1. Introduction

The goal of this document is to make the following calculation.

Theorem 1.1. Let $M$ be a closed 5-manifold and $B \in H^2(M; \mathbb{Z}/2)$. Then,

\begin{equation}
\langle BSq^1 B + Sq^2 Sq^1 B, [M] \rangle = \frac{1}{2} \langle \bar{w}_1 - \Psi(B), [M] \rangle.
\end{equation}

The right-hand side uses some unfamiliar notation, which we proceed to define.

Lemma 1.3. If $\mathcal{Z}_{w_1}$ denotes the orientation local system, $H^1(BO_1, \mathcal{Z}_{w_1}) \cong \mathbb{Z}/2$.

Indeed, this is the group cohomology $H^1(\mathbb{Z}/2, \mathbb{Z}_\sigma)$, where $\mathbb{Z}_\sigma$ denotes $\mathbb{Z}$ with the sign action.

Definition 1.4. The pullback of the nonzero element of $H^1(BO_1; \mathcal{Z}_{w_1})$ under the determinant map $B \det : BO_n \to BO_1$ is called the twisted first Stiefel-Whitney class $\bar{w}_1 \in H^1(BO_n; \mathcal{Z}_{w_1})$.

Hence this defines a twisted first Stiefel-Whitney class of any real vector bundle, which lives in cohomology twisted by the orientation bundle. Its mod 2 reduction is the usual first Stiefel-Whitney class in untwisted $\mathbb{Z}/2$-cohomology. In (1.2), we consider its reduction $\bar{w}_1 \in H^1(BO_n; \mathcal{Z}_{w_1})$, twisted mod 4 cohomology.

Next, $\Psi$ denotes the Pontrjagin square $\Psi : H^2(M; \mathbb{Z}/2) \to H^4(M; \mathbb{Z}/4)$ (though it exists in greater generality). It is the realization of the idea that if you know an element of $\mathbb{Z}/2$, you know $x^2$ mod 4.

On the right-hand side of (1.2), we use cup and cap products in twisted $\mathbb{Z}/4$-cohomology: if $[M]$ denotes the fundamental class in twisted $\mathbb{Z}/4$-cohomology, this is

\begin{equation}
H^1(M; (\mathbb{Z}/4)_{w_1}) \otimes H^4(M; \mathbb{Z}/4) \xrightarrow{\cong} H^5(M; (\mathbb{Z}/4)_{w_1}) \xrightarrow{[M]} \mathbb{Z}/4.
\end{equation}

However, since $2\bar{w}_1 = 0$, $\langle \bar{w}_1 - \Psi(B), [M] \rangle$ is even, and so it makes sense to divide by 2 and obtain an element of $\mathbb{Z}/2$, so we can compare with the left-hand side of (1.2).

We’ll prove Theorem 1.1 in three steps:

1. First, prove that both sides of (1.2) are cobordism invariants for a certain class of manifolds.
2. Then, determine generating manifolds for the group of cobordism classes of those manifolds.
3. Finally, verify (1.2) on the generators.

2. Cobordism-invariance

To capture the notion of cobordism of a manifold and a degree-2 cohomology class, we consider cobordism of manifolds with a $\mathbb{Z}/2$-gerbe, or equivalently, manifolds $M$ together with a degree-$2$ $\mathbb{Z}/2$ cohomology class $B$, where $(M, B)$ bounds if $M = \partial W$ for $W$ compact and $B$ extends over $W$. For the rest of this document, cobordism-invariant, cobordism groups, etc., refers to this kind of cobordism unless otherwise specified.

The classifying space for this structure is $BO_n \times K(\mathbb{Z}/2, 2)$, so the cobordism groups are the homotopy groups of the Thom spectrum of the virtual bundle $(V_n - \mathbb{R}^n) \to BO_n \times K(\mathbb{Z}/2, 2)$ (here $V_n \to BO_n$ is the tautological bundle).

Lemma 2.1. Let $E \to X$ and $F \to Y$ be virtual vector bundles. The Thom space of $E \oplus F \to X \times Y$ is $\text{Thom}(E) \wedge \text{Thom}(F)$.

We’re looking at $(V_n - \mathbb{R}^n) \oplus 0$, hence obtain $MO \wedge K(\mathbb{Z}/2, 2)_+$ (the Thom space of the zero bundle on $X$ is $X/\emptyset = X_+$). The cobordism group we want is $\pi_5$ of this spectrum.
**Proposition 2.2.** The quantity \( \langle \tilde{w}_1 \sim \Psi(B), [M] \rangle \) is a bordism invariant, and in particular a group homomorphism \( \Omega^0_c(K(\mathbb{Z}/2, 2)) \to \mathbb{Z}/2 \).

**Proof.** This quantity is additive under disjoint union, so it suffices to show that it vanishes when \( M \) bounds. Let \((M, B)\) bound, i.e. \( M \) is a closed 5-manifold, \( B \in H^3(M; \mathbb{Z}/2) \), and there’s a compact manifold \( W \) and a \( \tilde{B} \in H^2(W; \mathbb{Z}/2) \) such that \( M = \partial W \) and if \( i: M \to W \) is inclusion, \( B = i^* \tilde{B} \). Then, \( TW|_M \cong TM \oplus \mathbb{R} \) (using the outward normal vector field), so \( i^* \tilde{w}_1(W) = \tilde{w}_1(M) \). By naturality \( i^* \Psi(\tilde{B}) = \Psi(B) \). In the long exact sequence for \((W, M)\),

\[
\begin{align*}
H^n(W; (\mathbb{Z}/4)_{w_1}) &\xrightarrow{i^*} H^n(M; (\mathbb{Z}/4)_{w_1}) \xrightarrow{\delta} H^{n+1}(W, M; (\mathbb{Z}/4)_{w_1}),
\end{align*}
\]

so \( \tilde{w}_1(M) \Psi(B) \in \text{Im}(i^*) = \ker(\delta) \).

Let \([W, M] \in H_{n+1}(W, M; (\mathbb{Z}/4)_{w_1})\) denote the fundamental class of the pair, and \([M] \in H_n(M; (\mathbb{Z}/4)_{w_1})\) denote the fundamental class. Under the connecting morphism \( \partial: H_{n+1}(W, M; (\mathbb{Z}/4)_{w_1}) \to H_n(M; (\mathbb{Z}/4)_{w_1}) \), \([W, M] \mapsto [M]\). Lefschetz duality gives us a version of Stokes' theorem: if \( x \in H^n(M; (\mathbb{Z}/4)_{w_1}) \), then

\[
\langle x, \partial[W, M] \rangle = \langle \delta x, [W] \rangle.
\]

Hence

\[
\langle \tilde{w}_1(M) \Psi(B), [M] \rangle = \langle \tilde{w}_1(M) \Psi(B), \partial[W, M] \rangle = \langle \delta(\tilde{w}_1(M) \Psi(B)), [W, M] \rangle = 0.
\]

**Corollary 2.8.** The quantity \( \langle B \text{Sq}^1 B + \text{Sq}^2 \text{Sq}^1 B, [M] \rangle \) is a bordism invariant.

**Proof.** Let \( B \) denote the tautological class in \( H^2(K(\mathbb{Z}/2, 2); \mathbb{Z}/2) \). The quantity \( B \text{Sq}^1 B + \text{Sq}^2 \text{Sq}^1 B \) is a Whitney number for \( X = K(\mathbb{Z}/2, 2) \), \( a = B \text{Sq}^1 B + \text{Sq}^2 \text{Sq}^1 B \in H^5(K(\mathbb{Z}/2, 2); \mathbb{Z}/2) \), and \( p = 1 \), hence is a cobordism invariant.

3. Computing the cobordism group

**Proposition 3.1.** (Serre [Ser53]). \( H^*(K(\mathbb{Z}/2, 2); \mathbb{Z}/2) \) is generated by \( \text{Sq}^1 B \), where \( B \in H^2(K(\mathbb{Z}/2, 2); \mathbb{Z}/2) \) is the tautological class, and we consider admissible sequences \( I = (i_1, \ldots, i_m) \) such that \( i_j \geq 2i_{j+1} \) and \( 2i_j - \sum_{j \geq 1} i_j < 2 \).

**Corollary 3.2.** The low-dimensional mod 2 homology groups of \( K(\mathbb{Z}/2, 2) \) are:

\[
H_i(K(\mathbb{Z}/2, 2); \mathbb{Z}/2) \cong \begin{cases} 
\mathbb{Z}/2, & i = 0, 2, 3, 4 \\
(\mathbb{Z}/2)^{\oplus 2}, & i = 5 \\
0, & i = 1.
\end{cases}
\]

**Proof.** The low-degree generators of cohomology are 1 (degree 0), \( B \) (degree 2), \( \text{Sq}^1 B \) (degree 3) \( B^2 \) (degree 4), and \( \text{Sq}^2 \text{Sq}^1 B \) and \( B \text{Sq}^1 B \) (degree 5). To get homology, use the universal coefficient theorem over the field \( \mathbb{F}_2 \), so all Ext terms vanish and the homology and cohomology are isomorphic.

**Proposition 3.4.** \( \pi_5(MO \wedge K(\mathbb{Z}/2, 2)_+) \cong (\mathbb{Z}/2)^{\oplus 4} \).
This is a finite coproduct, hence equivalent to a finite product.

For the purpose of taking \(\pi_5\), we can 5-truncate to obtain

\[
H\mathbb{F}_2 \wedge K(\mathbb{Z}/2, 2) \wedge \Sigma^2 H\mathbb{F}_2 \wedge K(\mathbb{Z}/2, 2) \wedge \Sigma^4 H\mathbb{F}_2 \wedge K(\mathbb{Z}/2, 2) \wedge \Sigma^5 H\mathbb{F}_2 \wedge K(\mathbb{Z}/2, 2).
\]

This is a finite coproduct, hence equivalent to a finite product. \(\pi_5\) commutes with products, and finite products and sums of abelian groups are the same, so we now have

\[
\pi_5(MO \wedge K(\mathbb{Z}/2, 2)) = \pi_5(H\mathbb{F}_2 \wedge K(\mathbb{Z}/2, 2)) \oplus \pi_5(\Sigma^2 H\mathbb{F}_2 \wedge K(\mathbb{Z}/2, 2)) \oplus (\pi_5(\Sigma^4 H\mathbb{F}_2 \wedge K(\mathbb{Z}/2, 2)))^2
\]

(3.6a)

\[
= H_5(K(\mathbb{Z}/2, 2); \mathbb{Z}/2) \oplus H_3(K(\mathbb{Z}/2, 2); \mathbb{Z}/2) \oplus (H_1(K(\mathbb{Z}/2, 2); \mathbb{Z}/2))^2 \oplus H_0(K(\mathbb{Z}/2, 2); \mathbb{Z}/2).
\]

Plug in the results from Corollary 3.2 and we’re done.

\(\square\)

## 4. Finding the Generators

Proposition 2.6 tells us that we can use Whitney numbers to determine whether a candidate set of generators is linearly independent. Using Proposition 3.1, the relevant cohomology classes are \(1 \in H^0, B \in H^2, Sq^1 B \in H^3, B^2 \in H^4,\) and \(BSq^1 B\) and \(Sq^2 Sq^1 B \in H^5\). Thus we can write down the Whitney numbers, and some of them coincide.

- When \(a = 1\), we get ordinary Stiefel-Whitney numbers for 5-manifolds. Using the Wu formula, one can show they’re all either 0 or determined by \(w_2 w_3\).
- When \(a = B\), we get \(w_3 B, w_2 w_1 B,\) and \(w_1^3 B\). However, \(w_2 w_1 = v_3\) on any closed manifold, and \(v_3 = 0\) on any 5-manifold.
- When \(a = Sq^1 B\), we get \(w_2 Sq^1 B\) and \(w_1^2 Sq^1 B\). However,

\[
Sq^1(w_1^2 B) = Sq^1(w_1^2) B + w_1^2 Sq^1 B = w_1^2 Sq^1 B,
\]

and since \(v_1 = w_1\),

\[
Sq^1(w_1^2 B) = v_1 w_1^2 B = w_1^3 B,
\]

so this class isn’t anything new; similarly, because \(Sq^1 w_2 = w_3\) on a 5-manifold,

\[
Sq^1(w_2 B) = Sq^1 w_2 B + w_2 Sq^1 B = w_3 B + w_2 Sq^1 B,
\]

and because \(w_1 w_2 = 0\) as noted above,

\[
Sq^1(w_2 B) = v_1 w_1 B = w_1 w_2 B = 0.
\]

Thus \(w_2 Sq^1 B = w_3 B,\) and this isn’t anything new either.
- When \(a = B^2\), we get \(w_1 B^2\). Since \(w_1 = v_1\),

\[
w_1 B^2 = v_1 B^2 = Sq^1(B^2) = 0.
\]

- For \(a = Sq^2 Sq^1 B\), we must let \(p = 1\), giving the Whitney number \(Sq^2 Sq^1 B\). This is

\[
Sq^2 Sq^1 B = v_2 Sq^1 B = (w_2 + w_1) Sq^1 B,
\]

so this is a sum of terms we’ve already accounted for.
- Finally, we can take \(a = BSq^1 B\), again forcing \(p = 1\) and the Whitney number \(BSq^1 B\).

So there are four candidate Whitney numbers: \(w_3 w_2, w_3 B, w_1^3 B,\) and \(BSq^1 B\). We now use them to determine a generating set of \(\Omega^5_5(K(\mathbb{Z}/2, 2))\). The answer is in Table 1, and the calculations follow.

**Example 4.7.** The Wu manifold \(W := SU_3/SO_3\) has cohomology ring \(H^*(W; \mathbb{Z}/2) \cong \mathbb{F}_2[z_2, z_3]/(z_2^2, z_3^2)\) with \(w = 1 + z_2 + z_3, Sq(z_2) = z_2 + z_3 + z_2^2,\) and \(Sq(z_3) = z_3 + z_2 z_3\). Hence \((W, 0)\) and \((W, z_2)\) are linearly independent in \(\Omega^5_5(K(\mathbb{Z}/2, 2))\), giving us two of the four needed generators. \(\blacksquare\)
We've reduced the problem to verifying (1.2) on the four generators.

\( (4.10) \)

\[
\begin{array}{cccc}
\text{Example 4.9.} & \text{Consider } X = S^1 \times \mathbb{RP}^2 \times \mathbb{RP}^2, \text{ whose cohomology is } H^*(X; \mathbb{Z}/2) \cong \mathbb{F}_2[x, u, v]/(x^2, u^3, v^3), \text{ where } x \text{ generates } H^1(S^2; \mathbb{Z}/2), \text{ and } v \text{ generates } H^1 \text{ of the first } \mathbb{RP}^2, \text{ and } y \text{ generates } H^1 \text{ of the second copy. Then,}
\end{array}
\]

\[
\begin{equation}
(1 + u + u^2)(1 + v + v^2) = 1 + u + v + u^2 + uv + v^2 + u^2v + uv^2 + u^2v^2.
\end{equation}
\]

The Steenrod action is determined by \( \text{Sq}(u) = u + u^2 \), \( \text{Sq}(v) = v + v^2 \), and \( \text{Sq}(x) = x \).

When \( B = ux \), \( w_3B = u^2v^2x \neq 0 \), but \( B\text{Sq}^1B = u^3x^2 = 0 \), so \( (x, ux) \) is linearly independent from the previous three examples, and hence is the last generator.

\(
5. \text{ Checking on the generators}
\)

**Proposition 5.1** (Massey [Mas69]). Let \( m, n \in \mathbb{Z} \) be such that \( m \equiv n \mod 2 \) and \( X \) be a topological space.

If \( a \in H^m(X; \mathbb{Z}/2) \) and \( y \in H^n(X; \mathbb{Z}/2) \), then

\[
\Psi(ab) = \Psi(a)\Psi(b) + \theta((\text{Sq}^{m-1}u)v\text{Sq}^1v + u\text{Sq}^1u(\text{Sq}^{n-1}v)),
\]

where \( \theta: H^*(X; \mathbb{Z}/2) \to H^*(X; \mathbb{Z}/4) \) is induced by the multiplication by 2 map \( -2: \mathbb{Z}/2 \to \mathbb{Z}/4 \).

**Proof of Theorem 1.1.** We've reduced the problem to verifying (1.2) on the four generators.

1. When \( (M, B) = (W, 0) \), both sides are 0 because \( B = 0 \).
2. For \( (M, B) = (W, z_2) \), we have

\[
(5.2)
\]

\[
BSq^1B + Sq^2\Sigma^1B = z_2z_3 + z_2^2z_3 = 0,
\]

and since \( W \) is orientable, \( \bar{w}_1(W) = 0 \) and the right-hand side is also 0.

3. If \( (M, B) = (S^1 \times \mathbb{RP}^4, xy) \),

\[
(5.3a)
\]

\[
\text{Sq}^1(xy) = x\text{Sq}^1y + x\text{Sq}^1y = xy^2
\]

\[
(5.3b)
\]

\[
\text{Sq}^2\text{Sq}^1(xy) = (\text{Sq}^2x)y^2 + \text{Sq}^1x\text{Sq}^1y + x\text{Sq}^2(y^2) = xy^4
\]

\[
(5.3c)
\]

\[
xy\text{Sq}^1(xy) = x^2y^3 = 0.
\]

Hence

\[
(5.4)
\]

\[
\langle BSq^1B + Sq^2\Sigma^1B, [S^1 \times \mathbb{RP}^4] \rangle = \langle xy^4, [S^1 \times \mathbb{RP}^4] \rangle = 1.
\]

To compute the right-hand side of (1.2), apply Proposition 5.1 with \( m = n = 1 \):

\[
(5.5)
\]

\[
\Psi(xy) = \Psi(x)\Psi(y) + \theta(xy^3 + x^3y) = \theta(xy^3)
\]

by degree considerations.

One can check on the generators of \( H^3(S^1 \times \mathbb{RP}^4, \mathbb{Z}/2) \) to show that \( xy^3 \) is not in the image of the Bockstein, hence \( \theta(xy^3) \neq 0 \). It lands in the piece of \( H^4(S^1 \times \mathbb{RP}^4; \mathbb{Z}/4) \) that is \( H^1(S^1; \mathbb{Z}/4) \otimes H^3(\mathbb{RP}^4; \mathbb{Z}/2) \cong \mathbb{Z}/4 \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2 \), hence is the generator.

Now, \( \tilde{w}_1(S^1 \times \mathbb{RP}^4) = \tilde{w}_1(S^1) + \tilde{w}_1(\mathbb{RP}^4) \). \( \tilde{w}_1(S^1) = 0 \) because \( S^1 \) is orientable, and \( \tilde{w}_1(\mathbb{RP}^4) = \tilde{y} \), the generator of \( H^1(\mathbb{RP}^4; \mathbb{Z}/2) \), using that the inclusion \( \mathbb{RP}^4 \hookrightarrow BO_1 \) is cellular.

Hence, in \( H^3(S^1 \times \mathbb{RP}^4; (\mathbb{Z}/4)_{w_1}) \), \( \tilde{w}_1\Psi(xy) \) is nonzero, so must be twice the generator. Thus

\[
(5.6)
\]

\[
\langle \tilde{w}_1 \rightarrow \Psi(xy), [S^1 \times \mathbb{RP}^4] \rangle = 2,
\]

so (1.2) is valid.
Finally, let \((M, B) = (S^1 \times \mathbb{RP}^2 \times \mathbb{RP}^2, xy)\). We have
\begin{equation}
BSq^1 B + Sq^2 Sq^1 B = 0 + uux + uux = 0.
\end{equation}

Using Proposition 5.1,
\begin{equation}
\Psi(B) = \Psi(uux) = \Psi(ux)\Psi(x) + \theta(uxSq^1 x + xuSq^1 u),
\end{equation}
which vanishes by degree considerations: \(\Psi(x) \in H^2(S^1; \mathbb{Z}/4) = 0\), \(Sq^1 x \in H^2(S^1; \mathbb{Z}/2) = 0\), and \(uSq^1 u \in H^3(\mathbb{RP}^2; \mathbb{Z}/2) = 0\).

\[\Box\]

**References**

