aligned}$$

Define

$$\begin{aligned} \nabla_{\frac{\partial}{\partial u}} &= \frac{\partial}{\partial u} + u^{-2}(K_X \star_q \cdot) + u^{-1}\mathrm{Gr} \\ \nabla_{\frac{\partial}{\partial q}} &= \frac{\partial}{\partial q} - q^{-1}u^{-1}([\omega] \star_q \cdot) \end{aligned}$$

where K_X is the first Chern class of $T_X^{\vee}$ in any ω-adapted almost complex structure and

$$\mathrm{Gr}|_{\mathrm{H}^k} = \frac{k-d}{2}\,\mathrm{id}_{\mathrm{H}^k}\,.$$

What about $\mathscr{E}_B$? We can try

$$\mathscr{E}_B = \bigoplus_k \mathrm{H}^k(X;(2\pi i)^k\mathbf{Q}),$$

but *this doesn't work!* Instead, we have the following proposal:

$$\bigoplus_k \mathrm{H}^k(X;2\pi i\mathbf{Q}) \xrightarrow{(\cdot)\wedge\hat{\Gamma}(X)} \bigoplus \mathrm{H}^k(X;\mathbf{C})$$

Here $\hat{\Gamma}(X)$ is the Gamma class of X, defined by

$$\hat{\Gamma}(X) = \prod_{i=1}^{d} \Gamma(1+\delta_i)$$

where $\Gamma(\cdot)$ is the Gamma function and $\delta_1,\ldots,\delta_d$ are the Chern roots of T_X.

4. nc Hodge structures on symplectic manifolds and Landau-Ginzburg models

Last time, we described the following construction: given a symplectic manifold (X,ω) compact of dimension $2d$, there is a 1-parameter variation of nc Hodge structures (potentially). We have a triplet $((\mathscr{H},\nabla),\mathscr{E}_B,\mathbf{iso})$ where:

(1) $\mathscr{H}$ is a module over $\mathbf{C}\{u,q\}$.
(2) ∇ is an explicit meromorphic connection depending on $\star_q$ (the quantum product), the canonical bundle K_X, and $[\omega]$.
(3) $\mathscr{E}_B$ is a rational local system on $\mathbf{A}^1 \setminus \{0\}$ given by

$$\mathrm{H}^{\bullet}(X;\mathbf{Q}) \xrightarrow{(2\pi i)^{k/2}} \mathrm{H}^{\bullet}(X;\mathbf{C}) \xrightarrow{\wedge\hat{\Gamma}(X)} \mathrm{H}^{\bullet}(X;\mathbf{C})$$

Conjecture 4.1. This data gives a pure nc Hodge structure of exponential type. If X underlies a Calabi-Yau then this is a pure ordinary Hodge structure.

Some evidence for the conjecture is the following:

(1) The conjecture is true for X Calabi-Yau.
(2) The nc filtration axiom is always true. However, the exponential type is not clear in general; it is true for toric Fano and for general Fano which are complete intersections in toric varieties.
(3) The $\mathbf{Q}$-structure axiom is true for toric Fano Deligne-Mumford stacks.
(4) The opposedness axiom is true for Calabi-Yau and true for toric Fano and complete intersection Fano varieties for generic symplectic forms (Reichelt-Sevenheck).

This Hodge structure lives on the A-side of mirror symmetry. What is the analogue of this on the B-side? It turns out that the B-side analogue is a Hodge structure on the cohomology of a "holomorphic Landau-Ginzburg model." As far as geometric data is concerned, this is just a pair

$$Y \xrightarrow{w} \mathbf{C}$$

where Y is a complex manifold and w is a holomorphic function.

Before we describe the nc Hodge structure on the cohomology of (Y, w), we need a different interpretation of the data $(\mathscr{H}, \mathscr{E}_B, \mathbf{iso})$. We can forget everything but $\mathscr{E}_B$ (the Betti data) or $(\mathscr{H}, \nabla)$ (the de Rham data).

Theorem 4.2. *Consider the category of triples* $(\mathscr{H}, \mathscr{E}_B, \mathbf{iso})$ *satisfying the nc-filtration (exp) axiom and the* $\mathbf{Q}$*-structure (exp) axiom. This category is equivalent to the category of triples* $((\mathscr{H}, \nabla), \mathscr{F}_B, f)$ *where:*

(1) $(\mathscr{H}, \nabla)$ *still satisfies the (exp version of the) nc filtration axiom.*
(2) $\mathscr{F}_B \in \mathrm{Constr}(\mathbf{A}^1, \mathbf{Q})$ *such that* $R\Gamma(\mathbf{A}^1, \mathscr{F}_B) = 0$.
(3) $f : \mathscr{F}_B \otimes \mathbf{C} \to \mathrm{DR}(i_*(\widehat{\mathscr{H}, \nabla}))$. *is an isomorphism*

where

(1) i *is the map* $\mathbf{A}^1 \setminus \{0\} \to \mathbf{A}^1$.
(2) $(\widehat{-})$ *is the Fourier transform for* $\mathscr{D}$*-modules.*
(3) DR *is the de Rham complex of a* $\mathscr{D}$*-module.*

This gives a description of nc Hodge structures via gluing.

Theorem 4.3. *Specifying a nc Hodge structure* $(H, \mathscr{E}_B, \mathbf{iso})$ *of exponential type is equivalent to specifying:*

(1) *(regular type) a finite set*

$$S = \{c_1, \ldots, c_k\} \subset \mathbf{A}^1$$

and a collection $(R_i, \mathscr{E}_{B,i}, \mathbf{iso}_i)$ *of nc Hodge structures which are regular.*

(2) *(gluing data) a base point* $c_0 \in \mathbf{A}^1$ *and a collection of paths* $c_0 \to c_i$, $i > 0$ *together with* $\mathbf{Q}$*-linear isomorphisms*

$$T_{ij} : (\mathscr{E}_{B,j})_\infty \to (\mathscr{E}_{B,i})_\infty$$

Suppose now we have a holomorphic Landau-Ginzburg model $Y \xrightarrow{w} \mathbf{C}$ where Y is quasi-projective and the critical locus of w is proper. Make $S = \{c_1, \ldots, c_k\}$ the set of critical values of w. Fix $c_0 \in \mathbf{A}^1 \setminus S$ and paths $c_0 \to c_i$, $i > 0$ (actually, we should thicken these into discs D_i). Define

$$U_i = \mathrm{H}^m(w^{-1}(D_i), w^{-1}(c_0); \mathbf{Q})$$

and set

$$U = \bigoplus_{i=1}^{k} U_i = \mathrm{H}^m(Y, w^{-1}(c_0); \mathbf{Q})$$

Finally, let $T_{ii} : U_i \to U_i$ be the monodromy along ∂D_i. This gives $\mathscr{F}_B$: explicitly,

$$(\mathscr{F}_B)_z = \mathrm{H}^m(Y, w^{-1}(z); \mathbf{Q}),$$

and this has no global cohomology. The regular piece at c_i is

$$\mathscr{H}^\bullet = \mathbb{H}^{\bullet \mathrm{mod} 2}_{\mathrm{Zar}}(Y_i, (\Omega^\bullet_{Y_i}, ud + dw)) \otimes \mathbf{C}\{u\}$$

and we define $\mathrm{H}^\bullet_{\mathrm{dR}}((Y_i, w), \mathbf{C})$ to be this module (here $Y_i = w^{-1}(D_i)$). There is an isomorphism

$$\Phi : \mathrm{H}^\bullet(Y, w^{-1}(c_0), \mathbf{Q}) \otimes \mathbf{C} \xrightarrow{\sim} \mathrm{H}_{\mathrm{dR}}(Y, w)$$

given by oscillating integrals.

4.1. **Mirror symmetry.** The "pedestrian statement" of Mirror Symmetry is that a symplectic manifold (X, ω) is a mirror to a Landau-Ginzburg (Y, w) if the nc Hodge structures for the two are isomorphic. All this is supposed to be a corollary of Homological Mirror Symmetry. In fact, (X, ω) ill give rise to an A_∞ category of boundary theories (D-branes) and this is the Fukaya category $\mathrm{Fuk}(X, \omega)$. On the other hand, (Y, w) gives rise to an A_∞-category $\mathbb{D}^b(Y, w)$. What homological mirror symmetry asserts is the following: if (X, ω) and (Y, w) are mirror, then $\mathrm{Fuk}(X, \omega) \cong \mathbb{D}^b(Y, w)$ and $\mathbf{ncHS}(X, \omega) \cong \mathbf{ncHS}(Y, w)$.

4.2. **The main conjecture.** The idea is that the primary object is a category $\mathrm{Fuk}(X, \omega)$ and we should be doing geometry with that category. Let's make a simplifying assumption.

Definition 4.4. (1) A differential $\mathbf{Z}/2$-graded (d$\mathbf{Z}/2$g) category is called *affine* if it is of the form ${}_A\mathbf{Mod}$ where A is a d$\mathbf{Z}/2$g algebra.
(2) A *nc space* is a d$\mathbf{Z}/2$g graded category $\mathscr{C}$ which has all small limits and colimits.
(3) An object $E \in \mathscr{C}$ is called *perfect* if $\mathrm{Hom}_{\mathscr{C}}(E, -)$ commutes with colimits.
(4) A nc space $\mathscr{C}$ is *smooth* if $A \in \mathrm{Perf}({}_{A \otimes A^{\mathrm{op}}}\mathbf{Mod})$ and *compact* if $\dim_{\mathbf{C}} \mathrm{H}^\bullet(A) < \infty$.

If X is a scheme (quasi-compact and quasi-separated over $\mathbf{C}$) then $\mathbb{D}(\mathrm{Qcoh}(X))$ is a nc space; in fact it is an affine nc space and it is smooth (resp. compact) if and only if X is smooth (resp. compact). If $\mathscr{C}$ is an affine nc space we can define $\mathrm{H}^\bullet_{\mathrm{dR}}(\mathscr{C})$ to be $\mathrm{HP}_\bullet(A)$ (this is a module over $\mathbf{C}((u))$ and deforms naturally $\mathrm{HH}_\bullet(A)$) and $\mathrm{H}^\bullet_{\mathrm{Dol}}(\mathscr{C}) = \mathrm{HH}_\bullet(A)$.

Conjecture 4.5 (Main conjecture). $\mathrm{H}^\bullet_{\mathrm{dR}}(\mathscr{C})$ and $\mathrm{H}^\bullet_{\mathrm{Dol}}(\mathscr{C})$ fit into a pure nc Hodge structure when $\mathscr{C}$ is smooth and compact.

5. Mirror symmetry for Calabi-Yau threefolds and normal functions

A *mirror pair* of Calabi-Yau threefolds (CY_3's) is two one-parameter families. A *model family* is a pair (X, ω) where X is a compact real 6-manifold and ω is a (closed?) symplectic form that underlies a Ricci flat Kähler metric. This gives a family $(X, \log q\omega)$, where q is a complex parameter (the form ω is complexified). Finally, a *B-model family* is a holomorphic family $\hat{\mathscr{X}} \to D$ of projective CY_3's over the complex disc D (with coordinate q) so that

1. $\hat{\mathscr{X}}_q$ is smooth for $q \neq 0$.
2. $\hat{\mathscr{X}}_0$ is a normal crossing.
3. We have a global relative non-vanishing 3-form on $\hat{\mathscr{X}}$.
4. The monodromy $T : \mathrm{H}^3(\hat{\mathscr{X}}_q) \to \mathrm{H}^3(\hat{\mathscr{X}}_q)$ is maximally quasi-unipotent, i.e., if $d > 0$ then $(T^d - \mathrm{id})^3 = 0$.
5. We have a strong equivalence of the associated categories of D-branes, i.e., $\mathrm{Fuk}(X, \omega) \cong \mathscr{P}(\mathscr{X})$. Here $\mathscr{P}(\mathscr{X})$ is the dg enhancement of the derived category of coherent sheaves on $\mathscr{X}$ over $\Gamma(\mathscr{O}_D)$; and $\mathrm{Fuk}(X, \omega)$ is the Fukaya category, which is a linear A_∞ category over $\Gamma(\mathscr{O}_D)$. This isomorphism is required to be *strong,* i.e., it exchanges Calabi-Yau structures (both categories are Calabi-Yau A_∞ categories).

We are supposed to think of $\mathrm{Fuk}(X, \omega)$ as the category of topological A-branes and $\mathscr{P}(\mathscr{X})$ as the category of topological B-branes. The equivalence ought to identify moduli of branes; this follows for moduli problems that are defined intrinsically. In particular, homological mirror symmetry will give an iso between germs of moduli $a \in \mathrm{ob}\,\mathrm{Fuk}$ and $b \in \mathrm{ob}\,\mathscr{P}(\mathscr{X})$ as formal stacks. Unfortunately, in physics, moduli spaces are always critical loci of potentials. What we have are spaces

$$\begin{array}{ccc} \mathrm{Branes}^{A,\text{off-shell}} & & \mathrm{Branes}^{B,\text{off-shell}} \\ \downarrow{\scriptstyle w_A} & & \downarrow{\scriptstyle w_B} \\ \mathbf{C} & & \mathbf{C} \end{array}$$

and $\mathrm{crit}(w_A) = \mathrm{Topo}^A$ (which comes with a dg structure) is (weakly?) equivalent to $\mathrm{crit}(w_B) = \mathrm{Topo}^B$.

Let's first discuss the B-side, which is easier. Then $\mathscr{P}(\mathscr{X})$ is the category of all graded C^∞ complex vector bundles equipped with an integrable $(0, \bullet)$ superconnection. A typical object is $(\mathbf{E}, \boldsymbol{\nabla})$ with $\mathbf{E} = \bigoplus_i \mathbf{E}_i$ and

$$\boldsymbol{\nabla} : \mathbf{E} \to A^{0,\bullet}_{\mathscr{X}}(\mathbf{E})$$

so that $\boldsymbol{\nabla} \in (\mathrm{End}_{\mathbf{C}}\,\mathbf{E} \otimes A^{0,\bullet}_{\mathscr{X}})^1$ satisfies the graded Leibniz rule with respect to $\overline{\partial}$. If

$$E_1 \xrightarrow{d_1} E_2 \xrightarrow{d_2} \cdots \xrightarrow{d_{k-1}} E_k$$

is a complex of holomorphic vector bundles, then $\mathbf{E}_i$ is the underlying C^∞ vector bundle and $\boldsymbol{\nabla}$ is a deformation of d.Then

$$\mathrm{Branes}^{\text{off-shell},B} = \left\{ (\mathbf{E}, \boldsymbol{\nabla}) \;\middle|\; \boldsymbol{\nabla}\text{: we no longer require } \boldsymbol{\nabla}^2 = 0 \right\}$$

If $\mathrm{hms}(a) = (\mathbf{E}, \boldsymbol{\nabla})$, $\boldsymbol{\nabla}$ integrable, then

$$\mathrm{Branes}^{\text{off-shell},B} = \Gamma(\mathscr{X}, (\mathrm{End}\,\mathbf{E} \otimes A^{0,\bullet})^1)$$

and w_B is holomorphic Chern-Simons:

$$w_B(\mathbf{\nabla} + \mathbf{A}) = \int_{\mathscr{X}_q} \mathrm{tr}(\mathbf{\nabla}\mathbf{A} \wedge \mathbf{A} + \frac{2}{3}\mathbf{A}^3) \wedge \Omega_q.$$

Remark 5.1. Both the deformation theory of $(\mathbf{E}, \mathbf{\nabla})$ inside $\mathscr{P}(\mathscr{X})$ and the fact that deformations are critical points of w_B give derived structures on Topo^B.

Recall that if X is a scheme, a dg structure on X is the following data:

$$RX = (\underline{RX}, \mathscr{O}_{\underline{RX}})$$

where

(1) $\underline{RX}$ is a scheme and $X \subseteq \underline{RX}$ is a closed subscheme.
(2) $\mathscr{O}^{\bullet}_{\underline{RX}}$ is a sheaf of dg algebras on $\underline{RX}$ with quasi-coherent terms and coherent cohomology (differential $\mathscr{O}$-linear)
(3) $\mathscr{H}(\mathscr{O}_{\underline{RX}}) = \mathscr{O}_X$.

Here are some examples:

(1) Suppose M is a manifold and $w : M \to \mathbf{C}$ is a map. Then $X = \mathrm{crit}(w)$ has a natural derived structure as

$$RX = (M, (\bigwedge^{\bullet} T_M, \lrcorner dw))$$

(2) If $L^{\bullet}$ is a $\mathbf{Z}_{\geq 0}$-graded L_∞-algebra, we get

$$RX = \left[(\mathrm{Spec}(\mathrm{Sym}\, L^{\vee}_{\geq 1}[-1]), Q_-)/\exp L^0)\right]$$

and

$$X = [\mathrm{MC}(L)/\exp L^0].$$

6. Derived structures on the moduli of A-branes

The setup is the same as that of yesterday: we have a 6-dimensional symplectic manifold (X, ω) underlying a compact Calabi-Yau threefold. We get a family $(X, \log q\omega)$ parameterized by the punctured disc; this gives rise to the linear A_∞ Fukaya category $\mathrm{Fuk}(X, \omega)$ over $\mathbf{C}_q$. We make the assumption that it is really linear over $\Gamma(D, \mathscr{O}_D)$ (where D is a small disc near 0 in the q-line).

Locally, the moduli problem we're interested in solving is the following: we fix an object a in $\mathrm{Fuk}(X, \omega)$ and look at the germ of deformations of a: $\mathrm{Topo}^A \to D$. The issue is that Topo^A is a formal stack but also has a dg structure, in fact it has two natural ones:

(1) Topo^A has a dg structure coming from the deformation theory of a as an object of $\mathrm{Fuk}(X, \omega)$
(2) There is also a dg structure because Topo^A is the critical locus $\mathrm{crit}(w_A)$ of a potential

$$w_A : \mathrm{Branes}^{A,\text{off-shell}} \to \mathbf{C}.$$

Because of this, Topo^A gets a dg structure from the Koszul complex.

In fact, these are the same.

6.1. **The construction of the Fukaya category.** We will only address "restricted Fukaya categories" (and only in a hand-waving manner). These are built in stages:

(1) Construct a $\mathbf{C}$-linear dg category $\mathscr{F}_0$ (the large volume limit).
(2) Deform $\mathscr{F}_0$ to a "weak" A_∞ category $\mathscr{F}$ over $\Gamma(\mathscr{O}_D)$.
(3) Complete $\mathscr{F}$ algebraically (i.e., "Karoubi close and pass to twisted complexes") to produce $\mathrm{Fuk}(X, \omega)$.

For step 1, fix a collection of smooth pairwise transversal Lagrangians $\{L_i\}_{i=1}^k$. The union $L_1 \cup \cdots \cup L_k$ will be the skeleton. Let $\mathrm{ob}\,\mathscr{F}_0 = \{(L_i, \mathbf{V})$ where $\mathbf{V} = (V, \nabla)$ is a complex local system on L_i. We then let

$$\mathrm{Hom}\big((L_i, \mathbf{V}), (L_j, \mathbf{V}')\big) = \bigoplus_{p \in L_i \pitchfork L_j} \mathrm{Hom}(\mathbf{V}_p, \mathbf{V}'_p).$$

if $i \neq j$. There are some subtle grading issues here, but no differential. On the other hand,if $i = j$

$$\mathrm{Hom}\big((L_i, \mathbf{V}), (L_j, \mathbf{V}')\big) = \Gamma(L_i, (A^\bullet(\mathrm{Hom}(V, V')), d^{\nabla, \nabla'})).$$

For step 2, we must deform everything by counting pseudo-holomorphic discs. Note that our objects in $\mathscr{F}_0$, say support on L_i, are really dg modules over $(A^\bullet_{\mathbf{C}}(L), d, \cap)$, $\mathrm{tr} = \int_L$. The disc instantons correct the products

$$m_1^0 = d$$
$$m_2^0 = \wedge$$
$$m_3^0 = 0$$
$$\vdots$$

to get

$$m_1^q = m_1^0 + q \text{ corrections}$$
$$m_2^q = m_2^0 + q \text{ corrections}$$
$$m_3^q = m_3^0 + q \text{ corrections}$$
$$\vdots$$

What are the q corrections? Fix a Lagrangian L. We consider the moduli space $\mathscr{M}_{n+1,L}$ of

$$f : (D, x_1, \ldots, x_{n+1}) \to X$$

where $D = \{|z| \leq 1\} \subset \mathbf{C}$, $x_1, \ldots, x_{n+1} \in \partial D$, f is pseudo-holomorphic when restricted to the interior of D, and $f(\partial D) \subseteq L$.

Remark 6.1. (1) $\mathscr{M}_{n+1,L}$ exist for generic almost complex structures.
(2) $\mathscr{M}_{n+1,L}$ is non-compact in general.
(3) $\mathscr{M}_{n+1,L}$ has corners in general.
(4) For $\dim X = 6$ it has no corners.
(5) $\mathscr{M}_{n+1,L}$ modulo its boundary has a virtual fundamental class.
(6) $\dim \mathscr{M}_{n+1,L} = n + 1$.

We have an evaluation map $\mathrm{ev} : \mathscr{M}_{n+1,L} \to L^{\times(n+1)}$ given by

$$(D, x_1, \ldots, x_{n+1}, f) \mapsto (f(x_1), \ldots, f(x_n)).$$

We get a chain

$$\mathrm{ev}_*([\mathscr{M}_{n+1,L}]^{\mathrm{vir}}) \subseteq L^{\times(n+1)}$$

and then a functional

$$A^\bullet_{\mathbf{C}}(L^{\times(n+1)}) \to \mathbf{C} \quad \text{given by} \quad \alpha \mapsto \int_{\mathrm{ev}_A(\mathscr{M}_{n+1,L})} \alpha.$$

We rewrite as an operator

$$\varphi : A^\bullet_{\mathbf{C}}(L^{\times n}) \to A^\bullet_{\mathbf{C}}(L).$$

Note that $\mathscr{M}_{n+1,L}$ has connected components labelled by $\pi_2(X,L)$ (or more precisely, the set of connected components maps onto $\pi_2(X,L)$ with finite fibers). Note we get one φ_η for each $\eta \in \pi_0(\mathscr{M}_{n+1,L}) \sim \pi_2(X,L)$. For each map $(D, \underline{x}, f)$ we have $\int_D f^*\omega$ depending only on the connected component of $(D, \underline{x}, f)$. We then introduce the following definition:

$$m_k^q = m_k^0 + \sum_{\eta \in \pi_0(\mathscr{M}_{k+1,L})} q^{a_\eta} \varphi_\eta.$$

These m_k^q give also a correction to the Chern-Simons functional: let

$$\mathrm{Branes}^{A,\text{off-shell}} = \text{all complex (super-)connections on } L \otimes \Gamma(\mathscr{O}_D)$$

At $q = 0$,

$$w^A : \mathrm{Branes}^{A,\text{off-shell}} \to \mathbf{C}$$

is ordinary Chern-Simons:

$$(L, \mathbf{V}), \mathbf{V} = (V, \nabla^0 + A) \mapsto \int_L (A \wedge dA + \frac{2}{3} A^3)$$

At any q,

$$w^A : \mathrm{Branes}^{A,\text{off-shell}} \to \mathbf{C}$$

is

$$(L, \mathbf{V}), \mathbf{V} = (V, \nabla^0 + A) \mapsto \sum_{k \geq 0} \frac{1}{k+1} \int_L \mathrm{tr}(m_k^q(A, \ldots, A) \wedge A)$$

Now the objects in $\mathrm{Fuk}(X, \omega)$ coming from a at large volume are critical points of w^A.

Now we want to say that the two derived structures are the same. We can expand w_A as a function of q (in fact, add also m_{-1}^q, which is $\sum_\eta \# \mathrm{ev}_*(\mathscr{M}_{0,L}) q^{a_\eta}$). Assume we have a generic almost complex structure so that

$$\mathscr{M}_{0,L} = \{D_1, D_2, \ldots\}$$

with $\operatorname{sgn} D_i = \pm 1$ and $a_{D_i} \to \infty$ as $i \to \infty$ and with the ∂D_i disjoint. We then have

$$w_A(A) = \mathrm{CS}(A) + \sum_i \mathrm{sgn}(D_i) \sum_k \mathrm{tr}(I_i^k) \frac{q^{k a_i}}{k^2}.$$

where

$$\mathrm{tr}(I_i^k) = \sum_n \frac{1}{n+1} \mathrm{tr}(m_n^q(A, \ldots, A) \cdot A)$$

Rewriting as an iterated integral on $(S^1)^{n+1}$, we find

$$\mathrm{tr}(I_i^k) = \mathrm{tr}((\text{monodromy of } A \text{ on } \partial D_i)^k)$$

In particular, we take $I_i = \exp(\int_{\partial D_i} A_i)$ (up to some normalization). Computing $\partial w_A/\partial A$ we get critical points

$$A \in \Gamma_{L^2}(L, \operatorname{End} V \otimes \mathbf{A}^1)$$

such that A is smooth on $L \setminus \bigcup_i \partial D_i$ and

$$F_A = \sum_i \operatorname{sgn}(D_i) \log(1 - q^{a_i} T^A_{x,\partial D_i}) \delta_{\partial D_i}$$

where $T^A_{x,\partial D_i}$ is the holonomy of A based at x. In this way we get flat connections on $L \setminus (\bigcup_i D_i)$ with monodromy constrained so that

$$1 - q^{a_i} T^A_{\delta_i} = (T^A_{\sigma_i})^{\operatorname{sgn} D_i}$$

where $\delta_i \sim \partial D_i$ and σ_i is the linking circle. In fact,

$$\operatorname{Fuk}_{\operatorname{Supp} L}(X, \omega) = \operatorname{Rep} \frac{\mathbf{C}\pi_1\Big(L \setminus \bigcup_i \partial D_i\Big) \otimes \Gamma(\mathscr{O}_D)}{\langle 1 - q_i^a \delta_i = \sigma_i^{\operatorname{sgn} D_i} \rangle}$$

Theorem 6.2. $\operatorname{Rep} \to \operatorname{Fuk}_{\operatorname{Supp} L}(X, \omega)$ *is étale.*