QUANTUM TOPOLOGY AND CATEGORIFICATION SEMINAR, SPRING 2017

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These notes were taken in a learning seminar in Spring 2017. I live-TEXed them using `vim`, and as such there may be typos; please send questions, comments, complaints, and corrections to a.debray@math.utexas.edu. Thanks to Michael Ball for finding and fixing a typo.

CONTENTS

Part 1. Quantum topology: Chern-Simons theory and the Jones polynomial

1. THE JONES POLYNOMIAL: 1/24/17

Today, Hannah talked about the Jones polynomial, including how she sees it and why she cares about it as a topologist.

1.1. Introduction to knot theory.

Definition 1.1. A **knot** is a smooth embedding $S^1 \hookrightarrow S^3$. We can also talk about **links**, which are embeddings of finite disjoint unions of copies of S^1 into S^3.

One of the major goals of 20^{th}-century knot theory was to classify knots up to isotopy.

Typically, a knot is presented as a **knot diagram**, a projection of $K \subset S^3$ onto a plane with "crossing information," indicating whether the knot crosses over or under itself at each crossing. Figure 1.1 contains an example of a knot diagram.

Given a knot in S^3, there's a theorem that a generic projection onto $\mathbb{R}^2$ is a knot diagram (i.e. all intersections are of only two pieces of the knot).

Link diagrams are defined identically to knot diagrams, but for links.

Theorem 1.2. *Any two link diagram for the same link can be related by planar isotopy and a finite sequence of* ***Reidemeister moves****.*

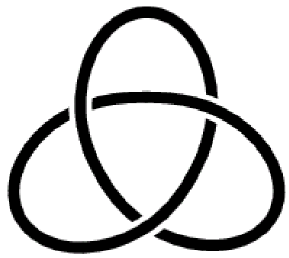

FIGURE 1.1. A knot diagram for the left-handed trefoil knot. Source: Wikipedia.

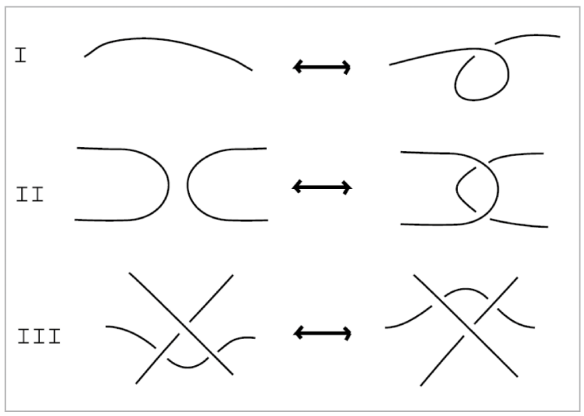

FIGURE 1.2. The three Reidemeister moves. Source: `https://www.computer.org/csdl/trans/tg/2012/12/ttg2012122051.html`.

1.2. **Polynomials before Jones.** The first knot polynomial to be defined was the Alexander polynomial $\Delta_K(x)$, a Laurent polynomial with integer coefficients that is a knot invariant, defined in the 1920s.

Here are some properties of the Alexander polynomial:

- It's symmetric, i.e. $\Delta_K(x) = \Delta_K(x^{-1})$.
- It cannot distinguish handedness. That is, if K is a knot, its **mirror** $\overline{K}$ is the knot obtained by switching all crossings in a knot diagram, and $\Delta_K(x) = \Delta_{\overline{K}}(x)$.[1]
- The Alexander polynomial doesn't detect the unknot (which is no fun): there are explicit examples of knots 11_{34} and 11_{42} whose Alexander polynomials agree with that of the unknot.[2]

So maybe it's not so great an invariant, but it's somewhat useful.

1.3. **The Jones polynomial.** The Jones polynomial was defined much later, in the 1980s. The definition we give, in terms of skein relations, was not the original definition. There are three local models of crossings, as in Figure 1.3.

FIGURE 1.3. The three local possibilities for a crossing in a knot diagram (technically, L_0 isn't a crossing). Source: `https://en.wikipedia.org/wiki/Skein_relation`.

[1]The mirror of the left-handed trefoil is the right-handed trefoil, for example.

[2]The notation for these knots follows Rolfsen.

The idea is that, given a knot K, you could try to calculate a knot polynomial for K in terms of knot polynomials on links where one of the crossings in K has been changed from L_- to L_+ (or vice versa), or **resolved** by replacing it with an L_0. A relationship between the knot polynomials of these three links is a **skein relation**. This is a sort of inductive calculation, and the base case is the unknot. In particular, you can use the value on the unknot and the skein relations for a knot polynomial to describe the knot polynomial!

Example 1.3. The Alexander polynomial is determined by the following data.

- On the unknot, $\Delta(U) = 1$.
- The skein relation is $\Delta(L_+) - \Delta(L_-) = t\Delta(L_0)$. $\blacktriangleleft$

Definition 1.4. The **Jones polynomial** is the knot polynomial v determined by the following data.

- For the unknot, $v(U) = 1$.
- The skein relation is

$$\left(t^{1/2} - t^{-1/2}\right)v(L_0) = t^{-1}v(L_+) - t v(L_-). \tag{1.5}$$

Example 1.6. Let's calculate the Jones polynomial on a Hopf link H, two circles linked together once. The standard link diagram for it has two crossings, as in Figure 1.4.

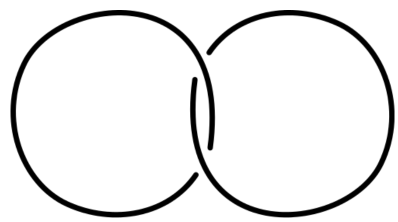

FIGURE 1.4. A Hopf link. Source: `https://en.wikipedia.org/wiki/Link_group`.

- Resolving one of the crossings produces an unknot: $v(L_0) = 1$.
- Replacing the L_- with an L_- produces two unlinked circles. One more skein relation produces the unknot, so $v(L_+) = -(t^{1/2} - t^{-1/2})$.

Putting these together, one has

$$(t^{1/2} - t^{-1/2}) \cdot 1 = t^{-1}(-(t^{1/2} + t^{-1/2}) - t v(H)),$$

so $v(H) = -t^{-1/2} - t^{-5/2}$. $\blacktriangleleft$

There are many different definitions of the Jones polynomial; one of the others that we'll meet later in this seminar is via the Kauffman bracket.

Definition 1.7. The **bracket polynomial** of an unoriented link L, denoted $\langle L\rangle$, is a polynomial in a variable A defined by the skein relations

- On the unknot: $\langle O\rangle = 1$.
- There are two ways to resolve a crossing C: as two vertical lines V or two horizontal lines H. We impose the skein relation
$$\langle C\rangle = A\langle V\rangle + A^{-1}\langle H\rangle.$$
- Finally, suppose the link L is a union of one unlinked unknot and some other link L' (sometimes called the **distant union**). Then,
$$\langle L\rangle = (-A^2 - A^{-2})\langle L'\rangle.$$

Example 1.8. Once again, we'll compute the Kauffman bracket for the Hopf link. (TODO: add picture).
The result is $\langle H\rangle = -A^4 - A^{-4}$. $\blacktriangleleft$

You can show that this bracket polynomial is invariant under type II and III Reidemeister moves, but not type I. We obviously need to fix this.

Definition 1.9. Let D be an *oriented* link, and $|D|$ denote the link without an orientation. The **normalized bracket polynomial** is defined by

$$X(D) := (-A^3)^{-\omega(D)}\langle |D|\rangle).$$

Here, $\omega(D)$ is the writhe of D, an invariant defined based on a diagram. At each crossing, imagine holding your hands out in the shape of the crossing, where (shoulder $\to$ finger) is the positively oriented direction along the knot. If you hold your left hand over your right hand, the crossing is a **positive crossing**; if you hold your right hand over your left, it's a **negative crossing**.

Let ω_+ denote the number of positive crossings and ω_- denote the number of negative crossings. Then, the **writhe** of D is $\omega(D) := \omega_+ - \omega_-$. For example, the writhe of the Hopf link (with the standard orientation) is 2, and $X(H) = -A^{10} - A^2$.

Thankfully, this is invariant under all types of Reidemeister moves. The proof is somewhat annoying, however.

Theorem 1.10. *By substituting $A = t^{-1/4}$, the normalized bracket polynomial produces the Jones polynomial.*

So these two invariants are actually the same.

Here are some properties of the Jones polynomial.

- $v_{\overline{K}}(t) = v_K(t^{-1})$. Since the Jones polynomial is not symmetric, it can sometimes distinguish handedness, e.g. it can tell apart the left- and right-handed trefoils.
- It fails to distinguish all knots: once again, 11_{34} and 11_{42} have the same Jones polynomial.[3]
- It's unknown whether the Jones polynomial detects the unknot: there are no known nontrivial knots with trivial Jones polynomial.
- Computing the Jones polynomial is #**P**-hard: there's no polynomial-time algorithm to compute it. (Conversely, the Alexander polynomial is one of very few knot invariants with a polynomial-time algorithm.)

If a knot does have trivial Jones polynomial, we know:

- it isn't an **alternating knot** (i.e. one where the crossings alternate between positive and negative).
- It has crossing number at least 18 (which is *big*).

One interesting application of what we'll learn in this seminar is that there are knots (9_{42} and 10_{11}) that can't be distinguished by the Jones or Alexander (or HOMFLY, or ...) but *are* distinguished by SU(2)-Chern-Simons invariants.

2. Introduction to quantum field theory: 1/31/17

Today, Ivan talked about quantum field theory (QFT), including what QFT is, why one might want to study it, how it relates to other physical theories, classical field theories, and quantum mechanics, and how to use canonical quantization to produce a QFT.

So, why should we study QFT? One good reason is that its study encompasses a specific example, the **Standard model**, the "theory of almost everything." This is a theory that makes predictions about three of the four fundamental forces of physical reality (electromagnetism, the weak force, and the strong force), leaving out gravity. These predictions have been experimentally verified, e.g. by the Large Hadron Collider.

Unfortunately, the mathematical theory of QFTs is not well formulated; **free theories** are well understood, but if you can rigorously formulate the mathematical theory of **interacting QFTs**, you'll win a million-dollar prize! Perhaps that's a good reason to study QFT.

There's also the notion of a topological quantum field theory (TQFT), which has been rigorously formulated as mathematics, but many of the most important QFTs, including the Standard Model, do not fit into this framework.

QFTs fit into a table with other physical theories: the theory you want to use depends on how fast your particles move and how big they are.

- If your particles are larger than atomic scale and moving considerably slower than the speed of light c, you use *classical mechanics*.
- If your particles are atomic-scale, but moving much slower than c, you use *quantum mechanics*.
- If your particles are larger than atomic-scale, but moving close to the speed of light, you use *special relativity* or *general relativity*: the latter if you need to account for gravity, and the former if you don't.
- If your particles are at atomic-scale and moving close to the speed of light, but you don't need to take gravity into account, you use *quantum field theory*. In this sense, QFT is the marriage of special relativity and quantum mechanics.

[3] This is ultimately for the same reason as for the Alexander polynomial: there's a technical sense in which they're **mutant knots** of each other. It's notoriously hard to write down knot polynomials that detect mutations, and the Jones polynomial cannot detect them.

- If your particles are small, but moving at about c, *and* you need to consider gravity, you end up in the domain of *string theory*. Here be dragons, of course: string theory hasn't been experimentally verified yet…

With the big picture in place, let's talk a little about classical field theory.

Let $\mathbb{R}^{1,3}$ denote **Minkowski spacetime**, $\mathbb{R}^4$ with the normal **Minkowski metric**

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

Definition 2.1. A **field** is a section of a vector bundle over $\mathbb{R}^{1,3}$, or a connection on a principal G-bundle over $\mathbb{R}^{1,3}$. In the latter case, it's also called a **gauge field**.

In this context, we'l care the most about trivial vector bundles and principal bundles!

Definition 2.2. A **classical field theory** is a collection of PDEs that specify the time evolution of a collection of fields.

Example 2.3. Electromagnetism is a famous example of a classical field theory: there are electric and magnetic fields $\vec{E}$ and $\vec{B}$, respectively, and the **Maxwell equations** govern how they evolve in time:

$$\begin{aligned} \nabla \cdot \vec{E} &= \rho \\ \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 \\ \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} &= J. \end{aligned}$$

Here, J is the **electric current** and ρ is the **charge density**. There may be some constants missing here. ◀

Usually (always?), you can present the evolution of the classical field theory as the "critical points" of a functional of the form[4]

$$S(\varphi_1, \dots, \varphi_n) = \int_{\mathbb{R}^4} \mathrm{d}^4 x \, \mathscr{L}(\varphi_1, \dots, \varphi_n, \partial_\mu \varphi_1, \dots, \partial_\mu \varphi_n).$$

The functional S is called the **action functional**, and the function $\mathscr{L}$ is called the **Lagrangian**. Using calculus of variations, this notion of critical points is placed on sound footing. Physicists sometimes call these critical points **minimizers**, but sometimes we want to maximize S, not minimize it.

In this context, one can show that the critical points of S are the solutions to the **Euler-Lagrange equations**. In the case of a single field φ, these equations take the form

$$\frac{\partial \mathscr{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi)} \right) = 0.$$

(We are using and will continue to use Einstein notation: any index μ that's both an upper and lower index has been implicitly summed over.) So the Lagrangian contains all the information about the dynamics of the system.

Example 2.4. Let's look at electromagnetism again: if $\rho = 0$ and $J = 0$, then let

$$A := A_\mu \, \mathrm{d}x^\mu$$

be the **electromagnetic potential**. If $F = \mathrm{d}A$, then

$$F = F_{\mu\nu} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu.$$

Then, the Lagrangian is

$$\mathscr{L}_{\text{Maxwell}} := -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{2.5}$$

[4] To be precise, we should say what space of functions this takes place on. The right way to do this is to consider distributions, but we're not going to delve into detail about this.

where $F^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}$ and $g^{\mu\nu}$ denotes the coefficients of the standard Minkowski metric. Then, the Euler-Lagrange equations and the fact that $\mathrm{d}F = 0$ (since F is already exact) directly imply the Maxwell equations, where

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}.$$

$\blacktriangleleft$

Definition 2.6. A **free field theory** is one whose Lagrangian is quadratic in the fields and their partial derivatives. A field theory which is not free is called **interacting**.

In a free field theory, the Euler-Lagrange equations become linear, making them much easier to solve.

Example 2.7. One example of a free field theory uses the **Dirac Lagrangian**

$$\mathscr{L}_{\mathrm{Dirac}} := \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi,$$

where $\mu\colon U \subset \mathbb{R}^{1,3} \to \mathbb{C}^4$, γ^μ are **Dirac matrices**, and m is a **mass parameter**, and $\overline{\psi} = \psi^\dagger\gamma^0$. This is used to describe the behavior of a free fermion (e.g. an electron). You can explicitly check this is quadratic in ψ and γ.

The Maxwell Lagrangian (2.5) also defines a free field theory. $\blacktriangleleft$

Example 2.8. Here's an example of an interacting field theory; its classical solutions don't represent anything physical, but we'll see it again.

$$\mathscr{L}_{\mathrm{QED}} := \mathscr{L}_{\mathrm{Dirac}} + \mathscr{L}_{\mathrm{Maxwell}} + ie\overline{\psi}\gamma^\mu\gamma A_\mu. \tag{2.9}$$

Here e is the charge of an electron, not ≈ 2.78. The first two terms are free, but then it's coupled to an interacting term. $\blacktriangleleft$

From classical to quantum. To understand how we move from classical field theory to quantum field theory, we'll learn about quantum mechanics, albeit very quickly. This formalism extracts three aspects of a physical system.

- The **states** are the configurations that the system can be in.
- The **observables** are things which we can measure/observe about a system.
- **Time evolution** describes how observables or states evolve with time.

In quantum mechanics:

- The states are unit vectors in some (complex) Hilbert space $\mathscr{H}$.
- The observables are self-adjoint operators $A : \mathscr{H} \to \mathscr{H}$. They are not necessarily bounded. The things you can measure for A are in its spectrum $\operatorname{Spec} A \subset \mathbb{R}$ (since A is self-adjoint). For example, if A represents the position in a coordinate you chose, the spectrum denotes the set of allowed positions in that coordinate.
- Time evolution has two equivalent formulations.
  - The **Schrödinger picture** describes time evolution of the states. There's a distinguished observable, usually representing the energy of the system, called the **Hamiltonian** $H : \mathscr{H} \to \mathscr{H}$. Then, a state $\psi \in \mathscr{H}$ in this system evolves as

    $$i\hbar\frac{\partial}{\partial t}\psi(t) = H\psi(t).$$

  - The **Heisenberg picture** describes time evolution of observables as satisfying the equation

    $$\frac{\mathrm{d}}{\mathrm{d}t}A(t) - = i\hbar[H, A(t)].$$

    These two perspectives predict the same physics.

Generally, quantum field theories are obtained by taking a classical field theory and quantizing it. This is a process creating a dictionary based on the one between classical mechanics and quantum mechanics:

- The states in classical mechanics are points in T^*M, where M is a smooth manifold; quantum mechanics uses a Hilbert space.

- The observables in classical mechanics are smooth functions $T^*M \to \mathbb{R}$. In coordinates $(q^1, \dots, q^n, p_n, \dots, p_n)$, we have relations
$$\{q^i, q^j\} = 0 \qquad \{p_i, p_j\} = 0 \qquad \{q^i, p_j\} = \delta^i_j,$$
where $\{-,-\}$ is the Poisson bracket coming from the symplectic structure on T^*M. Quantum mechanics replaces functions with self-adjoint operators. In quantum mechanics, if X_i and P_i are the position and momentum operators in coordinate i, they satisfy the relations
$$[X_i, X_j] = 0 \qquad [P_i, P_j] = 0 \qquad [X_i, P_j] = i\hbar\delta_{ij}.$$
- Time evolution in classical mechanics satisfies $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = \{H, \gamma(t)\}$; quantum mechanics assigns the Schrödinger or Heisenberg pictures as above.

So for a classical field theory, we want a way to get a Hilbert space, a Hamiltonian, and position and momentum operators.

Example 2.10 (One-dimensional harmonic oscillator)**.** The harmonic oscillator in one dimension satisfies the equation
$$H(x,p) = \frac{p^2}{2m} + n\omega^2 x^2.$$
Then, the Hilbert space is $\mathscr{H} = L^2(\mathbb{R})$, $Xf = x \cdot f$, and $Pf = -i\hbar\frac{\partial f}{\partial x}$.[5] These automatically satisfy the relations $[X,P] = 1$, $[X,X] = 0$, and $[P,P] = 0$. The Hamiltonian is
$$H(X,P) = \frac{P^2}{2m} + m\omega^2 X^2.$$
$\blacktriangleleft$

It's worth noting that quantization is not a deterministic process, more of an art: choosing the right position and momentum operators and showing why they satisfy the relations doesn't follow automatically from some general theory. But if you can get the commutation relations to work and it describes a physical system, congratulations! You've done quantization.

Next time, we'll discuss quantum field theory, where the fields are replaced with quantum fields.

3. Canonical quantization and Chern-Simons theory: 2/7/17

Today, Jay will say some more things about quantum field theory. First, he'll discuss a setup for QFT, including some handwaving about canonical quantization, and some path integrams. Then, there will be some discussion of gauge theory and connections on G-bundles, and a little bit about Chern-Simons theory.

3.1. **Quantum field theory and path integrals.** Just like for quantum mechanics, fix a Hilbert space $\mathscr{H}$ of states. Quantum fields are the things that we use to take measurements, more or less: operator-valued distributions over spacetime. It's helpful to think of them as simply operators on $\mathscr{H}$.

The general way canonical quantization works is that one desires quantum fields $\phi(x), \pi(x)$ which satisfy relations similar to the ones that positions and momenta do in classical mechanics.[6] For example, their commutator $[\phi(x), \pi(y)]$ is another operator-valued distribution, and the position-momentum constraint is that $[\phi(x), \pi(y)] = \delta(x-y) \cdot \mathbf{1}$, analogous to the Poisson bracket for classical mechanics.

QFT actually computes things called **scattering amplitudes**: if you throw a bunch of particles at a bunch of other particles, sometimes new particles come out. Scattering amplitudes encode the probability of getting a particular new particle from a particular collision of old particles. These are computed with **propagators**, distributions of the form $D(x-y) = \langle 0 \mid \Phi(x)\Phi(y) \mid 0\rangle$, where $|0\rangle$ is the vacuum state (the lowest-energy state), not the zero vector of $\mathscr{H}$. Here $x, y \in \mathbb{R}^{1,3}$, and we assume x is "after" y, in that $x_0 > y_0$. Physicists think of this as the creation of a particle at y, followed by its annihilation at x, and from this other things can be built, so if you want to compute anything, this is a good place to start.

[5] Well, not all L^2 functions are differentiable, but there are ways of working around this, especially since differentiable functions are dense in L^2.

[6] Some of what follows requires additional indices unless these are scalars (functions on the space), but thinking in terms of scalars is helpful for now.

The propagator can be computed in terms of a path integral, which has the advantage that the quantum fields are replaced with classical fields inside the integral:

$$\langle 0 \mid \phi(x)\phi(y) \mid 0\rangle = \frac{\int D\phi\ \phi(x)\phi(y)e^{iS(\phi)/\hbar}}{\int D\phi\ e^{iS(\phi)/\hbar}},$$

where

$$S(\phi) = \int \mathscr{L}(\phi, \partial_\mu \phi)\,\mathrm{d}^4 x$$

and $D\phi$ is a "measure" on the space of fields, which famously still hasn't been made rigorous. Nonetheless, there is a theory for calculating path integrals, which boils down to Feynman diagrams and Wick contractions.

Unfortunately, approaching this systematically is generally done heuristically. The quantum fields are supposed to encode the amplitudes of the Fourier transforms of the classical fields, but making this precise doesn't come easily.

The path integral is so nice because it allows you to avoid an explicit quantization — you can compute things such as expectation values in terms of the classical field theory. For example, if $\mathscr{M}$ is a space of (classical) fields and F is a functional on those fields, the path integral allows you to compute the vacuum expectation value:

$$\langle 0 \mid F \mid 0\rangle = \frac{\int D\phi\ e^{-iS(\phi)}F(\phi)}{\int D\phi\ e^{-iS(\phi)}},$$

where S is as before, written in terms of the Lagrangian.

Intuition for computing the path integral: for a zero-dimensional field theory, fields are numbers, and the path integral reduces to ordinary Gaussian integrals. The tricks that we use to compute these generalize to higher-dimensional theories, in some vague sense.

Another trick is to argue as to why the majority of the measure is concentrated at extremal points of the Lagrangian.

A third trick is to discretize the quantum field theory into a lattice model, which approximates the path integral by an ordinary integral which can be computed rigorously. The hope is to take a limit as the lattice approximation gets finer and finer, but this is mysterious in general.

3.2. **Gauge theory.** Gauge theories are those in which the fields are connections on principal G-bundles, where G is a compact Lie group.

That was a lot of words. Here's what some of them mean.

Definition 3.1. Let G be a compact Lie group.

- A G**-torsor** is a space with a simply transitive left action on G, necessarily a manifold diffeomorphic to G and with an isomorphic left action, but without an origin.
- A **principal** G**-bundle** is a fiber bundle $P \to M$, where the fibers are G-torsors in a smoothly varying way.
- A **connection** on a principal bundle $P \to M$ is a $\mathfrak{g}$-valued 1-form on the total space of P, where $\mathfrak{g}$ is the Lie algebra of G.

The definition of a connection isn't super intuitive, but it's a way of defining parallel transport between the fibers of P. The tangent space to a point in P splits as the direct sum of the tangent space to M and the tangent space of G, which is $\mathfrak{g}$, so the 1-form is a way of projecting down to $\mathfrak{g}$, which is a local parallel transport.

If $P \to M$ is trivial (meaning it's just the projection $G \times M \to M$), the zero section defines a map $t : M \to G \times M$, and pulling back the connection along t allows one to think of it as a $\mathfrak{g}$-valued 1-form on A. In physics terminology, fixing this trivialization is called **fixing a gauge**.

Example 3.2. Electromagnetism is a gauge theory, with $G = \mathrm{U}(1)$ and $\mathfrak{g} = \mathbb{R}$ with trivial Lie bracket. Physicists think of the vector potential as a covector field on spacetime, rather than on $\mathrm{U}(1) \times \mathbb{R}^{1,3}$; this means that the gauge has already been implicitly fixed. ◂

There's a natural way to associate a $\mathfrak{g}$-valued 2-form of a connection called its **curvature**. The curvature of A is defined to be

$$F_A := \mathrm{d}A + \frac{1}{2}[A, A].$$

In electromagnetism, this associates the electromagnetic 2-tensor F_{ij} to the vector potential.

The Lagrangian for a gauge theory usually depends only on the curvature of the connection, e.g. in Chern-Simons theory, which we'll discuss below.

Chern-Simons theory is motivated by a few theorems in pure mathematics. The **adjoint action** $\mathrm{Ad}\, G$ of G on $\mathfrak{g}$ is the derivative of the action of G on itself by conjugation.

Theorem 3.3 (Chern-Weil)**.** *Given an* $\mathrm{Ad}\, G$*-invariant polynomial* h *on* $\mathfrak{g}$ *of degree* k*, one can build a* $2k$*-form* $T(h,A) = h(F_A, \dots, F_A)$*, and this* 2*-form is always exact: there's a* $(2k-1)$*-form* $\mathrm{CS}(A)$ *such that* $\mathrm{dCS}(A) = T(h,A)$*, defined as*

$$\mathrm{CS}(h,A) = \int_0^1 h(A, F_A, F_A, \dots, F_A)\, \mathrm{d}t.$$

This form is called the ***Chern-Simons form****.*

You can do this basis-independently: the $\mathrm{Ad}\, G$-invariant polynomials on $\mathfrak{g}$ of degree k are the space $\mathrm{Sym}^k(\mathfrak{g}^*)$. If h is the quadratic polynomial associated to the Killing form, then the Chern-Simons form is

$$\mathrm{CS}(h,A) = \mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{1}{3} A \wedge [A,A]\right).$$

It's possible to show that $\int_M \mathrm{CS}(h,A)\, \mathrm{d}M$ is a topological invariant, which suggests that the field theory with **Chern-Simons action**

$$S(A) := \frac{\ell}{4\pi} \int_M \mathrm{CS}(h,A)\, \mathrm{d}M$$

should have interesting topological properties. The Euler-Lagrange equations for this classical field theory boil down to $F_A = 0$ (these connections are called **flat connections**); in other words, the classical fields are flat connections. This is very useful: the moduli space of connections on G-bundles over a manifold M is really messy, but restricting to flat connections, you obtain something finite-dimensional.

The last thing we'll talk about today are examples of observables called Wilson loop operators. Let K be a functional on the space of fields, and $L \subset M$ be a link with components L_i. To each L_i, choose a finite-dimensional real representation R_i of G. Let

$$W_{R_i}(K_i) := \mathrm{Tr} \exp \oint_{K_i} A.$$

Its expectation, called the **Wilson loop operator**, is a path integral

$$\langle K \rangle = \int DA \, \exp\left(\frac{1}{k} S(A)\right) \prod_{i=1}^{r} W_{R_i}(K_i).$$

The Jones polynomial will be one of these Wilson loop operators.

4. Chern-Simons theory and the Wess-Zumino-Witten model: 2/14/17

These are Arun's notes for his talk about Chern-Simons theory.

4.1. Functorial TQFT and CFT.

Mired as we were in physics, let's zoom out a little bit. Chern-Simons theory ought to be a topological quantum field theory, meaning it should be possible to understand it with pure mathematics. Though Witten [7] does not do this, Gill and Adrian will be speaking about a paper [2] which adopts a more mathematical approach, and this mathematical notion of TQFT will also come up in the second half of the seminar.

Informally, a TQFT is the categorified notion of a bordism invariant. That is, equivalence classes closed n-manifolds up to bordism form an abelian group Ω_n under disjoint union, and a bordism invariant is a group homomorphism from Ω_n into some other abelian group. For example, the signature of the intersection pairing is a bordism invariant $\Omega_4^{\mathrm{SO}} \to \mathbb{Z}$.

The categorified notion of an abelian group[7] is a **symmetric monoidal category**, a category $\mathscr{C}$ with a functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ that is (up to natural isomorphism) associative, commutative, and unital. A **symmetric monoidal functor** is a functor between symmetric monoidal categories that preserves the product.

Example 4.1.

(1) Complex vector spaces are a symmetric monoidal category under tensor product; the unit is $\mathbb{C}$.

[7] Technically of a commutative monoid.

(2) For any n, there is a **bordism category** $\mathscr{B}ord_n$ whose objects are closed n-manifolds and whose morphisms are bordisms between them (so a bordism X with incoming boundary M and outgoing boundary N defines a morphism $M \to N$). Its monoidal product is disjoint union, and the unit is the empty set, regarded as an n-manifold. ◂

There are several related bordism categories, where the manifolds are oriented, spin, etc. Another example is manifolds and bordisms with **conformal structure**, the data of a Riemannian metric up to scaling, which form a bordism category we'll call $\mathscr{B}ord_n^{\mathrm{conf}}$.

Definition 4.2. An n-dimensional **topological quantum field theory** (TQFT) is a symmetric monoidal functor $Z : \mathscr{B}ord_n \to (\mathscr{V}ect_{\mathbb{C}}, \otimes)$.

So for every n-manifold M, there's a complex vector space $Z(M)$; for every bordism there's a linear map of vector spaces; and disjoint unions are mapped to tensor products.

This concept is called **functorial TQFT**. The idea is it should encompass the properties of a QFT that only depend on topological information, e.g. $Z(M)$ is the space of states of the theory on the manifold M.

Definition 4.3. An n-dimensional **conformal field theory** (CFT) is a symmetric monoidal functor $Z : \mathscr{B}ord_n^{\mathrm{conf}} \to (\mathscr{V}ect_{\mathbb{C}}, \otimes)$.

This is basically the same; for CFT, though, the theory is allowed to depend on geometric information, as long as such information is scale-invariant.

Describing Chern-Simons theory as a functorial TQFT would be pretty cool, making all the calculations for the Jones polynomial much easier, and that's what Adrian and Gill are going to tell us about. But there's a wrinkle: Chern-Simons theory is **anomalous**, in that it depends on slightly more than a topological structure. There are a couple of approaches you can take here.

- One way to deal with this is to add structure that makes the anomaly vanish, so as a theory of manifolds with that structure, Chern-Simons theory is a TQFT. For example, the anomaly is trivial on framed manifolds, which is why Witten's calculations require playing with framings of knots. The anomaly vanishes on weaker structures, e.g. a signature structure or a trivialization of p_1 (the latter is the approach BHMV use), and these structures are easier to deal with.
- There's a sense in which Chern-Simons theory is "topological in one direction and conformal in the other two," and the anomaly arises from the conformal part. So you could choose a 3-manifold $M^3 = \Sigma^2 \times C^1$ such that the theory is topological in C and conformal in Σ and study each part separately. Witten uses this approach to compute some state spaces, which I'll tell you about in a little bit.

Today, though, we're going to continue the physics-based approach.

4.2. **Connections and the Chern-Simons form.** A principal G-bundle is a generalization of a covering space, together with its covering group. Let G be a finite group and $p : \widetilde{X} \to X$ be a covering space with deck transformation group G. Then, every $x \in X$ has a neighborhood U such that $p^{-1}(U) \cong U \times G$ in a way that commutes with projection to U and the G-action. If you replace "finite group" with "Lie group" in the previous sentence, you obtain the definition of a principal G-bundle.

We also defined a connection on a principal G-bundle $P \to X$ to be a $\mathfrak{g}$-valued one-form on the total space P. These can be used to define parallel transport, just like connections on vector bundles. Given a connection A, $A \wedge \mathrm{d}A + (1/3)A \wedge [A,A] \in \Omega_X^3(\mathfrak{g})$ (after pulling back along the zero section $X \to P$), so its trace is a real-valued 3-form that can be integrated. This is what the Chern-Simons action does:

$$S(A) := \frac{k}{2\pi} \int_M \mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{1}{3} A \wedge [A,A]\right), \tag{4.4}$$

where $k \in \mathbb{Z}$ is called the **level** of the theory. This defines a classical field theory as we discussed, and quantum Chern-Simons theory is its quantization. Working the quantization out is extremely difficult!

Anyways, assuming we can do that, we'd like to get some useful invariant out of the theory. If A is a connection on $P \to X$, it defines parallel transport. Just like parallel transport on a Riemannian manifold, this is locally unique but not globally unique, and given a curve $K \subset X$, we can ask what happens when we take something at a point, parallel-transport it around K, and compare the final result with the initial result.

Another analogy is to think about regular covering spaces (so for finite G): if you start with an x in the fiber $p^{-1}(y)$ and wind around K back to $p^{-1}(y)$, you might find yourself at $x' \neq x$. Since the action of G is transitive,

$x' = gx$ for some $g \in G$, and g turns out to depend only on the conjugacy class of x. This g is called the **holonomy** of x around K, denoted $\mathrm{hol}_K(x)$. Exactly the same definition applies to principal G-bundles in general (except that the connection is needed to define parallel transport).

It's easier to deal with numbers than with elements of G, so we'll take a trace: let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of G. Let

$$W_V(K) := \mathrm{Tr}(\rho(\mathrm{hol}_K(\mathrm{id}_G))).$$

That is: parallel-transport the identity element around G, take its action on V, and take the trace.

Let $K = \coprod_i K_i$ be a link, and $\vec{V} = \{\rho_i : G \to \mathrm{GL}(V_i)\}_i$ be a collection of (finite-dimensional, real) G-representations. Then, we define the **Wilson loop operator** associated to K and $\vec{V}$ to be

$$\langle K \rangle := \int DA\, e^{ikS(A)/4\pi} \prod_i W_{V_i}(K_i),$$

where $S(A)$ is the Chern-Simons action. This sketchy path integral integrates out the connection, so this is now an invariant of G, K, $\vec{V}$, and k. For judicious choices of $\vec{V}$, these will contain the data of the Jones polynomial.

From a physics perspective, Wilson loops encode generalized charge. If this were QED, a Wilson loop would measure the charge picked up by traveling around the loop in question, in the context of electromagnetic fields. It will sometimes be helpful to think of the representations as generalized charges.

4.3. **Connections to Wess-Zumino-Witten CFT.** Great, so how do you calculate anything? Witten's idea is to cut S^3 with an embedded link into pieces to derive the skein relation, meaning you just have to understand Chern-Simons theory on boundaries and calculate some state spaces for $\Sigma \times \mathbb{R}$ (a collar neighborhood for the Riemann surface Σ).

Herein lies a problem: the Chern-Simons action (4.4) is not gauge-invariant on a manifold with boundary. Oops! This is a manifestation of the anomaly mentioned earlier.

To fix this, you have to add a boundary term. This boundary term describes a 2D conformal field theory called the Wess-Zumino-Witten (WZW) model,[8] and the state spaces of Chern-Simons theory are identified with the conformal blocks of the WZW theory. The conformal blocks can then be explicitly calculated. This has actually been proven mathematically, though we won't go into the proof.

The Wess-Zumino-Witten model is a σ-model (meaning the fields are maps to some space) associated to a Lie group G (which we assume to be compact, simply connected, and simple). Let Σ be a Riemann surface and $\gamma : \Sigma \to G$ be smooth and $k \in \mathbb{Z}$.

Let $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be the Killing form, a symmetric bilinear form that is $x, y \mapsto 4\,\mathrm{Tr}(xy)$ for $\mathfrak{su}_2$. The WZW action is a sum of two terms: the first is the **kinetic term**

$$S^{\mathrm{kin}}(\gamma) := -\frac{k}{8\pi} \int_\Sigma B(\gamma^{-1}\partial^\mu \gamma, \gamma^{-1}\partial_\mu \gamma)\, \mathrm{d}A.$$

To define the second term, called the **Wess-Zumino term**, let X be a 3-manifold which Σ bounds and $\widetilde{\gamma} : X \to G$ be an extension of γ.[9] Let $\{e_i\}$ be a basis for $\mathfrak{g}$; then, the expression $B(e_i, [e_j, e_k])$ is alternating, so

$$\omega^{\mathrm{ZW}} := B(e_i, [e_j, e_k])\, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$$

is an invariant 3-form on G. The Wess-Zumino term is then

$$S^{\mathrm{WZ}}(\gamma) := \int_X \widetilde{\gamma}^* \omega^{\mathrm{WZ}}.$$

This depends on the choice of $\widetilde{\gamma}$, but is well-defined in $\mathbb{R}/\mathbb{Z}$. Exponentiating the action functional fixes this. Anyways, the WZW action is

$$S(\gamma) := S^{\mathrm{kin}}(\gamma) + 2\pi k S^{\mathrm{WZ}}(\gamma).$$

The relationship between Chern-Simons theory and the WZW model is an example of the **holographic principle**: that in many contexts, information about an n-dimensional QFT (the "boundary") determines and is determined by an $(n+1)$-dimensional TQFT (the "bulk"). In general, this is an *ansatz* rather than a theorem or even a physics-motivated argument, but in some cases there's good evidence for it. In our case, it's a theorem!

[8] There is also something called the Wess-Zumino model, and it is different!

[9] The obstruction to such an extension existing is $\pi_2(G)$, which vanishes because G is simply connected.

Theorem 4.5 (CS-WZW correspondence)**.** *There is an isomorphism between the state space of Chern-Simons theory for G on a surface Σ and the space of* ***conformal blocks*** *of the WZW model on Σ for G.*

This is understood, but the proof is complicated, and we won't go into it.

4.4. **Some calculations.** In this section, we assume the level k is large. The trivial representation is denoted $\underline{\mathbb{C}}$.

Witten calculates the Wilson loop operator for a link K in S^3 by breaking S^3 and K into pieces. This requires understanding the boundary, the state space for a punctured Riemann sphere S^2, with the punctures given by $S^2 \cap K$. Each puncture is labeled by the representation V that was associated to its loop, where we replace V with V^* if the orientation of the link disagrees with the direction of time.

The holographic principle identifies the state space of Chern-Simons theory on the punctured sphere with the space of conformal blocks of the WZW model, which is the space of possible charges assigned at each puncture. That is, at each puncture p_i, we want an element R_i, and the global state is the tensor product of the local states. However, the total charge of the space must be zero (just as the total electric charge of our universe is predicted to be 0), which means restricting to elements that are G-invariant. If S^2_m be the Riemann sphere with m punctures, this means that we want to calculate

$$\mathscr{H}_{S^2_m,\vec{V}} = \left(\bigotimes_i V_i\right)^G.$$

It's necessary to restrict to representations which are **positive-energy** considered as representations of the loop group LG, but all of the ones we use satisfy this property.

Lemma 4.6. *A G-representation $\underline{\mathbb{C}} \oplus V$, with V nonzero, is not positive-energy.*

Proposition 4.7. *For $m = 0$, $\mathscr{H}_{S^2,\varnothing} \cong \mathbb{C}$.*

Proof. We want to compute the G-invariants of a tensor product over $\varnothing$, i.e. $\underline{\mathbb{C}}^G = \mathbb{C}$. ⊠

Proposition 4.8. *For $m = 1$, $\mathscr{H}_{S^2_1,V}$ is 1-dimensional if V is trivial, and is otherwise 0.*

Proof. The proposition is proven when V is trivial, so assume V is nontrivial. By Maschke's theorem, V^G is a subrepresentation of G and a direct sum of copies of the trivial representation, so by Lemma 4.6, we must have $V^G = 0$. ⊠

Proposition 4.9. *For $m = 2$, $\mathscr{H}_{S^2_2,\{V_1,V_2\}}$ is 1-dimensional if $V_1^* \cong V_2$, and is otherwise 0.*

Proof sketch. If $V_1^* \cong V_2$, $V_1 \otimes V_2 = \mathrm{End}(V_2, V_2)$ as G-representations, and the diagonal matrices are an invariant subspace. Since V_2 is positive-energy, there can't be any more invariants.

If otherwise, the direct-sum decomposition of $V_1 \otimes V_2$ into irreducibles cannot have any copies of the trivial representation (as otherwise they would appear in either V_1 or V_2). ⊠

The three-point functions are tied to an interesting theory, but we won't need them to compute the Jones polynomial. The dimension of the state space attached to three *irreducible* representations V_i, V_j, and V_k is a coefficient N_{ij}^k expressed in terms of the S**-matrix** of the theory. This can be understood mathematically using modular tensor categories.

In principle, the 3-point functions determine all higher correlators in the theory, but we just need a specific 4-point function. Witten claims that if V is a G-representation and $V \otimes V$ decomposes into s distinct irreducible representations, then the Hilbert space associated to 4-punctured S^2 with points labeled by V, V, V^*, and V^* is s-dimensional. We will only need this for V the defining representation for SU_2, where we can prove it directly.

Proposition 4.10. *Let V denote the defining representation of SU_2. Then, $\mathscr{H}_{S^1_4,V,V,V^*,V^*}$ is one-dimensional.*

Proof. This is a fun calculation with the irreducible representations of SU_2. Recall that there's an irreducible $(n+1)$-dimensional representation P_n whose character on $\left(\begin{smallmatrix} z & 0 \\ 0 & \bar{z}\end{smallmatrix}\right)$ is

$$\chi_n(z) = z^n + z^{n-2} + \cdots + z^{-n+2} + z^{-n}.$$

The defining representation V and its dual are isomorphic to P_1, so

$$\begin{aligned}\chi_{V\otimes V\otimes V^*\otimes V^*}(z) &= \chi_1(z)^2\chi_1(z)^2 = (z + z^{-1})^4 \\ &= z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4} \\ &= \chi_4(z) + 3\chi_2(z) + 2\chi_0(z).\end{aligned}$$

Since P_0 is the trivial representation, then as SU_2-representations,

$$V\otimes V\otimes V^*\otimes V^*\cong P_4\oplus(P_2)^{\oplus 3}\oplus\underline{\mathbb{C}}^{\oplus 2}.$$

The Hilbert space is the space of G-invariants factors through the direct sum. For an irreducible representation, $P_n^G=0$ unless P_n is the trivial representation, so the G-invariants vanish on the copies of P_4 and P_2, leaving a 2-dimensional space. ⊠

5. The Jones polynomial from Chern-Simons theory: 2/21/17

"S^3 is much bigger than S^2."

Today, Sebastian spoke about the rest of Witten's paper [7], in particular how the Jones polynomial arises out of Chern-Simons theory.

Recall that by a topological quantum field theory (TQFT) we mean a symmetric monoidal functor (meaning taking disjoint unions to tensor products) $Z:\mathscr{B}ord_n\to\mathscr{V}ect_{\mathbb{C}}$. The objects of $\mathscr{B}ord_n$ are closed $(n-1)$-manifolds, which Z maps to vector spaces, which turn out to always be finite-dimensional, and the morphisms of $\mathscr{B}ord_n$ are (diffeomorphism classes of) bordisms $B:M\to N$, which are sent to $\mathbb{C}$-linear maps $Z(B):Z(M)\to Z(N)$. Bordisms compose by gluing along a common boundary, and this is mapped to composition of linear maps.

Generally, one cares about manifolds with a little extra structure, e.g. orientation. Today the extra structure will be a framing, though we won't do much in the way of explicit calculations with it.

The gluing-to-composition property makes it possible to calculate Z on complicated bordisms in terms of simpler ones: cut your bordism B into a sequence of simpler bordisms B_i that glue to form B, and therefore $Z(B)$ is the composition of these $Z(B_i)$.[10] The punchline is: *to understand a TQFT, you can cut things into smaller pieces, understand the smaller pieces, and then glue them together in a prescribed way.*

There's a similar way to think about knot polynomials. Polynomials such as the Jones polynomial and the Alexander polynomial are determined by skein relations, in that you can evaluate, say, the Jones polynomial on a complicated knot by cutting it into smaller pieces and determining the Jones polynomial on those pieces. It's a different kind of cutting, where the ends are glued back together in certain ways, but it suggests that there could be a relation. For example the Jones polynomial is determined by its value on the unknot $V(0)=1$ and the Skein relation is given in (1.5) (see Figure 1.3 for what L_+, L_0, and L_- mean).

So, is there a way to relate these two kinds of decomposition by cutting? Anything's possible for Witten. His paper didn't come out of nowhere, though; there was already a connection between the Jones polynomial and 2D conformal field theory, and Witten found a 3D TQFT which related to that conformal field theory.

The first step is to fix our bordism category. We want to compute the Jones polynomial (or other invariants) for knots in three-manifolds, e.g. S^3, and these are example morphisms. An example object would be a generic slice of this, and the intersection of the slice and the embedded knot is a set of points. Thus, we consider the following bordism category.

Objects: Closed, oriented 2-manifolds with isolated marked points.
Morphisms: Oriented 3-manifolds with embedded links.

This bordism category is no mathematical artifact: physicists think of embedded links as histories of particles interacting in fields. This suggests using a gauge theory, in this case Chern-Simons theory. Fix a Lie group G, which we'll assume is simple and simply-connected. Pick a principal G-bundle $P\to M$ together with a connection ∇ for P, which is locally $\mathrm{d}+A$ for some $A\in\Omega^1_M(\mathfrak{g})$. Then, the Chern-Simons action is

$$S^k_{\mathrm{CS}}(A)=\frac{k}{4\pi}\int_M\mathrm{Tr}\left(A\wedge\mathrm{d}A+\frac{2}{3}A\wedge A\wedge A\right).$$

The integer k is called the level, and though the classical theory was independent of k, the quantum theory is not. The Chern-Simons action is not gauge invariant (meaning $S^k_{\mathrm{CS}}(A)$ can change under the action of G), which is a little alarming, and is the reason that, as a theory of oriented manifolds, Chern-Simons theory is not a TQFT! This is the anomaly we discussed last time — and it makes sense that it shouldn't be topological, because it's closely related to the Wess-Zumino-Witten conformal field theory, which is also not a TQFT.

The partition function is defined by modding out by the gauge action and exponentiating. Then, there's the path integral, which is a huge headache for mathematicians (and there are a few other issues that should be addressed

[10] For example, any 2-dimensional oriented bordism can be cut into incoming and outgoing pairs of pants and discs.

in a formalization of this theory):

$$Z(M) = \int_{\mathscr{A}/\mathscr{G}} DA\, e^{iS^k_{\mathrm{CS}}(A)}.$$

Here, $\mathscr{A}/\mathscr{G}$ is the moduli space of connections modulo gauge invariance.

For a link K embedded in a 3-manifold M, choose a representation V_i of G for every component C_i of K. Then, the invariant associated with the theory is the Wilson line operator, the holonomy around each loop:

$$W_{V_i}(C_i) = \mathrm{Tr}_{V_i}\Big(Pe^{\oint_{C_i} A}\Big).$$

These operators get inserted into the path integral, producing the invariant

$$Z(M; \{(C_i, V_i)\}) = \int_{\mathscr{A}/\mathscr{G}} DA \prod_{i=1}^m W_{V_i}(C_i) e^{iS^k_{\mathrm{CS}}(A)}.$$

This is a number depending on M, C_i, and V_i, but not on the connection.

This leaves two questions, each significant:

- How do we compute this?
- How does it relate to the Jones polynomial?

Computing the Chern-Simons invariants looks daunting, but it behaves well under surgery.

Theorem 5.1 (Lickorish-Wallace)**.** *Every closed, connected, orientable* 3*-manifold can be obtained by performing Dehn surgery on a link in the* 3*-sphere.*

We'll hear more about this in the next two weeks, but the point is that, if you understand how the invariants change under Dehn surgery and you know the invariants on links in S^3, you know them everywhere.

The second step in computation is to determine skein relations for the Chern-Simons invariant. Thus, they can be used to compute the invariants on arbitrary links starting with those for unlinked unknots. Finally, cutting the sphere into a sequence of bordisms will mean that it suffices to understand what happens on the disc and on S^3 with a single embedded unknot.

The 3-sphere can be considered a composition of two bordisms: from the north pole to the equator, then the equator (an S^2) to the south pole. Thus, Chern-Simons theory sends these to maps $Z(\varnothing) = \mathbb{C} \to Z(S^2) \to Z(\varnothing) = \mathbb{C}$, which sends $1 \mapsto |\psi\rangle$ and then $|\psi\rangle \to \langle \chi \mid \psi\rangle$, called a **vacuum expectation value**.

Understanding what this is on S^2 involves delving into conformal field theory, but this is well-understood. On the sphere with marked points $p_1, \dots, p_r$ marked with representations $V_1, \dots, V_r$, then for sufficiently large k, the state space is

$$Z(S^2_{\{V_1,\dots,V_r\}}) = \left(\bigotimes_{i=1}^r V_i\right)^G.$$

That is, we take all the representations, tensor them together, and take G-invariants. For something we first described with a path integral, this is surprisingly concrete! For any k, the state space is a subspace of this.

Example 5.2.

- If there are no punctures, $\dim Z(S^2) = 1$.
- If there's one puncture, this corresponds to something going in but not out, and $Z(S^2)$ is trivial unless the puncture is labeled by the trivial representation, in which case you get $\mathbb{C}$.
- If there are 2 punctures labeled by V_1 and V_2, then $\dim Z(S^2_{V_1,V_2})$ is trivial unless $V_1 = V_2^*$, in which case it's 1-dimensional. This corresponds to a single Wilson line labeled by V_1 that's both incoming and outgoing.
- Let V be the defining representation of $G = \mathrm{SU}_n$, and consider S^2 punctured four times and labeled with V, V, V^*, and V^* (so for two loops). Then, the dimensions of $Z(S^2)$ is at most 2, since the SU_n-invariant space of $V \otimes V \otimes V^* \otimes V^*$ is two-dimensional. ◀

The last calculation is important: suppose all links are labeled with the defining SU_2-representation V you cut a ball out of S^3 and replace it with one of L_+, L_0, or L_-, then the values of the theory on L_+, L_-, and L_0 are states in $Z(S^2_{V,V,V^*,V^*})$, so they must be linearly dependent. That is, there's a skein relation

$$\alpha z(L_+) + \beta(L_0) + \gamma Z(L_-) = 0. \tag{5.3}$$

Determining the actual values of α, β, and γ requires some conformal field theory, however.

The next step is to determine what happens when you cut two disconnected loops apart. This is a connected sum of the two pieces, hence cutting across an S^2 with zero marked points. Thus, it's a one-dimensional Hilbert space, so the inner product is just multiplication, so $\langle a \mid b\rangle\langle c \mid d\rangle = \langle a \mid d\rangle\langle c \mid b\rangle$. Thus,

$$\begin{aligned} Z(M_1 \# M_2) Z(S^3) &= \langle \overline{M}_1 \mid \overline{M}_2\rangle\langle D^3 \mid D^3\rangle \\ &= \langle \overline{M}_1 \mid D^3\rangle\langle \overline{M}_2 \mid D^3\rangle \\ &= Z(M_1)\cdot Z(M_2). \end{aligned}$$

Rescaling, let $\langle M\rangle := Z(M)/Z(S^3)$, so $Z(M_1 \# M_2) = Z(M_1)Z(M_2)$. Similarly, let O denote the unknot in S^3, so we normalize for links by letting $\langle M, C\rangle := Z(M, C)/Z(S^3, O)$.

As a consequence $\langle S^3, 0\rangle = 1$, and for SU_n, we can fill in the constants in (5.3), which comes from a calculation in conformal field theory:

$$\begin{aligned} \alpha &= -\exp\left(\frac{2\pi i}{n(n+k)}\right) \\ \beta &= -\exp\left(\frac{\pi i(2-n-n^2)}{n(n+k)}\right) + \exp\left(\frac{\pi i(2+n+n^2)}{n(n+k)}\right) \\ \gamma &= \exp\left(\frac{2\pi i(1-n^2)}{n(n+k)}\right). \end{aligned}$$

If you multiply (5.3) by $\exp(\pi i(n^2-2)/n(n+k))$ and substitute

$$q := \exp\left(\frac{2\pi i}{n(n+k)}\right),$$

the result is the Skein relation

$$q^{-n/2}L_+ + (q^{1/2} - q^{-1/2})L_0 + q^{-n/2}L_- = 0, \tag{5.4}$$

at least for sufficiently high k (but this determines a polynomial for all k).

When $n = 2$, (5.4) is the Skein relation for the Jones polynomial. For general n, this is the HOMFLY polynomial (albeit with a different substitution). If you run a similar calculation with different groups, you get different invariants.

- For $G = \mathrm{SO}_n$, you get the Kauffman polynomial.
- For $G = \mathrm{U}_1$, in the large-k limit you get the linking number.

One interesting aspect of this derivation of the Jones polynomial is that it's manifestly a link invariant from this perspective, but it's hard to see that it's a polynomial. Conversely, the usual derivations make it obvious that it's a polynomial, but it's harder to see that it's a link invariant!

6. TQFTs and the Kauffman bracket: 2/28/17

Today, Gill spoke about Blanchet-Habegger-Masbaum-Vogel's paper [2], which is one of the papers that takes Witten's physical argument and turns it into a rigorous mathematical proof using surgery theory.

Today, we're working in the bordism category $\mathcal{B}ord_n$ of *space* dimension n, i.e. its objects are closed, oriented n-manifolds and its morphisms $\mathrm{Hom}_{\mathcal{B}ord_n}(N_1, N_2)$ are the (diffeomorphism classes of) compact, oriented bordisms $X : N_1 \to N_2$. Composition in $\mathcal{B}ord_n$ is by gluing of bordisms,[11] and the identity on N is $N \times [0,1] : N \to N$. There's an involution on this category defined by reversing orientations and morphisms, which is contravariant.

We care about a slighly different category $\mathscr{C} = \mathcal{B}ord_2^{p_1}(e)$. The p_1 means this is a category of manifolds with p_1-structure, which is a choice of trivialization of the first Pontrjagin class; this is akin to a spin structure, which is a choice of trivialization of the first two Stiefel-Whitney classes w_1 and w_2. The (e) bit means that we ask the objects (closed surfaces with p_1-structure) to have an even number of marked points and the morphisms (p_1-bordisms) $X : N_1 \to N_2$ to come with embeddings $C \times I \to X$, where C is a 1-manifold with boundary with an even number of components, and such that $\partial C \cap N_i$ is the marked points in N_i for $i = 1, 2$. We consider two bordisms equivalent if there's an orientation-preserving diffeomorphism between them fixing the boundary and preserving the p_1-structure.

[11] This is associative because we've taken equivalence classes of bordisms up to diffeomorphisms that are the identity on the boundary, which is why taking equivalence classes is important.

Quantization functors. In this section, $\mathscr{C}$ can be $\mathscr{B}ord_n^{p_1}(e)$ or $\mathscr{B}ord_n$ (or other bordism categories). By "n-manifold" we mean an object of $\mathscr{C}$ (e.g. oriented manifold, p_1-manifold), and by "$(n+1)$-manifold" we mean a morphism in $\mathscr{C}$ (oriented bordism, p_1-bordism, etc.).

Let A be a commutative ring (which we always assume has an identity) with a conjugation involution $c \mapsto \overline{c}$, and let $V : \mathscr{C} \to \mathscr{M}od_A$ be a functor sending $\varnothing \mapsto A$. In particular, given a bordism $X : M \to N$, we obtain a map $V(X) : V(M) \to V(N)$. For historical reasons, $V(X)$ is also denoted $Z(X)$. A 3-manifold X is a bordism $\varnothing \to X$, and therefore determines a map $A \to V(\partial X)$. This is equivalent to the data of where $1 \in A$ maps to, so $Z(X)$ is identified with the image of 1 in $V(\partial X)$. Similarly, a closed 3-manifold defines a bordism $\varnothing \to \varnothing$, hence a value $V(X) \in V(\varnothing) = A$.

Since A has involution, it makes sense to define when a bilinear form on A is sesquilinear and Hermitian: we ask that $\langle ax, by\rangle = a\overline{b}\langle x, y\rangle$ and $\langle y, x\rangle = \overline{\langle x, y\rangle}$, respectively.

Definition 6.1. A **quantization functor** is a functor V sending $\varnothing \to A$ together with a nondegenerate, sesquilinear, Hermitian form on $V(\Sigma)$ for all closed n-manifolds (objects of $\mathscr{C}$) Σ, and such that for all $(n+1)$-manifolds M_1 and M_2 with $\partial(M_1) \cong \partial(M_2) \cong \Sigma$,

$$\langle Z(M_1), Z(M_2)\rangle_{V(\Sigma)} = V(M_1 \cup_\Sigma M_2).$$

Often, the number (well, element of A) $V(M)$ for M a closed $(n+1)$-manifold is denoted $\langle M\rangle$, and called the **bracket**. Thus $\langle\varnothing\rangle = 1$ (as $\varnothing$ is the empty bordism $\varnothing \mapsto \varnothing$).

Definition 6.2. With V as above, V is **cobordism generated** if for all $\Sigma \in \mathscr{C}$, the collection $\{Z(M) \mid \partial M = \Sigma\}$ generates $V(\Sigma)$.

Definition 6.3.

- The bracket is **multiplicative** if for all closed $(n+1)$-manifolds M_1 and M_2, $\langle M_1 \amalg M_2\rangle = \langle M_1\rangle\langle M_2\rangle$.
- The bracket is **involutive** if for all closed $(n+1)$-manifolds M, $\langle -M\rangle = \overline{\langle M\rangle}$, where $-M$ denotes M with the opposite orientation.

If V is a quantization functor, then its bracket is automatically multiplicative and involutive:

$$\langle -M\rangle = \langle \varnothing \cup \varnothing - M\rangle = \langle Z(\varnothing), Z(M)\rangle_\varnothing = \overline{\langle M\rangle}\langle 1, 1\rangle_\varnothing = \overline{\langle M\rangle}.$$
$$\langle M_1 \amalg M_2\rangle = \langle M_1 \cup_\varnothing -M_2\rangle = \langle Z(M_1), Z(-M_2)\rangle_\varnothing = \langle M_1\rangle\overline{\langle -M_2\rangle}\langle 1, 1\rangle_\varnothing = \langle M_1\rangle\langle M_2\rangle.$$

What's particularly nice is that the converse is true.

Proposition 6.4. *Let $\langle -\rangle$ be an invariant of closed $(n+1)$-manifolds, i.e. a function of sets $\mathrm{Hom}_{\mathscr{C}}(\varnothing, \varnothing) \to A$. If $\langle -\rangle$ is multiplicative and involutive, then there is a unique quantization functor on $\mathscr{C}$ that extends it.*

Proof. The proof is by a universal construction.

(1) Let $N \in \mathscr{C}$. Then, let $\mathscr{U}(\Sigma)$ be the free A-module generated by diffeomorphism classes of $(n+1)$-manifolds M with $\partial M = N$.
(2) Let $\langle M, M'\rangle_N := \langle M \cup_N -M'\rangle$ whenever $\partial M \cong \partial M' \cong N$, and extend linearly; since $\langle -\rangle$ is multiplicative and involutive, this is Hermitian and sesquilinear.
(3) Take the quotient of $\mathscr{U}(N)$ by the left kernel of $\langle -, -\rangle_\Sigma$, the elements x such that $\langle x, -\rangle = 0$, and let $V(N)$ be this quotient module, so that $\langle -, -\rangle_N$ is nondegenerate.
(4) Now we need to define morphisms obtained from bordisms. Let $M : N_1 \to N_2$ be a bordism in $\mathscr{C}$, and suppose $\partial M' = N_1$, so $Z(M') \in V(N_1)$. Then, we say that $Z_M(Z(M')) := Z(M' \cup_{N_1} M) \in V(N_2)$. ⊠

This is pretty cool, but you might ask for yet more axioms on your quantization functor: that it's symmetric monoidal.

I: $V(-N) \cong V(N)^*$ for all $N \in \mathscr{C}$.
M: $V(N_1 \amalg N_2) \cong V(N_1) \otimes V(N_2)$.
F: For all N, $V(N)$ is free of finite rank and the bracket is unimodular.

Definition 6.5. A cobordism-generated quantization functor satisfying the latter two axioms above is called a **topological quantum field theory** (TQFT).

This agrees with our usual definition of the word.

Vaguely, this allows us to define our main theorem: for each $p \geq 3$ in $\mathbb{N}$, there's a ring $k_p := \mathbb{Z}[a, k, d^{-1}]/(\varphi_{2p}(a), k^6 - u)$ (where φ_{2p} is the $2p^{\text{th}}$ cyclotomic polynomial) and a quantization functor V_p on $\mathscr{B}ord_2^{p_1}(e)$, and this V_p is a TQFT. Moreover, V_p satisfies the Kauffman bracket relations and surgery axioms Definition 1.7, and every cobordism generated quantization functor over an integral domain which satisfies the Kauffman bracket and surgery axioms is obtained from some V_p by a change of coefficients.

Recall that the Kauffman bracket relations were $\langle C \rangle = A\langle V \rangle + A^{-1}\langle H \rangle$, when resolving a crossing C as two vertical lines V or two horizontal lines H. Moreover, a link plus an unlinked unknot satisfies $\langle L \cup O \rangle = (-A^2 - A^{-2})\langle L \rangle$.

Definition 6.6. Let A be a commutative ring and $a \in A$ be a unit. Let M be a compact 3-manifold and L be a banded link in ∂M. Then, the **Jones-Kauffman skein module** $K(M, L)$ is the A-module generated by isotopy classes of $\widetilde{L} \subset M$ such that $\widetilde{L} \cap \partial M = L$, quotiented by the Kauffman bracket relations.

If $\mathscr{L}(M, L)$ denotes the free A-module before we quotiented, and V is a quantization functor, then $V(\partial M, L)$ is a quotient of it (under the map $L \mapsto Z(M, L)$).

Definition 6.7. A quantization functor V **satisfies the Kauffman bracket relations** if the quotient $V(\partial M, L)$ factors through $K(M, L)$.

Let's say a little about surgery. Since $\partial(S^p \times D^q) \cong \partial(D^{p+1} \times S^{q-1}) \cong S^p \times S^{q-1}$, it's possible to excise an $S^p \times D^q \subset M$ and replace it with a $D^{p+1} \times S^{q-1}$. You can express the Kauffman bracket relations in terms of surgery.

(0) $\langle S^3 \rangle \in A$ is invertible.
(1) $Z(S^0 \times D^3) = \eta Z(D^1 \times S^2) \in V(S^0 \times S^2)$ for some $\eta \in A$ (coming from index 1 surgery).
(2) $Z(D^2 \times S^1)$ lies in the submodule generated by links in $-(S^1 \times D^2)$ (coming from index 2 surgery).
(3) There are more axioms corresponding to higher-index surgery.

The important takeaway is:

Proposition 6.8. *If V satisfies the surgery axioms and M is connected with boundary Σ, then the map $\mathscr{L}(M, L) \to V(\Sigma, L)$ is surjective. Moreover, if M' is a connected manifold such that $\partial M' = \Sigma$, then the left kernel of this map is the left kernel of the forms*
$$\langle L, L' \rangle_{(M,M')} = \langle Z(M, L), Z(M', L') \rangle_{V(\Sigma)}.$$

Anyways, the point is that this quantization functor, whose definition may be ugly, but it has really nice properties which imply that it's a TQFT. This leads us to the Jones polynomial: a banded link defines an element of A, and this element is a polynomial! More on this next time.

7. TQFTs and the Kauffman bracket, II: 3/7/17

Today, Adrian talked about the paper [2] again. The paper can be confusing to read, so Adrian's going to provide a structuralist overview of what they're showing, what the different parts are, and so on. There won't even be time to delve into the proofs, unfortunately.

The goal of this paper was to rigorously construct a TQFT equivalent to Witten's Chern-Simons theory. The partition functions, as invariants of 3-manifolds, had already been rigorously constructed by Reshetikhin-Turaev, but not the rest of the field theory.

Last time, we defined TQFTs in a somewhat loquacious manner: a TQFT is a functor $V : \mathscr{B}ord_2^* \to \mathscr{V}ect_k$, where k is an integral domain and $*$ represents some tangential structure we'll specify later, satisfying the following axioms.

- **Monoidality:** $V(M \amalg N) = V(M) \amalg V(N)$.
- **Unitality:** $V(\varnothing) = k$. These two axioms imply V is symmetric monoidal.
- **Cobordism generated:** For all surfaces $\Sigma \in \mathscr{B}ord_2^*$, $V(\Sigma)$ is generated by $\{Z_M(1)\}$ where M ranges over all 3-manifolds with boundary Σ.
- **Q2:** V is compatible with the inner product: for all M_1 and M_2 such that $\partial M_1 = \partial M_2 = \Sigma$, $\langle M_1 \cup_\Sigma (-M_2) \rangle = \langle Z_{M_1}(1), Z_{M_2}(1) \rangle_\Sigma$.
- **Unimodularity:** For all surfaces Σ and $v \in V(\Sigma)$, $\langle v, - \rangle$ defines an isomorphism $V \to V^*$.

These are not arbitrary axioms: rather, they are things that people generally envisioned a good TQFT as satisfying. There is a caveat: [2] doesn't construct an isomorphism from the theory they constuct to compute the Jones polynomial and Witten's Chern-Simons theory, but they do prove a uniqueness result forcing such an isomorphism to exist.

To obtain the Jones polynomial, we need to consider the cobordism category described last lecture, where 3-manifolds have embedded links and 2-manifolds have marked points representing the boundaries of these links. We want these to satisfy the Kauffman bracket relations (Definition 6.7), and to also follow some surgery axioms.

(1) We'd like $\eta := \langle S^3 \rangle$ to be invertible, so that we can normalize by it.
(2) If M' is obtained from M via index 1 surgery, then $\langle M' \rangle = \eta \langle M \rangle$.
(3) There exist $\sum \lambda_i L_i$ in $V(S^1 \times D^2)$ such that if M' is obtained from M via index 2 surgery, then $\langle M' \rangle = \sum_i \lambda_i \langle M_i \rangle$, where M is glued by the links L_i.

All 3-manifolds can be obtained from sufficiently interesting surgeries on the sphere, so what this says is that the theory on the sphere (with embedded links) determines the values of the theory.

Finally, for all surfaces Σ, we'd like $\operatorname{rank} V(\Sigma)$ to agree with the rank that Witten gives, and finally a universality condition will cause uniqueness. We discussed last time that a multiplicative and involutive invariant is uniquely given by a TQFT (Proposition 6.4).

There's a frustrating issue with this theory, and that's its dependence on a p_1-structure. The paper isn't very explicit about why this structure in particular is necessary: in fact, there are a few choices, all relaxations of a framing. In any case, the theory we wrote down depends on the p_1-structure: there are multiple p_1-structures on S^3, and they produce different values.

So what is a p_1-structure? Let $B\mathrm{O}$ denote the classifying space of the infinite orthogonal group, so the first Pontrjagin class $p_1 \in H^4(B\mathrm{O})$. Thus, it is equivalent data to a homotopy class of maps $p_1 : B\mathrm{O} \to K(\mathbb{Z}, 4)$. Since $B\mathrm{O}$ is the classifying space for stable vector bundles, the (stabilized) tangent bundle to a manifold M defines a homotopy class of maps $t : M \to B\mathrm{O}$.

Definition 7.1. Let X be the homotopy pullback of $p_1 : B\mathrm{O} \to K(\mathbb{Z}, 4)$ along a point ("killing p_1"). Then, a **p_1-structure** on M is a homotopy class of maps $t' : M \to X$ lifting t.

Homotopy pullbacks are somewhat abstract, but here's another way: instead of considering $p_1(M)$ inside the set $H^4(M) \cong [M, K(\mathbb{Z}, 4)]$, consider the space of maps $\mathrm{Map}(M, K(\mathbb{Z}, 4))$. A p_1-structure is a path from $p_1 \circ t$ to a constant map, i.e. in the groupoid $\mathscr{G} := \pi_{\le 1} \mathrm{Map}(M, K(\mathbb{Z}, 4))$, an element of $\mathrm{Hom}_{\mathscr{G}}(p_1 \circ t, *)$.[12] When you glue together manifolds with p_1-structures, it's useful to remember the path, not just its homotopy class.

In Proposition 6.4, we learned that given a multiplicative and involutive 3-manifold invariant, there's a unique TQFT yielding that invariant. Here's the way we'll turn this into the result we want.

(1) First, choose the invariant (and structure on p_1-manifolds, and coefficient ring) suitably from the Kauffman bracket.
(2) Then, show the additional properties are satisfied, first by splitting 3-manifolds into a composition of bordisms.
(3) Then, split 2-manifolds into compositions of bordisms with corners to check the rest of the axioms. This is interesting because it's one of the earlier instances of **extended TQFT**, associating a (k-linear) category to a 1-manifold, shortly after the initial proposal [3]. Extending the theory is ultimately necessary to obtain monoidality and to check that the theory has the correct dimension on surfaces by splitting a surface into spheres.
(4) Finally, use the Verlinde formula to obtain the correct dimensions. This is basically what it was invented for, which is nice.

There's also something to say about universality: one of the paper's results is the construction of a ring k_p for every prime p such that for any TQFT Z defined over k that satisfies the other axioms, there's a ring map $k_p \to k$ that carries the invariant in question to Z. So this is almost unique, though it does depend on p.

Part 2. Categorification: Khovanov homology

8. Khovanov homology and computations: 3/21/17

Today, Jacob talked about the construction of Khovanov homology and did some computations. Throughout today's lecture, D will denote a link.

[12] Though this definition is very confusing, the idea is to replace the set $H^4(M)$ with a groupoid, so it's possible to choose a witness to $p_1 = 0$ inside it. The same approach can be used to define orientations as a choice of trivialization of w_1, spin^c structures as trivializations of W_3, etc.

Khovanov himself called Khovanov homology a categorification of the Jones polynomial. What does that mean? The clearest example is the categorification of the Euler characteristic χ of nice spaces (say, finite simplicial complexes). This is a useful invariant, but isn't very powerful, e.g. not distinguishing spheres of different dimensions. Replacing it with the homology groups, a graded abelian group, produces a stronger invariant whose Euler characteristic (for chain complexes, i.e. the alternating sum of dimensions) recovers the Euler characteristic for topological spaces.

In the same way, the normalized Jones polynomial $\widehat{J}$ is categorified into the Khovanov homology $\mathrm{Kh}^{*,*}$, a bigraded homology theory for links.

8.1. **Some recollections about the normalized Jones polynomial.** Let D be a diagram of an oriented link L with n crossings, n_+ of which are positively oriented and n_- of which are negatively oriented. Let X denote the crossing with \ over /. Let S_H denote the smoothing into two horizontal segments and S_V denote the smoothing into two vertical segments. Then, the Kauffman bracket is a quantity valued in $\mathbb{Z}[q,q^{-1}]$ satisfying the skein relation $\langle X\rangle = \langle S_H\rangle + q\langle S_V\rangle$, and where the value on the unlink with k components is $(q+q^{-1})^k$. This isn't invariant under Reidemeister moves, but can be fixed into the Jones polynomial

$$\widehat{J}(D) := (-1)^{n_-}q^{n_+-2n_-}\langle D\rangle.$$

Example 8.1. The Hopf link has two crossings X_1 and X_2 (from top to bottom), so we get four smoothings. Use 0 to denote smoothing horizontally, and 1 to denote smoothing vertically.

- If you smooth both horizontally (00), you get an unlink with two components.
- If you smooth 10 or 01, you get an unknot.
- If you smooth 11, you get an unlink with two components.

If $\alpha \in \{0,1\}^n$, r_α denotes the total number of 1s in α, and k_α is the number of circles in the smoothing defined by α, the Jones polynomial satisfies the formula

$$\widehat{J}(D) = \sum_{\alpha\in\{0,1\}^n} (-1)^{r_\alpha+n_-}q^{r_\alpha+n_+-2n_-}(q+q^{-1})^{k_\alpha}. \tag{8.2}$$

This equation will be useful when we show the graded Euler characteristic of Khovanov homology is the Jones polynomial. But for now, let's use it to compute the Jones polynomial of the Hopf link:

- The 00 resolution contributes $q^2(q+q^{-1})$.
- The 01 and 10 resolutions each contribute $-q^3(q+q^{-1})$.
- The 11 resolution contributes $q^4(q+q^{-1})^2$.

Thus, the Jones polynomial of the Hopf link H is

$$\widehat{J}(H) = q^2(q+q^{-1})(1-q+q^3).$$ ◂

8.2. **The construction of Khovanov homology.** Recall that a graded vector space W is a direct sum $W = \bigoplus_{m\in\mathbb{Z}} W^m$. If $\ell\in\mathbb{Z}$, $W[\ell]$ denotes the graded vector space whose m^{th} component is $W^{m-\ell}$.

Definition 8.3. The **quantum dimension** or **graded dimension** of a graded vector space W is

$$\mathrm{qdim}(W) := \sum_m q^m \dim(W^m) \in \mathbb{Z}[q,q^{-1}].$$

Lemma 8.4. *Let W and W' be graded vector spaces.*

(1) $\mathrm{qdim}(W\otimes W') = \mathrm{qdim}(W)\,\mathrm{qdim}(W')$.
(2) $\mathrm{qdim}(W\oplus W') = \mathrm{qdim}(W) + \mathrm{qdim}(W')$.
(3) $\mathrm{qdim}(W[\ell]) = q^\ell\,\mathrm{qdim}(W)$.

These should be reminiscent of the behavior of the ordinary dimension for ungraded vector spaces.

Definition 8.5. If $W^{*,*}$ is a bigraded vector space, its **graded Euler characteristic is**

$$\chi(W) := \sum_{i,j\in\mathbb{Z}} (-1)^i\,\mathrm{qdim}(W^{i,*}) \in \mathbb{Z}[q,q^{-1}].$$

Though you can define Khovanov homology over $\mathbb{Z}$, we'll work over $\mathbb{Q}$ for now, so as to not have to address torsion. Let V be the graded $\mathbb{Q}$-vector space with basis $\{1,x\}$, where $\deg 1 = 1$ and $\deg x = -1$. Given an $\alpha\in\{0,1\}^n$, let

$$W_\alpha := V^{\otimes k_\alpha}[r_\alpha + n_+ - 2n_-].$$

Thus we can begin to define the chain complex: let

$$C^{i,*}(D) := \bigoplus_{\alpha\colon r_\alpha = i+n_-} V_\alpha.$$

This is only supported in gradings $-n_- \le i \le n_+$. In particular, it's finite.

Example 8.6. Let's see what this looks like for the Hopf link.

- The 00 smoothing contributes a term of $V^{\otimes 2}[2]$, so $C^{0,*}(H) = V^{\otimes 2}[2]$.
- The 01 and 10 terms each contribute a $V[3]$, so $C^{1,*}(H) = V[3]^{\oplus 2}$.
- The 11 term contributes a term of $V^{\otimes 2}[4]$, so $C^{2,*}(H) = V^{\otimes 2}[4]$. $\blacktriangleleft$

This is actually bigraded: if $v \in V_\alpha \subset C^{i,j}(D)$, the **homological grading** is $i := r_\alpha - n_-$, and the **quantum grading** is $j := \deg(v) + i + n_+ - n_-$. Notice that the quantum grading resembles the exponent in (8.2).

The $\alpha \in \{0,1\}^n$ are a hypercube: each edge between α and α' preserves all digits but one; we'll let the edge sending $i_1 i_2 \cdots i_j \mathbf{0} i_{j+2} \cdots i_n$ to $i_1 i_2 \cdots i_j \mathbf{1} i_{j+2} \cdots i_n$ be denoted $i_1 i_2 \cdots i_j \star i_{j+2} \cdots i_n$ (e.g. for the Hopf link, $0\star$, $\star 0$, $1\star$, and $\star 1$). The key is that at the crossing whose index the $\star$ is in, passing from one smoothing to another is a cobordism where you excise a neighborhood of the crossing and glue in a saddle, and the rest of the link is unchanged. Thus, the cobordisms will be either cylinders or pairs of pants.

To construct the differential, we need linear maps associated with these cobordisms, as the cobordism passes from a smoothing with homological grading i to one of homological grading $i+1$. We'll generate them by defining a commutative Frobenius algebra, where the cylinder is sent to the identity map, the outgoing pair of pants is sent to the **multiplication** map, and the incoming pair of pants is sent to the **comultiplication** map. Precisely, the multiplication map is the map $m\colon V \otimes V \to V$ sending

$$\begin{aligned} 1 \otimes 1 &\longmapsto 1 \\ 1 \otimes x, x \otimes 1 &\longmapsto x \\ x \otimes x &\longmapsto 0. \end{aligned}$$

The comultiplication map is the map $\Delta\colon V \to V \otimes V$ sending

$$\begin{aligned} 1 &\longmapsto 1 \otimes x + x \otimes 1 \\ x &\longmapsto x \otimes x. \end{aligned}$$

To define a commutative Frobenius algebra, we also need either a unit or a counit (and get the other for free): the **unit** is the map $i\colon \mathbb{Q} \to V$ sending $1 \mapsto 1$, and the **counit** is the map $\epsilon\colon V \to \mathbb{Q}$ sending $1 \mapsto 0$ and $x \mapsto 1$. You can check that (m, Δ, i, ϵ) define a commutative Frobenius algebra structure on V.[13]

Now we can define the differential. Let (α, α') be a pair of strings differing in a single index $\star$, $w(\alpha, \alpha')$ be the number of 1s to the left of $\star$ (which is the same for α and α'), and $\operatorname{sign}(\alpha, \alpha') = (-1)^{w(\alpha,\alpha')}$.

We associated a cobordism to (α, α'): let $d_{\alpha,\alpha'} : V_\alpha \to V_{\alpha'}$ act by the identity on crossings where the bordism is a cylinder, m where it's an outgoing pair of pants, and Δ where it's an incoming pair of pants. Then, the differential $d^i\colon C^{i,*}(D) \to C^{i+1,*}(D)$ for a $v \in V_\alpha \subset C^{i,*}(D)$ is

$$d^i * (v) := \sum_{\alpha' \text{ such that } (\alpha,\alpha')} \operatorname{sign}(\alpha, \alpha') d_{\alpha,\alpha'}.$$

Proposition 8.7. $d^{i+1} \circ d^i = 0$.

This is a chore to work out, but the key is the $\operatorname{sign}(\alpha, \alpha')$ terms, which force everything to cancel out. Thus we may take homology:

Definition 8.8. The **Khovanov homology** of D, denoted $\mathrm{Kh}^{*,*}(D)$, is the homology of the chain complex $(C^{*,*}, d)$ (in the first index of $C^{*,*}$).

Proposition 8.9. *Khovanov homology is invariant under the Reidemeister moves, and therefore is a link invariant.*

This proof is also not fun, but is done well in Turner's notes "Five lectures on Khovanov homology."

Hom. degree	0	1	2
Cycles	$\{x\otimes x, 1\otimes x - x\otimes 1\}$	$\{(1,1),(x,x)\}$	$\{1\otimes 1, 1\otimes x, x\otimes 1, x\otimes x\}$
Boundaries		$\{(1,1),(x,x)\}$	$\{x\otimes x, 1\otimes x + x\otimes 1\}$
Homology	$\{x\otimes x, 1\otimes x - x\otimes 1\}$	0	$\{1\otimes 1, 1\otimes x\}$
q-degree	$0,2$		$6,4$

TABLE 1. Computation of the Khovanov homology of the Hopf link.

8.3. **Computation for the Hopf link.** The first thing is to check what the cobordism does: for $0\star$ and $\star 0$, we get outgoing pairs of pants, hence m, and for $1\star$ and $\star 1$, we get incoming pairs of pants, hence Δ.

See Table 1 for the calculations, and Table 2 for the final answer. The graded Euler characteristic is $q^6+q^4+q^2+1$, which is the Jones polynomial for the Hopf link, as desired.

$j\downarrow \backslash i \rightarrow$	0	1	2
6			$\mathbb{Q}$
5			
4			$\mathbb{Q}$
3			
2	$\mathbb{Q}$		
1			
0	$\mathbb{Q}$		

TABLE 2. The Khovanov homology of the Hopf link.

This is a geometric way to construct Khovanov homology: there's also a more combinatorial version that was constructed recently. Also, the Khovanov homology is cohomologically graded, but it was called a homology theory at first and that's the name that stuck.

9. KHOVANOV HOMOLOGY AND LOW-DIMENSIONAL TOPOLOGY: 3/28/17

Today, Allison talked about applications of Khovanov homology in low-dimensional topology, including Milnor's conjecture, Khovanov-Lee homology, and Rasmussen's s-invariant. The goal is for this to be a jumping-off point into different papers and topics.

Here's a lightning overview of some applications.

- A contact structure is a particular kind of 2-plane field on a 3-manifold. Khovanov homology has applications to contact topology, which Christine will discuss next time.
- Khovanov homolology can be used to detect symmetries of a knot.
- Perhaps Khovanov homology will be useful for disproving the smooth Poincaré conjecture in dimension 4, e.g. as suggested by [4].
- Khovanov homology can be used to find obstructions to realizing a given 3-manifold as surgery on a knot K.[14]
- There are relations to other homology theories, e.g. Kronheimer-Mrowka [5] derive a relationship between Khovanov homology (diagrammatic and relatively computable) and instanton Floer homology (more powerful, harder to compute) to show that Khovanov homology detects the unknot. Typically one wants to relate Khovanov homology to some sort of Heegaard Floer homology.

Milnor's conjecture. There are a lot of things called Milnor's conjecture; we're going to look at a single one.

Let $f:\mathbb{C}^2\to\mathbb{C}$ be a complex analytic function such that $f(0)=0$ and 0 is an isolated critical point, e.g. $f_{p,q}(z_1,z_2)=z_1^p+z_2^q$. For a small $\varepsilon>0$, consider the surface $V_\varepsilon^f := f^{-1}(\varepsilon)$, and consider the link $K_\varepsilon^f := f^{-1}(\varepsilon)\cap S_\varepsilon^3$, i.e. intersecting V_ε^f with a small 3-sphere. For ε sufficiently small, this is a link in S^3, e.g. you can obtain torus links $T_{p,q}$ in this way from $f_{p,q}$. Knots that arise in this way are called **algebraic knots**.

[13] You can't do this with just any Frobenius algebra: the Khovanov hypercube puts strong constraints on which ones you can use. There is essentially one other choice to be made.

[14] The Lickorish-Wallace theorem shows that every 3-manifold arises as surgery on some link, but we're asking about a fixed link.

Every knot K in S^3 bounds a surface, called a **Seifert surface**, and its **3-genus** $g_3(K)$ is the minimal genus of a surface it bounds. It's not hard to prove that $g_3(T_{p,q}) = (p-1)(q-1)/2$. If you extend to the 4-ball, you can allow surfaces that the knots bound to extend into the ball, and the minimal-genus surface in this case is called the **smooth 4-genus** g_4^{sm}.

Conjecture 9.1 (Milnor)**.** The smooth 4-genus of $T_{p,q}$ is equal to $g_3(T_{p,q})$.[15]

This was first proven by Kronheimer-Mrowka using a lot of heavy machinery, including gauge theory. But there's a combinatorial proof using Khovanov homology, or a modified variant called Khovanov-Lee homology.

Let's recall the setup of Khovanov homology. Let $L \subset S^3$ be a link and D be a diagram of L with n crossings. Choose an orientation for D; then, we get a cube of resolutions for the diagram, indexed on $\{0,1\}^n$. Then, the edges in the hypercube are realized as oriented cobordisms, so we can apply a 2-dimensional TQFT $\mathscr{A}$ to obtain a bigraded vector space $C\mathrm{Kh}^i_j(D)$ with a differential that increases index i. Applying homology to this complex, we obtain the Khovanov homology of L.

For Khovanov-Lee homology, we work rationally, i.e. the TQFT is determined by the Frobenius algebra $\mathbb{Q}\langle 1, x\rangle$, where $\deg(1) = 1$ and $\deg(x) = -1$. The multiplication map sends $1 \otimes 1 \mapsto 1$, $1 \otimes x, x \otimes 1 \mapsto x$, and $x \otimes x \to 0$, and the comultiplication sends $x \mapsto x \otimes x$ and $1 \mapsto x \otimes 1 + 1 \otimes x$.

We're quite restricted as to which Frobenius algebra we can choose, because we want the differential to preserve the quantum grading, which comes from V. But it's unsatisfying in other ways: 1 and x do not play symmetric roles.

Khovanov-Lee homology symmetrizes the roles of 1 and x, which means that (well, after a basis change) they have something to do with opposite orientations. The construction is scarcely different: you use a slightly different TQFT $\mathscr{A}'$, where V is the same vector space, but with different multiplication: this time, we let $m(x \otimes x) = 1$ and $\Delta(x) = x \otimes x + 1 \otimes 1$. Then, you can form the chain complex and differential as before, though you have to use the new multiplication and comultiplication to define d.

It's important to check that $d^2 = 0$, but the proof is similar to that of Proposition 8.7. Thus we can take homology to obtain a graded vector space $\mathrm{Kh}'(L)$, called the **Khovanov-Lee homology** of L. In a similar way, one proves it's invariant under Reidemeister moves.

However, this differential does not preserve the quantum grading j! Thus, $\mathrm{Kh}'(L)$ is only a singly graded homology theory, though since the Khovanov-Lee differential never decreases j-gradings, we have a filtration

$$\cdots \subseteq C\mathrm{Kh}\{j \le 2\} \subseteq C\mathrm{Kh}\{j \le 1\} \subseteq \cdots$$

Corollary 9.2. *There is a spectral sequence* $\mathrm{Kh}(L) \Rightarrow \mathrm{Kh}'(L)$*.*

This is extremely important in computations.

We mentioned that Khovanov-Lee homology has something to do with orientations; here's an example.

Theorem 9.3 (Lee)**.** *Let L be a link and u be an orientation of L. Then, we can associate a class $\mathfrak{s}_u \in C\mathrm{Kh}^\bullet(D)$, called the **canonical orientation generator**, such that*

$$\mathrm{Kh}'(L) \cong \bigoplus_{\text{orientations } u} \mathbb{Q}\langle \mathfrak{s}_u \rangle,$$

and Reidemeister moves send $\mathfrak{s}_u \mapsto \lambda \mathfrak{s}_0$.

If K is a knot, it only has two orientations, so the Khovanov-Lee homology is

$$\mathrm{Kh}'(L) = \mathbb{Q}\langle \mathfrak{s}_u \rangle \oplus \mathbb{Q}\langle \mathfrak{s}_{\underline{i}} \rangle,$$

both in grading $i = 0$.

You can use this to write down some more invariants: given an $x \in \mathrm{Kh}'(L)$, let

$$\underline{\underline{q}}(x) := \max\{j \mid \text{there exists } a \in C\mathrm{Kh}\{\le j\} \text{ with } [a] = x\}.$$

There's a way to restate this using spectral sequences.

[15] In fact, one can ask whether this is true for any algebraic knot, which is sometimes called the **Thom conjecture**. This can also be proven with Khovanov-Lee homology. The general question of when the 3-genus equals the smooth 4-genus is known for large classes of knots, e.g. all positive knots.

Definition 9.4. Let K be a knot. Then, the **Rasmussen s-invariant** is

$$s(K) := \frac{\underline{\underline{q}}([\mathfrak{s}_u]) + \underline{\underline{q}}([\mathfrak{s}_{\overline{u}}])}{2}.$$

This will allow us to prove Milnor's conjecture (Conjecture 9.1).

We can use Khovanov-Lee homology to get smooth invariants (which is somewhat magical: usually you need things like gauge theory to do this). If Σ is a smooth surface embedded in $S^3 \times I$, $\partial\Sigma = K_0 \amalg K_1$ that defines a cobordism between knots $K_0 \subset S^3 \times \{0\}$ and $K_1 \subset S^3 \times \{1\}$, then $\mathscr{A}'$ defines a linear map $C\mathrm{Kh}'(K_0) \to C\mathrm{Kh}(K_1)$ by decomposing it into multiplications, comultiplications, caps, and cups, and you can check how this interacts with the j-grading: this determines a map $f_\Sigma \colon \mathrm{Kh}'(K_0) \to \mathrm{Kh}'(K_1)$.

Theorem 9.5 (Rasmussen). *Let K be a knot in S^3.*

(1) $|s(K)| \le g_4^{\mathrm{sm}}(K)$.

(2) *If K is a positive knot, in particular including torus knots, then $\mathfrak{s}_u$ lives in the maximal j-filtration on $C\mathrm{Kh}'(K)$, and therefore*
$$s(K) = g_3(K) = g_4^{\mathrm{sm}}(K).$$

The first part is already almost provable from what we've done already.

You can use this to cut about half of the hard work (namely, the gauge theory) out of showing there are exotic $\mathbb{R}^4$s; the other half you can't avoid so easily.

10. Khovanov homology for tangles and Frobenius algebras: 4/4/17

Today, Yixian talked about $(1+1)$-dimensional TQFTs and Frobenius algebras, as well as Khovanov homology for tangles. Thanks to Christine Lee for taking these notes, as I was out of town. I transcribed them into LaTeX, so any mistakes are on me.

10.1. (1+1)-dimensional TQFTs and Frobenius algebras.

Let R be a commutative ring with 1 and $\mathscr{B}ord_2$ denote the 2-dimensional *oriented* cobordism category, whose objects are closed 1-manifolds and whose morphisms are surfaces with boundary interpreted as cobordisms. $\mathscr{B}ord_2$ is symmetric monoidal under disjoint union.

Definition 10.1. A **(2-dimensional, oriented) topological quantum field theory** is a symmetric monoidal functor $Z \colon \mathscr{B}ord_2 \to \mathscr{M}od_R$.

In particular, this assigns to every closed 1-manifold T an R-module $Z(T)$. In our application to Khovanov homology, $Z(S^1)$ will be $R\{1, x\}$, and will have extra structure that we'll elucidate.

A TQFT is a symmetric monoidal functor. What does that actually tell us?

- It's *monoidal*, so if $T = T_1 \amalg T_2$, then there is an isomorphism $m_{T_1T_2} \colon Z(T) = Z(T_1) \otimes Z(T_2)$. It also means that $Z(\varnothing) = R$, regarded as an R-module. This allows composition of maps, coming from gluing of cobordisms, to be well-defined.
- The structure of the cobordism category tells us that if $\Sigma \colon T_1 \to T_2$ is a surface with boundary, thought of as a cobordism from T_1 to T_2, it is sent to an R-linear map $Z(\Sigma) \colon Z(T_1) \to Z(T_2)$. Moreover, if Σ and Σ' are homeomorphic cobordisms (through a homeomorphism fixing the boundary), then $Z(\Sigma) = Z(\Sigma')$.

  It also tells us about gluing cobordisms: let $\Sigma_1 \colon T_1 \to T_2$ and $\Sigma_2 \colon T_2 \to T_3$ be two cobordisms, so that $\Sigma_1 \amalg_{T_2} \Sigma_2$ is a cobordism from T_1 to T_3. Then, the linear map $Z(\Sigma_1 \amalg_{T_2} \Sigma_2) = Z(\Sigma_2) \circ Z(\Sigma_1)$.
- Z is *symmetric*. That is, if $\tau_1 \colon T_1 \amalg T_2 \to T_2 \amalg T_1$ is the natural isomorphism and $\tau_2 \colon V \otimes W \to W \otimes V$ is the natural isomorphism, then the following diagram commutes:
$$\begin{tikzcd} Z(T_1 \amalg T_2) \arrow[r, "m_{T_1T_2}"] \arrow[d, "Z(\tau_1)"] & Z(T_1) \otimes Z(T_2) \arrow[d, "\tau_2"] \\ Z(T_2 \amalg T_1) \arrow[r, "m_{T_2T_1}"] & Z(T_2) \otimes Z(T_1). \end{tikzcd}$$
  These τ_1 and τ_2 are called **twist maps**.

From a few simple cobordisms we also get explicit maps.

- Let D_i denote the disc regarded as a cobordism $\varnothing \to S^1$. Applying Z determines an R-linear map $R \to V$, where $V := Z(S^1)$.

- The cylinder $S^1 \times [0,1]$ is the identity in $\mathscr{B}ord_2$, hence $Z(S^1 \times [0,1]) = \mathrm{id}_V$.
- Let P_i denote the pair of pants regarded as a cobordism $S^1 \to S^1 \amalg S^1$, so $Z(P_i)\colon V \to V \otimes V$.
- Let P_o denote the pair of pants in the other direction, i.e. a cobordism $S^1 \amalg S^1 \to S^1$, which produces a map $V \otimes V \to V$.
- Let D_o denote the disc regarded as a cobordism $S^1 \to \varnothing$. Applying Z produces a map $V \to R$.

These actually capture the whole structure of a 2-dimensional TQFT, as we'll explain below.

Theorem 10.2. *There is a bijection between the isomorphism classes of 2-dimensional oriented TQFTs valued in $\mathscr{M}od_R$ and the isomorphism classes of commutative Frobenius algebras over R.*[16]

Definition 10.3. A **Frobenius algebra** over R is a unital, associative R-algebra V together with a map $\varepsilon\colon R \to V$ such that

- V is projective and finite type as an R-module,
- $\langle v, w\rangle := \varepsilon(vw)$ is a nondegenerate bilinear form.

The nondegeneracy of $\langle -,-\rangle\colon V \otimes V \to V$ allows us to define a **comultiplication** $\Delta\colon V \to V \otimes V$, which satisfies $m(\Delta(v)) = m(m(\Delta(1)), v)$, because these both arise from cobordisms diffeomorphic to $P_o \circ P_i$.

11. TQFT for tangles: 4/11/17

Today, Richard talked about how Khovanov homology relates to TQFT. The reference for this is Dror Bar-Natan's notes [1]. It's an interesting paper, and some of the topics potentially could be pushed forward.

Last time, Yixian told us how to go from a tangle T to its Khovanov bracket $[[T]] \in \mathscr{C}h(\mathscr{M}at(\mathscr{C}ob(\partial T)))$: the category of chain complexes of matrices of cobordisms on the boundary of the manifold. This is scary, but we'll make sense of each piece.

There are a few problms with this approach:

(1) This category sucks: it's complicated and hard to picture. We'll endeavor to replace it with $\mathscr{C}h(\mathscr{A}b)$, chain complexes of abelian groups.
(2) The Khovanov bracket isn't a topological invariant. That's awkward. We'll "quotient out" the relations that are a problem. This will actually work well for us, in that it will explain how to generalize Khovanov homology by switching out the TQFT used to define it.

First let's explain the pieces in the category where the Khovanov bracket lives. Let T be a tangle with n crossings.

- $\mathscr{C}ob(\partial T)$ is the category whose objects are the 2^n smoothings of T, and whose morphisms are cobordisms between them. This is preadditive under disjoint union. You can think of this as cobordisms relative to ∂T.
- Given a preadditive category $\mathscr{C}$, its **additive closure** $\mathscr{M}at(\mathscr{C})$ is the category whose objects are formal finite direct sums of objects in $\mathscr{C}$, and whose morphisms are matrices of morphisms: if $(x_1, \dots, x_m), (y_1, \dots, y_n) \in \mathscr{M}at(\mathscr{C})$, then a morphism between them is an $n \times m$ matrix (φ_{ij}) where $\varphi_{ij} \in \mathrm{Hom}_{\mathscr{C}}(x_i, y_j)$, and composition of morphisms is matrix multiplication. This is also preadditive.
- Finally, we take chain complexes on this category: the objects are finite-length chain complexes

$$\cdots \longrightarrow C_0 \longrightarrow C_1 \longrightarrow C_2 \longrightarrow \cdots$$

and the morphisms are chain maps, i.e. commuting diagrams

$$\begin{tikzcd} \cdots \arrow[r] & C_0 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & D_0 \arrow[r] & D_1 \arrow[r] & D_2 \arrow[r] & \cdots \end{tikzcd}$$

So this is a complicated, but explicit, description of the category where the Khovanov bracket lives. Now, how do you actually define this bracket? Given a presentation of a tangle, consider its cube of smoothings. The i^{th} graded component is the formal sum (disjoint union) of the smoothings with i equal to the number of 1s minus the number of positive crossings. Thus we obtain a chain complex in $\mathscr{M}at(\mathscr{C}ob(\partial T))$ as needed.

Today, we'll turn this into a reasonable topological invariant. We'll start by quotienting $\mathscr{C}ob(\partial T)$ by some relations.

[16] It's possible to promote this to an equivalence of symmetric monoidal categories.

- The first relation, which we call S, says that if your cobordism contains a sphere as one of its connected components, it's equivalent to the zero cobordism (which, somewhat confusingly, is the identity cobordism whenever this makes sense).
- The next relation, called T, says that if your cobordism $X = T \amalg X'$ for some other X', where T is a torus, then $X \sim X' \oplus X'$.
- Finally, if you can intersect the cobordism with a disc and get 4 circles, joining the top two plus joining the bottom two is equivalent to joining the left two plus joining the right two. We'll call this U.

This is a really weird set of relations, but it works, which is pretty cool.

We'll also need to consider chain complexes up to homotopy. This is okay, because homology of a chain complex is a homotopy invariant. Homotopy is a very similar notion to homotopy in algebraic topology.

Definition 11.1. Let C and D be chain complexes and $f, g\colon C \rightrightarrows D$ be two chain maps. A **homotopy** from f to g is a collection of maps $h_i\colon C_i \to D_{i-1}$ such that $f_i - g_i = h_{i+1} \circ d_C + d_D \circ h_i$. In this case we write $f \sim g$. We say C and D are **homotopic** if there are maps $F\colon C \to D$ and $G\colon D \to C$ such that $F \circ G \sim \mathrm{id}$ and $G \circ F \sim \mathrm{id}$.

You can prove that a homotopy equivalence induces an isomorphism on homology, which is good: two different presentations for a tangle, modulo the specified relations, product chain homotopic complexes.

Example 11.2. TODO: picture of explicit example ◁

This feels like "stacking cobordisms on top of each other" (taking disjoint unions). Then, you can use the relations to simplify things: S gives you the identity cobordism, which is helpful when you want things to be homotopy invariants! Some of these are left as exercises in Bar-Natan's paper, and they can be kind of fun.

You can do this on each Reidemeister move: I and II are pretty reasonable, but III has a lot of crossings. Instead, Bar-Natan appeals to homotopy theory: the idea is that the bracket of a tangle is a cone of a certain map in the category of chain complexes, and this is invariant under strong deformation retracts. Then, the bracket associated to the third Reidemeister move is a cone of a certain map, and obtain a strong deformation retract and conclude that the cones are the same.

So now, we have a tangle invariant in the homotopy category of chain complexes, which is cool, and can take its homology groups to obtain an actual invariant.

Gradings. Grading is a condition on maps, so we're going to grade cobordisms.

Definition 11.3. Let C be a cobordism. Then, its **degree** is

$$\deg(C) := \chi(C) - \frac{1}{2}|\pi_0(\partial C)|.$$

That is, the Euler characteristic minus half the number of boundary components.

This is a somewhat defective notion: it's not additive. The zero cobordism S^2 has degree 2, but it should be equal to 0 for $\deg(f \circ g) = \deg f + \deg g$. One way to fix this is to track four marked points on ∂T, corresponding to ends of the tangle, but what about knots, which should be tangles with empty boundary?

Anyways, what you end up getting on chain complexes is the bracket complex with a grading $r + n_+ - n_-$ as in our explicit construction.

TQFT. We want to think of Khovanov homology as a functor $F\colon \mathscr{C}ob_{/\{R,S,U\}} \to \mathscr{A}b$, equivalent to a functor $F\colon \mathscr{C}ob \to \mathscr{A}b$ satisfying our three relations. The idea is that you could choose any functor that satisfies these relations. These functors are TQFTs, which is nice, so we're looking for TQFTs satisfying certain relations. When we take chain complexes on both sides, this will give us a functor from the category where the Khovanov bracket lives in to $\mathscr{C}h(\mathscr{A}b)$, and since the relations are met, it's well-defined up to homotopy (so we obtain homology groups as an actual tangle invariant).

A 2D TQFT is the same thing as a Frobenius algebra, so we can describe such TQFTs through the Frobenius algebras they assign to the circle.

Example 11.4. The TQFT that defines Khovanov homology is $\mathbb{Z}\{v_+, v_-\}$, where $|v_+| = 1$ and $|v_-| = -1$.

- The counit $\mathbb{Z} \to V$ sends $1 \mapsto v_+$.
- The unit $V \to \mathbb{Z}$ sends $v_+ \mapsto 0$ and $v_+ \mapsto 1$.
- The multiplication $V \otimes V \to V$ sends $v_+ \otimes v_+ \mapsto v_+$, $v_+ \otimes v_-$, $v_- \otimes v_+ \mapsto v_-$, and $v_- \otimes v_- \mapsto 0$.
- The comultiplication $V \to V \otimes V$ sends $v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+$ and $v_- \mapsto v_- \otimes v_-$.

You can check that this satisfies the relations we defined, and the chain complex it defines is exactly the Khovanov homology. ◁

A slightly different TQFT gives you Lee homology, which we discussed, and there are other possibilities. This leads to some interesting questions: can you classify the knot invariants that give these TQFTs? There's a lot we don't know, e.g. since $\mathscr{C}ob$ is enriched over $\mathscr{A}b$, we can consider the TQFT $\mathrm{Hom}_{\mathscr{C}ob}(X,-)$. Do these give you all of the information?

Over rings where 2 is invertible, you can choose different relations, e.g. the **cutting neck relation**, so you have other options.

Another question is: Khovanov's other invariant C hasn't been fit into this framework. Can this be brought into the fold somehow?

Finally, this is all totally local, so what other contexts can it work in? Knots in surfaces? 2-knots (embeddings $S^2 \hookrightarrow S^4$)? There's a lot of good questions that nobody knows the answer to.

12. The long exact sequence, functoriality, and torsion: 4/18/17

Today, Tom's going to talk about the long exact sequence, functoriality, and torsion in Khovanov homology, paving the way for computations in future talks.

The ideas behind these three somewhat separate ideas aren't hard. The proofs may be hard, but we're not going in depth about that.

12.1. **Long exact sequences.** To get an idea of what's going on, consider an unknot with two crossings as in Figure 12.1. Its four Khovanov soothings are given in Figure 12.2.

Figure 12.1. An unknot.

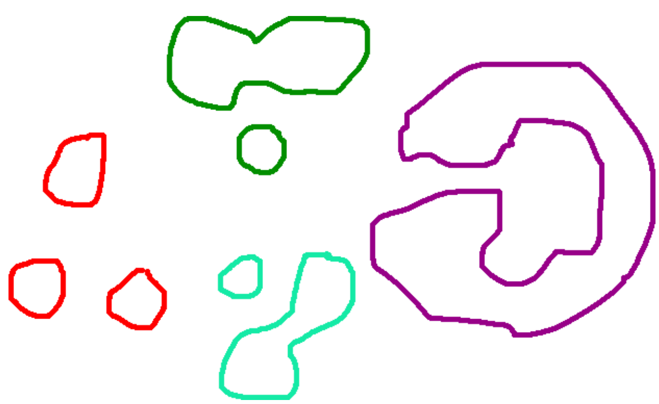

Figure 12.2. The four smoothings of the unknot in Figure 12.1.

If you look at the smoothings of the trefoil in Figure 12.3, you obtain some from a Hopf link and some from this version of the unknot. Therefore, if you follow the construction of Khovanov homology, there is a direct sum of *ungraded* vector spaces

$$C^{**}(D) \cong C^{**}(D_0) \oplus C^{**}(D_1),$$

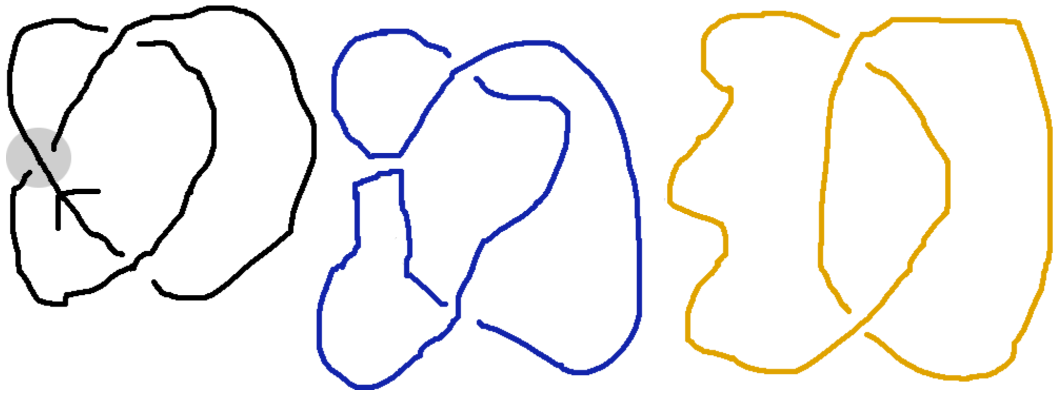

FIGURE 12.3. The trefoil (left) and the two smoothings at the gray crossing (center: D_0, right: D_1).

where D_0 is the unknot diagram in Figure 12.1 and D_1 is the standard diagram for a Hopf link, Figure 1.4. In particular, we have a short exact sequence

$$0 \longrightarrow C(D_1) \longrightarrow C(D) \longrightarrow C(D_0) \longrightarrow 0.$$

This was cool. Less fun is putting the gradings back in.

Now, let D be a link diagram with a designated crossing, and let D_0 and D_1 be the diagrams obtained by resolving this as 0 or as 1, respectively. There are a few cases here:

(1) Suppose our designated crossing is negatively oriented. Then, D_1 admits an induced orientation, but D_0 doesn't. Fix a quantum degre j, nd let c be the number of regular crossings of D_0 (ignoring sign) minus that of D (again, ignoring sign). Then, look carefully at the gradings in D_0 and D_1 (the proper way to do this is actually work through it, especially because I missed the argument). It turns out the part of $C^{i,*}(D_1)$ contained within $C^{i,j}(D)$ is the part with quantum degree $j+1$. Passing to the quotient, the i-grading changes too, and so what you end up with is

$$0 \longrightarrow C^{i,j+1}(D_1) \longrightarrow C^{i,j}(D) \longrightarrow C^{i-c,i-3c+1}D_0 \longrightarrow 0.$$

This is a short exact sequences of chain complexes in the i-degree, so we obtain a long exact sequence for Khovanov homology. However, since Khovanov homology is actually cohomologically graded, the differential raises the degree:

$$\cdots \longrightarrow \mathrm{Kh}^{i,j+1}(D_1) \longrightarrow \mathrm{Kh}^{i,j}(D) \longrightarrow \mathrm{Kh}^{i-c,j-3c+1}(D_0) \xrightarrow{\delta} \mathrm{Kh}^{i+1,j+1}(D_1) \longrightarrow \cdots$$

You can use this to quickly compute the Khovanov homology of the Hopf link.

(2) If the designated crossing is positively oriented, there's a similar but slightly different argument to show that you get a similar long exact sequence; this time, it goes from D_0 to D to D_1, and the gradings are slightly different. See Turner's notes for details.

12.2. **Functoriality.**

Definition 12.1. A **link cobordism** (Σ, L_0, L_1) is a smooth, compact, oriented surface Σ generically embedded into $\mathbb{R}^3 \times I$ such that $\partial\Sigma = \overline{K}_0 \cup L_1 \subset \mathbb{R}^3 \subset \{0,1\}$.

Here, $\overline{L}_0$ means L_0 with the opposite orientation. So this is a cobordism, but realized in ambient space rather than abstractly.

Definition 12.2. A **movie** for a link cobordism is how to "play it" along $t \in [0,1]$. That is, it's a sequence of **frames**, where each frame is given by doing a Reidemeister move on the previous frame or attaching a 0, 1, or 2-handle.

So in a sense these are local movies, which may have interesting global behavior. Just like Austin's independent film scene.[17]

[17]The local music scene doesn't have anything to do with Khovanov homology; rather, they're more about LP spaces.

Proposition 12.3. *Any link cobordism can be expressed as a finite movie, and the nonuniqueness of movies can be expressed with a short list of "movie moves."*

We'd like link cobordisms to induce maps on Khovanov homology. By the previous proposition, it suffices to understand what's happening on each frame.

- Reidemeister moves cannot change Khovanov homology, so we'll map it to the identity.
- Adding a 0-handle (an unlinked circle), will be action by the unit of the Frobenius algebra: $\mathrm{id} \otimes i$.
- Adding a 1-handle is a cobordism whose trace is the surgery gluing in a saddle. Apply the TQFT to this cobordism, and that's the map we associate to this movie.
- A 2-handle will correspond to the counit, tensored with the identity.

This is pretty cool, except that there may be link cobordisms with different movie representations which induce different morphisms. However, if you projectivize the target category, everything ends up OK.

Theorem 12.4. *Let M and M' be two movie representations for a link cobordism and ϕ_M, $\phi_{M'}$ be the induced maps on Khovanov homology. Then, $\phi_M = \pm\phi_{M'}$.*

12.3. **Torsion.** We've implicitly been working over $\mathbb{Q}$, but it's possible to define Khovanov homology over other rings, and in particular $\mathbb{Z}$. This leads to the possibility of torsion, and in particular in the trefoil (specifically, the diagram in Figure 12.3, left).

Consider the $(-2,-7)$ bigrading. The cycles are

$$\langle z_1 = (x \otimes x, 0, 0), z_2 = (0, x \otimes x, 0), z_3 = (0, 0, x \otimes x)\rangle.$$

The boundaries coming from $(-3,-7)$ are

$$\langle c_1 = 1 \otimes x \otimes x, c_2 = x \otimes 1 \otimes x, c_3 = x \otimes x \otimes 1\rangle,$$

and their differentials are

$$\begin{aligned} dc_1 &= z_1 + z_2 \\ dc_2 &= z_1 + z_3 \\ dc_3 &= z_2 + z_3. \end{aligned}$$

In particular, $d(c_1 + c_2 - c_3) = 2z_1$. One can show that z_1 isn't a cycle in $\mathbb{Z}$-coefficients, but $2z_1$ is, because we can't divide $c_1 + c_2 - c_3$, so if D denotes the trefoil, $K^{-2,-7}(D) \cong \mathbb{Z}/2$.

In fact, for diagrams of alternating links, if your diagram isn't a distant union of unknots or Hopf links, it will have $\mathbb{Z}/2$ torsion.

There is torsion over other primes, but it's hard and even the 2-torsion is poorly understood.

13. A transverse link invariant from Khovanov homology: 4/25/17

Today, Christine told us about a transverse link invariant coming from Khovanov homology.

To define transverse link invariants, one needs the geometry of contact structures.

Definition 13.1. Let M be a smooth, $(2n-1)$-dimensional manifold.

- A smooth 1-form α on M is a **contact form** if $\alpha \wedge (\mathrm{d}\alpha)^{n-1}$ is nonvanishing.
- A smooth tangent hyperplane field C is a **contact structure** if there's an open cover $\mathfrak{U}$ of M and contact forms $\alpha_U \in \Omega^1(U)$ for each $U \in \mathfrak{U}$ such that $C|_U = \ker(\alpha_U)$ for all $U \in \mathfrak{U}$.

Example 13.2. The **standard contact structure** on $\mathbb{R}^3$ is

$$\xi_{\mathrm{std}} := \ker(\mathrm{d}z - y\,\mathrm{d}x + x\,\mathrm{d}y).$$

See Figure 13.1 for a depiction. ◀

Definition 13.3.

(1) A **transverse knot** is a knot in $(\mathbb{R}^3, \xi_{\mathrm{std}})$ which is everywhere transverse to the contact planes.
(2) A **Legendrian knot** is a knot in $(\mathbb{R}^3, \xi_{\mathrm{std}})$ which is everywhere tangent to the contact planes.
(3) Two transverse knots K and K' are **transversely isotopic** if there exists a one-parameter family of transverse knots K_t (so a smooth map $K\colon [0,1] \times S^1 \to \mathbb{R}^3$ such that $K_t := K(t,-)$ is an embedding for each t) such that $K_0 = K$ and $K_1 = K'$.

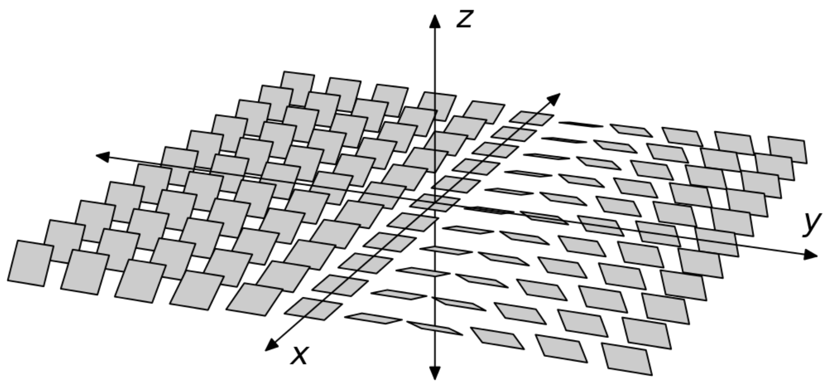

FIGURE 13.1. The standard contact structure on $\mathbb{R}^n$, as in Example 13.2, used for defining transverse knots. Source: `https://en.wikipedia.org/wiki/Contact_geometry`.

Transverse and Legendrian knots are opposites to each other, but there are ways to obtain Legendrian knots from transverse ones, and vice versa, which are useful when studying them.

Theorem 13.4 (Bennequin). *Every transverse knot in $(\mathbb{R}^3, \xi_{\mathrm{std}})$ is transversely isotopic to a closed braid.*

Hence it suffices to study closed braids. The following theorem is very hard (so we won't prove it), but it's useful.

Theorem 13.5 (Transverse Markov theorem, Wrinkel, Orekov-Shevchishin). *Let L_1 and L_2 be transverse links that are closed braids. Then, they're transversely isotopic iff one can be obtained from the other by positive stabilizations and braid isotopies.*

Positive stabilizations are Reidemeister moves except you're not allowed to straighten an over-under self-loop (but under-then-over is okay), This is a restriction only on the first Reidemeister move.

The relationship with Khovanov homology was worked out by Olga Plamenevskaya [6]. Consider a closed braid L of index n, and consider the resolution of L giving n parallel strands, by sending crossed braids to uncrossed ones (resolve all oriented crossings). This defines a $\psi \in C\mathrm{Kh}(L)$ which is $v_-^{\otimes n}$.

Theorem 13.6.

(1) *$\psi(L)$ is a cycle in $C\mathrm{Kh}(L)$.*
(2) *The homology class $[\psi(L)]$ is a transverse invariant.*

The way this is proven is by considering a "categorified bracket" description of Khovanov homology. Namely, if D and its resolutions D_0 and D_1 are as in Figure 13.2, $[\![D]\!] = [\![D_0]\!] \oplus [\![D_1]\!]\{1\}$.

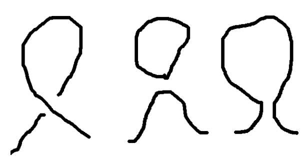

FIGURE 13.2. From the left: D, D_0, and D_1.

Now we want to describe the differential.

- If both pieces of D_0 are positive, we go to D_1 with positive orientation, which is acyclic.
- If the circle of D_0 is positive and the rest is negative, we map to D_1 with negative orientation, which is acyclic.
- If the circle is negative and the rest is positive, we get D_1 with negative orientation, but we get $C' = [\![\frown]\!]$.
- If all of D_0 is negatively oriented, we get 0.

The proof of invariance under the second and third Reidemeister moves is similar.

The key question is, *what is this invariant ψ measuring?* Since Plamanevskaya's paper, there's mostly been computations. Many results involve the use of a stronger transverse invariant coming from Heegaard Floer homology, which is more effective in general but seems to be not very related. This is another mystery.

Definition 13.7. A braid is **quasipositive** if it's a product of conjugates of positive powers of generators of the braid group.

Theorem 13.8. *If L is represented by a quasipositive braid, then $\psi(L) \neq 0$.*

Theorem 13.9. *Let L' be obtained from L by removing a right-over-left crossing, M be the cobordism induced by removing the crossing, and f be the map F induces on Khovanov homology. Then, $f(\psi(L)) = \pm\psi(L')$.*

By Theorem 12.4, even though f isn't uniquely defined, it's defined up to sign, so this is OK.

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