SUMMER 2016 ALGEBRAIC GEOMETRY SEMINAR

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JULY 28, 2016

Contents

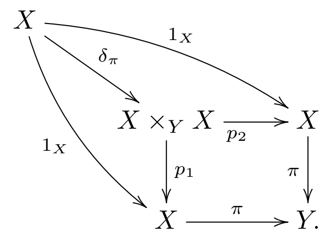
1. Separability, Varieties and Rational Maps: 5/16/16

Today's lecture was given by Tom Oldfield, on the first half of chapter 10.

This seminar has a website, located at
https://www.ma.utexas.edu/users/toldfield/Seminars/Algebraicgeometryreading.html.

The first half of Chapter 10 is about separated morphisms and varieties; it only took us 10 chapters! Vakil writes that he was very conflicted about leaving a proper treatment of algebraic varieties, a cornerstone of classical algebraic geometry, to so late in the notes. But from a modern perspective, our hands are tied: varieties are defined in terms of properties, which means building those properties out of other properties and out of the large amount of technology you need for modern algebraic geometry. With that technology out of the way, here we are.

One of these properties is separability. Let $\pi : X \to Y$ be a morphism of schemes; then, the **diagonal** is the induced morphism $\delta_\pi : X \to X \times_Y X$ defined by $x \mapsto (x, x)$; this maps into the fiber product because it fits into the diagram



(1.1)

Here, p_1 nad p_2 are the projections onto the first and second components, respectively, and 1_X is the identity map on X.

The diagonal has a few nice properties. Suppose $V \subset Y$ is open, and $U, U' \subset \pi^{-1}(V)$ are open subsets of X. Then, $U \times_V U' = p_1^{-1}(U) \cap p_2^{-1}(U')$: we constructed fiber products such that they send open embeddings to intersections. In particular, if $U \cong \operatorname{Spec} A$, $U' \cong \operatorname{Spec} A'$, and $V \cong \operatorname{Spec} B$ are affine, $U \times_V U' \cong \operatorname{Spec}(A \otimes_B A')$. Therefore $\delta_\pi^{-1}(U \times_V U') = \delta_\pi^{-1}(p_1^{-1}(U) \cap p_2^{-1}(U')) = U \cap U'$. That is, the diagonal turns intersections into fiber products.

This argument feels like it takes place in Set, but goes through word-for-word for schemes.

Definition 1.2. A morphism $\pi : X \to Y$ of schemes is a **locally closed embedding** if it factors as $\pi = \pi_1 \circ \pi_2$, where π_2 is a closed embedding and π_1 is an open embedding.

Proposition 1.3. *For any $\pi : X \to Y$, δ_π is locally closed.*

Proof. Let $\{V_i\}$ be an affine open cover of Y, so $V_i \cong \operatorname{Spec} B_i$ for each B, and $\mathfrak{U}_i = \{U_{ij}\}$ be an affine open cover of $\pi^{-1}(V_i)$ for each i. Then, $\{U_{ij} \times_{V_i} U_{ij'} : i, j, j'\}$ covers $X \times_Y X$. More interestingly, $\{U_{ij} \times_{V_i} U_{ij} : i, j\}$ covers $\operatorname{Im}(\delta_\pi)$: this is because if $x \in U_{ij}$, then $\delta_\pi(x) \in p_1^{-1}(U_{ij})$ and in $p_2^{-1}(U_{ij})$, and $p_1^{-1}(U_{ij}) \cap p_2^{-1}(U_{ij}) = U_{ij} \times_{V_i} U_{ij}$.

Now, it suffices to show that $\delta_\pi : \delta_\pi^{-1}(U_{ij} \times_{V_i} U_{ij}) \to U_{ij} \times_{V_i} U_{ij}$ is closed, since the property of being a closed embedding is affine-local. Since each $U_{ij} \cong \operatorname{Spec} A_{ij}$ is affine, then it suffices to understand what's happening ring-theoretically: the diagonal map corresponds to the ring morphism $A_{ij} \otimes_{V_i} A_{ij} \to A_{ij}$ sending $a \otimes a' \mapsto aa'$. This is clearly surjective, which is exactly the criterion for a morphism of schemes to be a closed embedding. ⊠

Corollary 1.4. *If X and Y are affine schemes, then δ_π is a closed embedding.*

Corollary 1.5. *If Δ denotes $\operatorname{Im}(\delta_\pi)$, then for any open $V \subset Y$ and $U \subset \pi^{-1}(V)$, $\Delta \cap (U \times_V U') \cong U \cap U'$ is a homeomorphism of topological spaces.*

This follows because a locally closed embedding is homeomorphic onto its image.

These will all be super useful once we define separability, which we'll do now.

Definition 1.6. A morphism $\pi : X \to Y$ is **separated** if $\delta_\pi : X \to X \times_Y X$ is a closed embedding.

This is weird upon first glance: why do we look at the diagonal to understand things about a morphism? The answer is that the diagonal has nice category-theoretic properties, so we can prove some useful properties by doing a few diagram chases.

More geometrically, separability corresponds to the Hausdorff property in topological spaces, and there's a criterion for this in terms of the diagonal.

Proposition 1.7. *If T is a topological space, then T is Hausdorff iff the diagonal morphism $T \to T \times T$ is a closed embedding.*

Equivalently, the image $\Delta \subset T \times T$ is a closed subspace.

Remark. Since schemes are topological spaces, you might think this proves separated schemes are Hausdorff, but this is untrue: fiber products of schemes are generally not fiber products of underlying spaces, and therefore closed embeddings of schemes are not the same as closed embeddings of their underlying spaces.

Separability is a nice property, and is good to have. But like Hausdorfness, we generally won't need to use schemes that aren't separated.

Example 1.8.

(1) By Corollary 1.4, all morphisms of affine schemes are separated.
(2) If we can cover $X \times_Y X$ by the sets $U_{ij} \times_{V_i} U_{ij}$ (with these sets as in the proof of Proposition 1.3), then π is separated.
(3) For a counterexample, let $X = \mathbb{A}^1_{(0,0)}$ be the "line with two origins" over a field k. This isn't a separated scheme: the diagonal is a "line with four origins," and these cannot be separated topologically: every open set containing one contains all of them. So take one affine piece of X, which contains exactly one origin, and therefore its image ought to contain all four, but it doesn't, so $X \to \operatorname{Spec} k$ isn't closed. This might feel a little imprecise, but one can make it fully rigorous.

We want separated morphisms to be nice: we'd like them to be preserved under base change and composition, and we'd like locally closed embeddings to be separated.

Proposition 1.9. *Locally closed embeddings are separated.*

This is the only example of a hands-on proof of a property; it's not hard, but the rest will be less abstract and easier. First, though, let's reframe it:

Proposition 1.10. *Any monomorphism of schemes is separated.*[1]

Proof. By point (2) of Example 1.8, it suffices to prove that fiber products $U_{ij} \times_{V_i} U_{ij}$ cover $X \times_Y X$ for our affine covers. So let's look at the fiber diagram (1.1) again; it tells us that $\pi \circ p_1 = \pi \circ p_2$. But since π is a monomorphism, then $p_1 = p_2$, so for any $z \in X \times_Y Z$, $p_1(z) = p_2(z)$; call this point x_z. Then, if $x_z \in U_{ij}$, $z \in p^{-1}(U_{ij})$ and $z \in p_2^{-1}(U_{ij})$, and their intersection is the fiber product. ⊠

Since locally closed embeddings are monomorphisms, Proposition 1.9 follows as a corollary.

At this point, we can define varieties, and Vakil does so, but can't do anything with them, so we'll come back to them in a little bit.

Proposition 1.11. *If A is a ring, $\mathbb{P}^n_A \to \operatorname{Spec} A$ is separated.*

The idea of the proof is to compute: we already know a cover of $\mathbb{P}^n_A$ by $n+1$ affine schemes, and can check that the induced map on rings is surjective.

The following proposition gives us an important geometric property of separability.

Proposition 1.12. *If A is a ring and $X \to \operatorname{Spec} A$ is separated, then for any affine open subsets $U, V \subset X$, $U \cap V$ is also affine.*

Proof. The diagonal is a closed embedding, so $\delta : U \times V \to U \times_A V$ is also a closed embedding. Therefore $U \times V$ is isomorphic to a closed subscheme of an affine scheme, and therefore is affine. ⊠

It's surprising how useful these arguments with the diagonal are: we got a useful and nontrivial result in one line! In general, you can prove a weirdly large amount of things by factoring them through the diagonal. In fact, le'ts use it to define another property.

Definition 1.13. A morphism $\pi : X \to Y$ is **quasiseparated** if δ_π is quasicompact.

This isn't the same as the other definition we were given, that for all affine $V \subset Y$ and $U, U' \subset \pi^{-1}(V)$, $U \cap U'$ is quasicompact. But it turns out to be equivalent.

Proposition 1.14. *$\pi : X \to Y$ is quasiseparated in the sense of Definition 1.13 iff it's quasiseparated in the sense we defined previously.*

The proof is a diagram chase involving the "magic diagram" for fiber products. This states that if $X_1, X_2 \to Y \to Z$ are maps in some category and the relevant fiber products exist, the diagram

$$\begin{array}{ccc} X_1 \times_Y X_2 & \longrightarrow & X_1 \times_Z X_2 \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\ \delta\ } & Y \times_Z Y \end{array}$$

[1] More is true in general; all you need is that $p_1 = p_2$ in the diagram (1.1), which is analogous to an injectivity condition on π. Hence, it suffices that π is injective as a map of sets, but this is a weird notion for schemes, so we generally phrase it in terms of monomorphisms.

is a fiber diagram; the proof is a diagram chase following from the associativity of products, or checking the universal property. This diagram is also very ubiquitous for proofs like these.

Proposition 1.15. *Separability and quasiseparability are preserved under base change.*

Proof. Suppose $\pi : X \to Y$ is separated and $\varphi : S \to Y$ is another map of schemes, so there's an induced morphism $\pi' : Z = X \times_Y S \to S$ fitting into the diagram

$$\begin{CD} Z @>{\pi'}>> S \\ @V{p_1}VV @VV{\varphi}V \\ X @>{\pi}>> Y. \end{CD}$$

The magic diagram for this is the fiber diagram

$$\begin{CD} Z @>{\delta_{\pi'}}>> Z \times_S Z \\ @VVV @VVV \\ X @>{\delta_\pi}>> X \times_Y X. \end{CD}$$

If π is separated, δ_π is closed, and therefore $\delta_{\pi'}$ is closed (since closed embeddings are preserved under base change), so π' is separated. The same argument works with π quasiseparated and δ_π quasicompact. ⊠

There are a few related properties that we won't prove, but whose proofs are very similar to the previous one.

Proposition 1.16. *Separability and quasiseparability are*

(1) local on the target,
(2) closed under composition, and
(3) closed under taking products: if $\pi : X \to Y$ and $\pi' : X' \to Y'$ are separated morphisms of schemes over a scheme S, then $\pi \times \pi' : X \times_S X' \to Y \times_S Y'$ is separated; if π and π' are merely quasiseparated, so is $\pi \times \pi'$.

Each of these is a diagram chase with the right diagram, and not a particularly hard one; the last one follows as a general categorical consequence of the others.

Now, though, we can define varieties.

Definition 1.17. Let k be a field. A ***k*-variety** is a k-scheme $X \to \operatorname{Spec} k$ that is reduced, separated, and of finite type. A **subvariety** of a given variety X is a reduced, locally closed subscheme.

Reducedness is a property of X, but the others are properties of the structure morphism $X \to \operatorname{Spec} k$. Notice that the affine line with doubled origin is reduced and of finite type, so separability is important for avoiding pathologies.

It's nontrivial that a subvariety $Y \subset X$ is itself a variety. X is finite type over $\operatorname{Spec} k$, so it's covered by finitely many affine opens that are schemes of finitely generated k-algebras, which are Noetherian, so X is Noetherian. Hence, $Y \hookrightarrow X$ is a finite-type morphsism into a Noetherian scheme, so Y is finite type; but we do need separability to be preserved under composition, which we just saw how to prove.

We did not require varieties to be irreducible; irreducibility doesn't behave as well as we would like, unless k is particularly nice.

Proposition 1.18. *The product of irreducible varieties over an algebraically closed field k is an irreducible k-variety.*

This follows from the nontrivial fact that if A and B are k-algebras that are integral domains, then $A \otimes_k B$ is an integral domain.

The last important thing we'll discuss today is a big meta-theorem about classes of morphisms.

Theorem 1.19 (Cancellation theorem)**.** *Consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ & {\scriptstyle\tau}\searrow \quad \swarrow{\scriptstyle\rho} & \\ & Z, & \end{array}$$

i.e. $\tau = \rho \circ \pi$, and let P be a property of morphisms preserved under base change and composition. If τ has P and δ_ρ has P, then π also has P.

The name is because we're "cancelling" ρ out of the composition.
The proof uses the notion of the graph of a morphism.

Definition 1.20. Let X and Y be schemes over a scheme S, and $\pi : X \to Y$ be a map of S-schemes. Then, the **graph** of π is the morphism $\Gamma_\pi : X \to X \times_S Y$ defined by $\Gamma_\pi ; (1_X, \pi)$.

That is, this sends a point to its image on the graph. We use this because any morphism factors through its graph. Then, since δ_ρ has P, so must Γ_π, which is useful. It seems weirdly abstract and pointless, but the idea is that the nice properties of the diagonal, including locally closed embeddings, can be canceled off. In fact, if Y is separated, we can cancel off properties of closed embeddings, and if Y is quasiseparated, we can cancell off properties of quasicompact morphisms.

Rational Maps. Let's talk about rational maps, which are rational maps defined almost everywhere, and up to almost everywhere agreement. Rational maps are usually only defined on reduced varieties, since it's nearly impossible to get a hold on them otherwise; they're inherently geometric, and geometry tends to involve varieties.

Definition 1.21. A **rational map** $\pi : X \dashrightarrow Y$ is an equivalence class of morphisms $f : U \to Y$, where $U \subset X$ is a dense open subset; (f, U) and (f', U') are considered equivalent if there's a dense open set $V \subset U \cap U'$ if $f|_V = f'|_V$. One says π is **dominant** if its image is dense, or equivalently, for all nonempty opens $V \subseteq Y$, $\pi^{-1}(V) \neq \varnothing$.

Notice that dominance is well-defined, as it's independent of choice of representative.

Proposition 1.22. *Let X and Y are irreducible schemes, then $\pi : X \dashrightarrow Y$ is dominant iff the generic point of X maps to the generic point of Y.*

Proof. In the reverse direction, the generic point η_Y of Y is contained in every open subset of Y, so the preimage contains the generic point η_X of X, and in particular is nonempty.

In the other direction, suppose $\pi(\eta_X) \neq \eta_Y$; let $U = Y \setminus \overline{\pi(\eta_X)}$, which is an open subset. Thus, $\eta_X \notin \pi^{-1}(U)$, which is an open set. Since η_X is dense, it meets every nonempty open, so $\pi^{-1}(U)$ is empty, and therefore π isn't dominant. ⊠

This is a pretty useful characterization of dominance. But why do we care about dominance? Because of composition.

Remark. Let $\pi : X \dashrightarrow Y$ and $\rho : Y \dashrightarrow Z$ be rational maps. If π is dominant and X is irreducible, it's possible to make sense of $\rho \circ \pi : X \dashrightarrow Z$ as a rational map, which is dominant iff ρ is.

This is nontrivial: if π isn't dominant, one might discover that the domain of ρ doesn't intersect the image of π; if they do, however, π^{-1} of the domain of definition of ρ is a nonempty open of X; since X is irreducible, it must be dense.

Definition 1.23. A rational map $\pi : X \dashrightarrow Y$ is **birational** if it's dominant and there exists a dominant $\psi : Y \dashrightarrow X$ such that as rational maps, $\pi \circ \psi \sim 1_X$ and $\psi \circ \pi \circ 1_Y$. In this case, one says π and ψ are **birational(ly equivalent)**.

Proposition 1.24. *Let X and Y be reduced schemes; then, X and Y are birational iff there exist dense open subschemes $U \subset X$ and $V \subset Y$ such that $U \cong V$.*

The idea is that we can let U and V be the domains of definition for our rational maps.

The notion of rationality is very specific to algebraic geometry; in the differentiable category, it's complete nonsense. Since any manifold can be triangulated, any two manifolds of the same dimension are birationally equivalent: remove the edges of the triangles, and you get a dense open set; clearly, any two triangles are birational. However, there exist algebraic varieties of the same dimension that aren't birationally equivalent.

Definition 1.25. A variety X over k is **rational** if it's birational to $\mathbb{A}_k^n$ for some n.

For example, $\mathbb{P}_k^n$ is rational. Rationality loses some information, but what it keeps is interesting. Finally, let's see what dominance means in terms of ring morphisms.

Definition 1.26. Let $\varphi : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes and $\varphi^\sharp : B \to A$ be the induced map on global sections. Then, φ is dominant (i.e. as a rational map) iff $\ker(\varphi^\sharp) \subset \mathfrak{N}(A)$.

Here, $\mathfrak{N}(A)$ denotes its nilradical, the intersection of all prime ideals of A (equivalently, the ideal of nilpotent elements). That is, if A and B are reduced, dominance is equivalent to injectivity! Interestingly, this also corresponds to an inclusion of function fields, i.e. a field extension! We've reduced a geometric problem to a problem about algebra. Often, we can go in the other direction, e.g. for varieties. In this setting, birationality means isomorphism on the function fields.

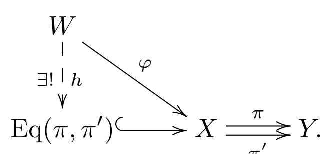
2. Proper Morphisms: 5/19/16

These are Arun's lecture notes on rational maps to separated schemes and proper morphisms, corresponding to sections 10.2 and 10.3 in Vakil's notes. I'm planning on talking about the following topics:

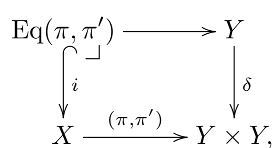
- Rational maps to separated schemes, including the reduced-to-separated theorem and some corollaries.
- The definition of proper morphisms, and that they form a nice class of morphisms. Projective A-schemes are proper over A.

Throughout this lecture, S is a scheme, which will often be the base scheme.

Rational Maps to Separated Schemes. If X and Y are spaces and $\pi, \pi' : X \rightrightarrows Y$ are continuous, it's sometimes useful to talk about the locus where they agree, $\{x \in X : \pi(x) = \pi'(x)\}$. Categorically, this is the equalizer $\operatorname{Eq}(\pi, \pi') \hookrightarrow X$, which is characterized by the property that if $\varphi : W \to X$ is a continuous map such that $\pi \circ \varphi = \pi' \circ \varphi$, then it factors through $\operatorname{Eq}(\pi, \pi')$, i.e. there's a unique $h : W \to \operatorname{Eq}(\pi, \pi')$ such that the following diagram commutes.



So if we can do this for schemes, we'll have a subscheme where two morphisms agree, rather than just a set. The universal property for the equalizer is the same as for the fiber product



(2.1)

where δ is the diagonal morphism. We know fiber products of schemes exist, so equalizers do too.

Lemma 2.2 (Vakil ex. 10.2.A)**.** *If $\pi, \pi' : X \rightrightarrows Y$ are two morphisms of schemes over S, then $i : \operatorname{Eq}(\pi, \pi') \hookrightarrow X$ is a locally closed subscheme of X. If Y is separated over S, $\operatorname{Eq}(\pi, \pi')$ is a closed subscheme.*

Proof. Since we're over S, the product in (2.1) should be replaced with $Y \times_S Y$, the product in Sch_S. Since δ is a locally closed embedding, and this is a property preserved under base change, then i is too. If $Y \to S$ is separated, then δ is a closed embedding, and this is also preserved by pullbacks. ⊠

Remark. The locus where two maps agree does not need to be reduced, e.g. if $\pi, \pi' : \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ are defined by $\pi(x) = 0$ and $\pi'(x) = x^2$, then they agree "to first order" at 0, and $\operatorname{Eq}(\pi, \pi') = \operatorname{Spec} \mathbb{C}[x]/(x^2)$.

The central result about these is the reduced-to-separated theorem.

Theorem 2.3 (Reduced-to-separated theorem (Vakil Thm. 10.2.2))**.** *Let $\pi, \pi' : X \rightrightarrows Y$ be two morphisms of S-schemes. If X is reduced, Y is separated over S, and π and π' agree on a dense open subset, then $\pi = \pi'$.*

This is equality in the sense of morphisms of schemes, which is stronger than pointwise equality.

Proof. By Lemma 2.2, $\mathrm{Eq}(\pi, \pi') \hookrightarrow X$ is a closed subscheme, but it contains a dense open set. Since X is reduced, its only closed subscheme containing a dense open set is itself. $\boxtimes$

Corollary 2.4. *If X is reduced, Y is separated, and $\pi : X \dashrightarrow Y$ is a rational map, then there is a maximal $U \subset X$ such that $\pi|_U : U \to Y$ is an honest morphism. In particular, this is true for rational functions on reduced schemes.*

This U is called the **domain of definition** of π; its complement is sometimes called the **locus of indeterminacy**.

Proof. We can choose U to be the union of all domains of representatives of π. If $f_1 : V_1 \to Y$ and $f_2 : V_2 \to Y$ are two morphisms representing π, then f_1 and f_2 agree on a dense open subset of $V_1 \cap V_2$, so by the reduced-to-separated theorem agree on all of $V_1 \cap V_2$. Thus, we can glue representing morphisms on their intersection and therefore define π on all of U. $\boxtimes$

Next, we need to digress slightly to understand the image of a locally closed embedding. This is from section 8.3 of the notes.

If $\pi : X \to Y$ is a morphism of schemes, it's in particular a continuous function, so its image $\pi(X) \subset Y$ is a subspace. This will be referred to as the **set-theoretic image**. As usual, the topological version of a thing tends to be less well-behaved than the scheme-theoretic one, so we'll define an image of π that's a subscheme of Y. Schemes are locally cut out by equations, so it seems reasonable to say that a closed subscheme $i : Z \hookrightarrow Y$ **contains the image** of π if functions in $\mathscr{O}_Y$ that vanish on Z also vanish when pulled back to X. That is, the composition $\mathscr{I}_{Z/Y} \to \mathscr{O}_Y \to \pi_*\mathscr{O}_X$ is zero, where $\mathscr{I}_{Z/Y} = \ker(i^\sharp : \mathscr{O}_Y \to i_*\mathscr{O}_Z)$ is the sheaf of ideals associated to the closed embedding of Z into Y.

Definition 2.5. The **scheme-theoretic image** $\mathrm{Im}(\pi)$ of π is the intersection of all closed subschemes containing the image of π.[2] If π is a locally closed embedding, $\mathrm{Im}(\pi)$ is also called the **scheme-theoretic closure** of π.

That is, $\mathrm{Im}(\pi)$ is the smallest closed subscheme of Y such that locally vanishing on $\mathrm{Im}(\pi)$ implies locally vanishing when pulled back to X.

Theorem 2.6 (Vakil cor. 8.3.5)**.** *Let $\pi : X \to Y$ be a morphism of schemes. If X is reduced or Y is quasicompact, the closure of the set-theoretic image of π is the underlying set of* $\mathrm{Im}(\pi)$.

We lack the time to prove this, but it follows from the defining properties of closed embeddings.

Just like we defined the graph of a morphism of S-schemes $\pi : X \to Y$ to be $\Gamma_\pi = (\mathrm{id}, \pi) : X \to X \times_S Y$, we can define the graph of a rational map in nice situations.

Definition 2.7. Let $\pi : X \dashrightarrow Y$ be a rational map over S, where X is reduced and Y is separated over S. For any representative morphism $f : U \to Y$ of π, the **graph of the rational map** π, denoted Γ_π, is the scheme-theoretic closure of the map $\Gamma_f \hookrightarrow U \times_S Y \hookrightarrow X \times_S Y$. (The first map is a closed embedding, and the second is an open embedding.)

The following diagram might make this definition clearer.

$$\begin{array}{ccccc} \Gamma_\pi & \xrightarrow{\text{cl.}} & X \times_S Y & & \\ \uparrow & \swarrow & & \searrow & \\ X & & \overset{\pi}{\dashrightarrow} & & Y. \end{array}$$

A priori this definition depends on the choice of representative, but fortunately, this isn't actually the case.

Proposition 2.8 (Vakil ex. 10.2.E)**.** *The graph of a rational map π is independent of choice of representative.*

[2] There's something to prove here, that containing the image of π is well-behaved under intersections.

Proof. Let $\xi' : U \to Y$ and $\xi : V \to Y$ be two representatives of π. Without loss of generality, we can assume V is the maximal domain of definition for π, so $U \subset V$ and $\xi' = \xi|_U$. Thus, we have a bunch of embeddings fitting into the diagram

$$\begin{array}{ccccc} \Gamma_{\xi'} & \xhookrightarrow{\text{cl.}} & U \times Y & \xhookrightarrow{\text{op.}} & X \times Y \\ \big\downarrow & & \big\downarrow \text{op.} & \nearrow_{\text{op.}} & \\ \Gamma_{\xi} & \xhookrightarrow{\text{cl.}} & V \times Y. & & \end{array}$$

Thus, $\Gamma_{\xi'}$ factors as a subset of a closed subset of $V \times Y$, so its scheme-theoretic closure, which is just the closure of its underlying set by Theorem 2.6, must factor through this. In particular, the graph of π as defined with respect to ξ' embeds into $V \times Y$. Thus, we can assume $V = X$, since everything takes place inside V. In this case, Γ_π as defined by ξ is just Γ_ξ, and $\Gamma_\xi \cong X$ by projection onto the first factor. This projection restricts to an isomorphism $\Gamma_{\xi'} \cong U$, and carries the embedding $\Gamma_\xi \hookrightarrow \Gamma_{\xi'}$ to the embedding $U \hookrightarrow X$. Finally, to form the graph of π with respect to ξ', we take the closure, and since U is a dense open subset, we get X, or all of Γ_ξ. ⊠

Finally, we discuss one application to effective Cartier divisors. (This is actually an excuse to introduce effective Cartier divisors, since they show up again and again.)

Definition 2.9. A closed embedding $\pi : X \hookrightarrow Y$ is an **effective Cartier divisor** if $\mathscr{I}_{X/Y}$ is locally generated by a single non-zerodivisor. That is, there's an affine open cover $\mathfrak{U}$ of Y such that for each $U_i = \operatorname{Spec} A_i \in \mathfrak{U}$, there's a $t_i \in A$ that is not a zerodivisor and such that $\mathscr{I}_{X/Y}(U) = A_i/(t_i)$.

Proposition 2.10 (Vakil ex. 10.2.G). *Let X be a reduced S-scheme and Y be a separated S-scheme. If $i : D \hookrightarrow X$ is an effective Cartier divisor, there is at most one way to extend an S-morphism $\pi : X \setminus D \to Y$ to all of X.*

Proof. This is true if we know it on an affine cover, so without loss of generality assume $X = \operatorname{Spec} A$ is affine and $D = V(t)$ for some $t \in A$ that isn't a zerodivisor. If $D(t) = X \setminus D$ is dense in X, then we're done by Theorem 2.3. Since X is reduced, then by Theorem 2.6 this is equivalent to the scheme-theoretic closure of $D(t)$ being all of X. Given a closed subscheme $Z \hookrightarrow X$, we want to understand when functions vanishing on Z pull back to the zero function on $D(t)$. The map $\Gamma(X, \mathscr{O}_X) \to \Gamma(D(t), \mathscr{O}_X)$ is also $A \to A_t$; since t isn't a zerodivisor, this is injective, so a function pulls back to 0 on $D(t)$ iff it vanishes on all of X. Hence, $\operatorname{Im}(D(t) \hookrightarrow X) = X$ as desired. ⊠

Proper Morphisms. The next topological notion we introduce to algebraic geometry is that of a proper map. Recall that a continuous map of topological spaces is proper if the preimage of any compact set is compact. Compactness doesn't really behave the same way in algebraic geometry, so we'll have to define properness in a different way, which will satisfy similar properties.

Proper maps are closed maps, meaning the image of a closed set is closed. This would be a reasonable starting point, except that closed maps are not preserved by fiber products. It turns out the right way to fix this is just to pick the ones that behave well.

Definition 2.11. A morphism $\pi : X \to Y$ of schemes is **universally closed** if for all morphisms $Z \to Y$, the pullback $Z \times_Y X \to Z$ is a closed map.

That is, it remains closed under arbitrary base change.

Lemma 2.12. *Universal closure is a "nice" property of schemes, i.e. local on the target, closed under composition, and preserved by base change.*

Proof. Clearly, universal closure is closed under composition, and by definition, it's preserved by fiber products. Being a closed map is local on the target, and therefore so is universal closure. ⊠

We use universal closure to define the property we really care about.

Definition 2.13. A morphism $\pi : X \to Y$ is **proper** if it's separated, finite type, and universally closed. If A is a ring, an A-scheme X is said to be **proper over** A if the structure morphism $X \to \operatorname{Spec} A$ is proper.

Example 2.14. Closed embeddings are our first example of proper morphisms: they're affine, and therefore separated. Closed embeddings are closed maps, and since the pullback of a closed embedding is a closed embedding, a closed embedding is universally closed. Finally, closed morphisms are finite type (which boils down the fact that if $B \twoheadrightarrow A$ is a surjective ring map, A is a finitely generated B-algebra).[3]

This agrees with our intuition for topological spaces, which is good.

Proposition 2.15 (Vakil prop. 10.3.4)**.**

(1) Properness is a "nice" property of schemes (in the sense of Lemma 2.12).
(2) Properness is closed under products: if $\pi : X \to Y$ and $\pi' : X' \to Y'$ are proper morphisms of S-schemes, then $\pi \times \pi' : X \times_S X' \to Y \times_S Y'$ is proper.
(3) Given a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ & {\scriptstyle\tau}\searrow \quad \swarrow{\scriptstyle\rho} & \\ & Z, & \end{array}$$

if τ is proper and ρ is separated, then π is proper.

For example, by (3), any morphism from a proper k-scheme to a separated k-scheme is proper (let $Z = \operatorname{Spec} k$).

Proof. Everything in this proposition comes nearly for free. We already knew finite type and separability to be nice properties of schemes, and by Lemma 2.12, so is universal closure; since properness is having all three at once, it too must be a nice property. (2) is a formal consequence of (1), which is proven for any nice class of morphisms in Vakil's ex. 9.4.F. Finally, since closed embeddings are proper, the cancellation theorem from last lecture applies to prove (3). ⊠

According to Vakil, the next example is the most important example of proper morphisms.

Theorem 2.16 (Vakil thm. 10.3.5)**.** *If A is a ring and X is a projective A-scheme, $X \to \operatorname{Spec} A$ is proper.*

Proof. Since X is projective, the structure morphism factors as $X \hookrightarrow \mathbb{P}^n_A \to A$, a closed embedding followed by the structure map for $\mathbb{P}^n_A$. Since closed embeddings are proper (Example 2.14), it suffices to show $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper, because proper morphisms are closed under composition. Projective schemes are finite type, and we proved last time that $\mathbb{P}^n_A \to \operatorname{Spec} A$ is separated, so it remains to check universal closure.

If $\varphi : X \to \operatorname{Spec} A$ is an arbitrary morphism, we would like for the map $\mathbb{P}^n_A \times_A X \to X$ to be closed. Since $\mathbb{P}^n_A = \mathbb{P}^n_{\mathbb{Z}} \times_A \operatorname{Spec} \mathbb{Z}$, then we have the following commutative diagram, in which both squares are pullback squares:

$$\begin{array}{ccc} \mathbb{P}^n_A \times_A X & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle\varphi} \\ \mathbb{P}^n_A & \longrightarrow & \operatorname{Spec} A \\ \downarrow & & \downarrow \\ \mathbb{P}^n_{\mathbb{Z}} & \longrightarrow & \operatorname{Spec} \mathbb{Z}. \end{array}$$

By checking the universal property, we see that the outer rectangle is a pullback square too: in other words, $\mathbb{P}^n_A \times_A X = \mathbb{P}^n_X$, so it suffices to show that the structure map $\mathbb{P}^n_X \to X$ is closed for arbitrary X. Being a closed map is a local condition, so we can check on an affine cover of $\mathbb{P}^n_X$; pulling back by $\operatorname{Spec} B \hookrightarrow X$ gives us $\mathbb{P}^n_B \to \operatorname{Spec} B$, so it suffices to know that the structure map is closed for all rings B. This is precisely the fundamental theorem of elimination theory (Thm. 7.4.7 in Vakil's notes), so we're done. ⊠

[3]The same line of reasoning shows that finite morphisms are proper, which is a generalization: they're affine, hence separated, and closed maps; since they're preserved under base change, they must also be universally closed. Finally, finite morphisms are finite type.

Perhaps surprisingly, the converse is almost true: it's difficult to come up with examples of schemes that are proper, but not projective.

The last thing we'll prove about proper schemes is another analogue of compactness. Recall that if M is a compact, connected complex manifold, all holomorphic functions on M are constant. We'll be able to prove a scheme-theoretic analogue of this.

Proposition 2.17 (Vakil 10.3.7)**.** *Let k be an algebraically closed field and X be a connected, reduced, proper k-scheme. Then $\Gamma(X, \mathscr{O}_X) \cong k$.*

Proof. First, we can naturally identify $\Gamma(X, \mathscr{O}_X)$ with the ring of k-scheme maps $X \to \mathbb{A}^1_k$: using the $(\Gamma, \operatorname{Spec})$ adjunction, $\operatorname{Hom}_{\mathsf{Sch}_k}(X, \mathbb{A}^1_k) = \operatorname{Hom}_{\mathsf{Alg}_k}(k[t], \Gamma(X, \mathscr{O}_X)) = \Gamma(X, \mathscr{O}_X)$, so functions on X are actually a ring of functions, which is nice.

Let $f \in \Gamma(X, \mathscr{O}_X)$, so f corresponds to a morphism $\pi : X \to \mathbb{A}^1_k$. If $i : \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$ is the usual open embedding, let $\pi' = i \circ \pi$. Since X is proper and $\mathbb{P}^1_k$ is separated over k, then π' must be proper, by Proposition 2.15, part 3 (let $Z = \operatorname{Spec} k$). Thus, π' is closed, so the set-theoretic image of π' is a closed, connected subset of $\mathbb{P}^1_k$. Since $\mathbb{P}^1_k$ has the cofinite topology, then $\operatorname{Im}(\pi)$ must be a single closed point p or all of $\mathbb{P}^1_k$, but if the latter, it can't factor through i. Since π' factors through $\mathbb{A}^1_k$, p is a closed point in $\mathbb{A}^1_k$, hence identified with an element of k.

The underlying set of the scheme-theoretic image of π is the closure of the set-theoretic image, so it's just p again; since X is reduced, so is its scheme-theoretic image. Thus, $\pi : X \to \mathbb{A}^1_k$ is a constant map of schemes $x \mapsto p$, and tracing through the adjunction, this corresponds to the constant function $f = p \in \Gamma(X, \mathscr{O}_X)$. ⊠

3. Dimension: 5/23/16

Today's lecture was given by Gill Grindstaff, on the first half of Chapter 11. This section relies on a lot of commutative algebra, which can make it difficult.

There are equiuvalent topological and algebraic formulations of the definition of the dimension of a scheme. We'd like this to agree with the intuitive notions of dimension: $\mathbb{A}^n$ should be n-dimensional, for example.

To motivate these definitions, recall that the dimension of a vector space V is the cardinality of some (and therefore any) basis for V. However, it's equivalent to say that the dimension of V is the supremum of lengths of nested chains of subspaces $0 \subsetneq W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_n = V$ (so we don't count 0). The scheme-theoretic definition will resemble this.

Definition 3.1.

- The **Krull dimension of a topological space** X is the supremum of lengths of chains $X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_n = X$ in which each X_i is a closed, irreducible subset of X.
- The **Krull dimension of a ring** A is the supremum of lengths of chains $0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ of prime ideals in A.

Fact. If A is a ring, then $\dim \operatorname{Spec} A = \dim A$. This is because $\mathfrak{p}_i \mapsto V(\mathfrak{p}_i)$ defines an inclusion-reversing bijection between the poset of prime ideals of A and the poset of closed irreducible subspaces of $\operatorname{Spec} A$, and in particular sends chains of nested subsets to chains of nested subsets (in the other direction).

Example 3.2.

(1) Every prime ideal of $\mathbb{Z}$ is of the form $\mathfrak{p} = (p)$ for a prime number p, or is the zero ideal. Hence, the longest chain we can make is $(p) \supset (0)$, so $\dim \mathbb{Z} = 1$.
(2) Similarly, for any field k, in $k[t]$ the longest chains we can make are $(f(t)) \supset (0)$ for f irreducible, so $\dim \mathbb{A}^1_k = 1$.
(3) In $k[x]/(x^2)$, (0) is the only prime ideal, so $\dim k[x]/(x^2) = 0$.

Dimension is *not* local, unlike the dimension of manifolds: consider the space $Z \subset \mathbb{A}^3$ consisting of the union of the xy-plane and the z-axis, which is not irreducible. Then, the dimension of the xy-plane is 2 but the dimension of the z-axis is 1. We do know that $\dim(Z)$ is the maximum of the dimensions of its irreducible subsets, however.

Definition 3.3. A scheme X is **equidimensional** if each of its irreducible components has the same dimension.

An equidimensional scheme of dimension 1 is called a **curve**; an equidimensional scheme of dimension 2 is called a **surface**; and so forth.

In order to get a handle on dimension, we'll need to do come commutative algebra.

Theorem 3.4. *Let* $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$ *be induced from an integral extension* $A \to B$ *of rings. Then,* $\dim \operatorname{Spec} A = \dim \operatorname{Spec} B$.

This follows from an algebraic result.

Theorem 3.5 (Going-up theorem)**.** *Let* $A \hookrightarrow B$ *be an integral extension,* $\mathfrak{p}_1 \subset \mathfrak{p}_2$ *be prime ideals of* A*, and* $\mathfrak{q}_1$ *be a prime ideal of* B *such that* $\mathfrak{q}_1 \cap A = \mathfrak{p}_1$.[4] *Then, there is a prime ideal* $\mathfrak{q}_2 \subset B$ *such that* $\mathfrak{q}_1 \subset \mathfrak{q}_2$ *and* $\mathfrak{q}_2 \cap A = \mathfrak{p}_2$.

This can be inductively extended to chains of prime ideals, which proves one half of Theorem 3.4.

The proof of Theorem 3.5 depends on the following lemma.

Lemma 3.6. *Let* $A \hookrightarrow B$ *be an integral extension. If* B *is a field, then* A *is a field.*

The following exercise is another nice property.

Exercise 3.7. Let $\nu : \widetilde{X} \to X$ be the normalization. Then, $\dim \widetilde{X} = \dim X$.

The **normalization** of X replaces rings with their integral closures on an affine cover; after checking that this behaves well, it defines a nice scheme that X embeds into as an open dense subset. The key to the proof is that the dimension of a ring is the same as the dimension of its integral closure, which follows from Theorem 3.4.

The next thing we'd like to define is codimension, but there are some weird pathologies: recalling Z, the union of the xy-plane and the z-axis, what's the codimension of the z-axis in Z? Should it be 0, since there's nothing above it? Or is it 1, since Z is 2-dimensional and the z-axis is 1-dimensional? There's no good answer, and as a result one only defines codimension inside irreducible schemes.

Definition 3.8. Let X be an *irreducible* topological space and $Y \subset X$. Then, the **codimension** $\operatorname{codim}_X Y$ is the supremum of lengths of chains of irreducible closed subsets $\overline{Y} \subsetneq Z_1 \subsetneq Z_2 \subsetneq \cdots \subsetneq Z_n = X$.

Since Y might not be closed, we must start with $\overline{Y}$. This is satisfactory: a closed point has codimension 2 inside $\mathbb{A}^2$.

There's also an algebraic analogue of codimension.

Definition 3.9. The **codimension of a prime ideal** $\mathfrak{p}$ in a ring A, written $\operatorname{codim}_R \mathfrak{p}$, is the supremum of lengths of *decreasing* chains of prime ideals $\mathfrak{p} \supsetneq \mathfrak{q}_1 \supsetneq \mathfrak{q}_2 \supsetneq \cdots \supsetneq \mathfrak{q}_n \supsetneq (0)$.

In particular, this implies that $\operatorname{codim}_R \mathfrak{p} = \dim R_{\mathfrak{p}}$.

Here are some useful results about dimension.

Proposition 3.10. *Let* R *be a UFD and* $\mathfrak{p} \subset R$ *be a codimension-one prime ideal. Then,* $\mathfrak{p}$ *is principal.*

Theorem 3.11. *Let* A *be a finitely generated* k*-algebra that's an integral domain. Then,* $\dim \operatorname{Spec} A = \operatorname{tr.deg.} K(A)/k$.

Here, $K(A)$ denotes the field of fractions of A, and $\operatorname{tr.deg.} K/k$ is the **transcendence degree** of a field extension K/k; the idea is: how many transcendental elements do you need to adjoin to get K? This intuition turns out to be correct.

Lemma 3.12 (Noether normalization)**.** *Let* S *be a finitely generated* k*-algebra that's an integral domain, such that* $\operatorname{tr.deg.} K(A)/k = n$. *Then, there exist* $x_1, \dots, x_n \in A$*, algebraically independent*[5] *over* k*, such that* A *is a finite extension of* $k[x_1, \dots, x_n]$.

These are very useful because the transcendence degree is much easier to understand than all prime ideals of a ring.

Just as we have a going-up theorem, there's also one in the pther dimension.

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[4]One often says that $\mathfrak{q}_1$ **lies over** $\mathfrak{p}_1$.

[5]Just as linear independence means not satisfying any nonzero linear relation, **algebraic independence** means not satisfying any nonzero polynomial relation.

Theorem 3.13 (Going-down theorem)**.** *Let $\phi : B \hookrightarrow A$ be a finite extension of rings (i.e. A is finitely-generated as a B-module), where B is an integrally closed domain and A is an integral domain. Suppose $\mathfrak{q}_1 \subset \mathfrak{q}_2$ are prime ideals of B and $\mathfrak{p}_2 \subset A$ is a prime ideal such that $\phi^{-1}(\mathfrak{p}_2) = \mathfrak{q}_2$ (so it lies over $\mathfrak{q}_2$). Then, there exists a prime $\mathfrak{p}_1 \subset \mathfrak{p}_2$ such that $\mathfrak{p}_1$ lies over $\mathfrak{q}_1$.*

This requires more assumptions and is harder to prove.

At this point, we talked about a few exercises.

Lemma 3.14. *Let X be a topological space and $U \subset X$ be open. Then, there's a bijection between the irreducible closed subsets of U and the irreducible closed subsets of X meeting U.*

This resembles a theorem from commutative algebra establishing a bijection between prime ideals of B/I and prime ideals of B containing I (where $I \subset B$ is an ideal).

The proof of Lemma 3.14 sets up the bijection by sending an irreducible closed subset $F \subset U$ to $\overline{F} \subset X$, and sends an irreducible $F' \subset X$ to $F \cap U \subset U$.

Exercise 3.15 (Vakil ex. 11.1.B)**.** Show that a scheme has dimension n iff it can be covered by affine open subsets of dimension at most n, where equality is achieved for some affine scheme in the cover.

4. Codimension One: 5/26/16

Today, Richard spoke about sections 11.3 and 11.4, on codimension 1 miracles. We'll skip the last section of chapter 11, because it provides solely algebraic proofs of some of these theorems, and doesn't assist one's geometric intuition.

Definition 4.1. A scheme X is **locally Noetherian** if it has a cover by affine opens $\operatorname{Spec} A_i$ such that each A_i is a local ring. If in addition X is quasicompact, it's called **Noetherian**.

All varieties are locally Noetherian and even Noetherian

One of the codimension 1 miracles is Krull's principal ideal theorem. There are a couple versions.

Theorem 4.2 ((Geometric) Krull's principal ideal theorem)**.** *Let X be a locally Noetherian scheme and $f \in \Gamma(X, \mathscr{O}_X)$. Then, the irreducible components of $V(f)$ are codimension 0 or 1 in X.*

Recall that $V(f)$ is the set of points $x \in X$ such that the stalk $[f] \in \mathscr{O}_{X,x}$ is equal to 0.

Theorem 4.2 follows from the algebraic version.

Theorem 4.3 ((Algebraic) Krull's principal ideal theorem)**.** *Let A be a Noetherian ring and $f \in A$. Then, every prime ideal $\mathfrak{p} \subset A$ minimal among those containing f has codimension at most 1. If f isn't a zerodivisor, the codimension is exactly 1.*

Since we can pass between (co)dimension of prime ideals and (co)dimension of schemes, these two formulations of the theorem are equivalent.

Definition 4.4. Let $X \hookrightarrow Y$ be a closed embedding. Then, X is **locally principal** if there is an affine open cover $\mathfrak{U}$ of Y such that for every $\operatorname{Spec} A \in \mathfrak{U}$, $X \cap \operatorname{Spec} A$ is cut out by a principal ideal of A.

That is, X is locally cut out by a single equation.

Corollary 4.5. *A locally principal closed subscheme has codimension 0 or 1.*

There are a lot of interesting exercises that derive further consequences of this theorem: here are a few.

Proposition 4.6 (Vakil ex. 11.3.C)**.** *Let X be a closed subset of $\mathbb{P}^n_k$ of dimension at least 1. Then, every nonempty hypersurface intersects X.*

This tells us, for example, that there are no parallel hypersurfaces in projective space; in particular, this is not true of affine space. Finding nice hypersurfaces is often a good way to reduce the dimensionality of a question.

Proposition 4.7 (Vakil ex. 11.3.E)**.** *Let $X, Y \subset \mathbb{A}^d_k$ be equidimensional subvarieties of codimensions m and n, respectively. Then, $X \cap Y$ has codimension at most $m + n$.*

Proposition 4.8 (Vakil ex. 11.3.G)**.** *Let A be a Noetherian ring and $f \in A$ be such that f isn't contained in any prime ideal of codimension 1. Then, f is invertible.*

The idea is to consider the dimension of the quotient $A/\mathfrak{p}$ if $f \in \mathfrak{p}$.

Example 4.9. Sometimes, codimension behaves pathologically. Let k be a field and $A = k[x]_{(x)}[t]$: elements of A are expressions of the form

$$\Phi = \sum_{i=1}^{n} \frac{f_i(x)}{g_i(x)} t^i,$$

where $x \nmid g_i(x)$. The ideal $\mathfrak{p} = (xt - 1)$ is prime, and $A/(xt-1) = k[x]_{(x)}[1/x] \cong k(x)$, so $(xt-1)$ is maximal, and hence has dimension 0. By Theorem 4.3, since $xt - 1$ is not a zerodivisor, then $\operatorname{codim}_A \mathfrak{p} = 1$.

Naïvely, we might expect this implies $\dim A = 1$, but in fact there's an irreducible chain of length 2: $(0) \subsetneq (t) \subsetneq (x, t)$, so $\dim A \geq 2$ (and in fact is exactly 2). So codimension is not just the difference in dimension.

Another cool application of dimension is to characterize UFDs (at least among Noetherian rings).

Proposition 4.10 (Vakil 11.3.5)**.** *Let A be a Noetherian integral domain. Then, A is a UFD iff all codimension 1 prime ideals are principal.*

Proof. The forward direction is Proposition 3.10: if $\mathfrak{p}$ is codimension 1, then for any $f \in \mathfrak{p}$, if g is an irreducible prime factor of f, then $(g) \subset \mathfrak{p}$, but since $\operatorname{codim} \mathfrak{p} = 1$, this forces $(g) = \mathfrak{p}$.

Conversely, we want to show that an $a \in A$ is irreducible iff it's prime. If a is irreducible, then by Theorem 4.2, $V(a)$ has a codimension 1 point $[(p)]$, so $a = a'p$ for some a'. Thus, a' must be a unit, so $(a) = (p)$, and hence a is prime. The other direction uses the Noetherian hypothesis. ⊠

The next great property of codimension 1 is a generalization of Krull's principal ideal theorems.

Theorem 4.11 (Krull height theorem)**.** *Let $X = \operatorname{Spec} A$, where A is a Noetherian ring and $Z = V(r_1, \ldots, r_\ell)$ be an irreducible subset. Then, $\operatorname{codim}_X Z \leq \ell$.*

Though this looks like it should follow inductively from Theorem 4.2, it's more subtle.

Another nice result is algebraic Hartogs' lemma, analogous to Hartogs' lemma in several complex variables, which is about poles or singularities of holomorphic functions.

Theorem 4.12 (Algebraic Hartogs' lemma)**.** *Let A be an integrally closed Noetherian integral domain. Then, if P denotes the set of prime ideals of A of codimension 1, then $A = \bigcap_{\mathfrak{p} \in P} A_\mathfrak{p}$.*

This intersection is understood to take place in the fraction field $K(A)$. The relation to the complex-analytic version is that if $f \in K(A)$, it can be interpreted as a rational function. If $f \notin A_\mathfrak{p}$, it's thought of as having a pole at $\mathfrak{p}$, and if it's in $\mathfrak{p}A_\mathfrak{p}$, it has a zero at $\mathfrak{p}$. Hartogs' lemma states that we can extend over singularities of codimension 2 or higher, but not necessarily codimension 1.

Dimension of fibers of morphisms of varieties. Recall that the fundamental theorem of elimination theory (which we used to prove Proposition 2.16) states that for every ring A, $\mathbb{P}_A^n \to \operatorname{Spec} A$ is a closed map. This tells us that closed subsets of projective space are cut out by inhomogeneous equations in $n+1$ variables over A.

One therefore wonders about the locus where the solution of the system of $n+1$ inhomogeneous equations is dimension at least d, for some d. This is a closed condition on the coefficients (just as in linear algebra), and therefore this locus is closed (just like in linear algebra).

Proposition 4.13 (Vakil ex. 11.4.A)**.** *Let $\pi : X \to Y$ be a morphism of locally Noetherian schemes, $p \in X$, and $q = \pi(p)$. Then, $\operatorname{codim}_X p \leq \operatorname{codim}_Y q + \operatorname{codim}_{\pi^{-1}(q)} p$.*

Example 4.14. For this, it's good to have a picture. Suppose X is the union of the xy-plane and the z-axis inside $\mathbb{A}^3$, and $Y = \mathbb{A}^2$. Let $\pi : X \to Y$ crush the z-axis down to 0, $p = (0,0,1)$, and $q = (0,0)$. Then, $\operatorname{codim}_X p = 1$, $\operatorname{codim}_Y q = 2$, and $\operatorname{codim}_{\pi^{-1}(q)} p = 1$ (since $\pi^{-1}(q)$ is the z-axis). Indeed, $1 \leq 2 + 1$.

Now, we have a result akin to the regular value theorem in differential topology: if $f : X \to Y$ is a smooth map of manifolds and $y \in Y$ is a regular value, then $f^{-1}(y) \subset X$ has codimension equal to the difference of their dimensions, or is empty.

Theorem 4.15 (Vakil 11.4.1)**.** *Let $\pi : X \to Y$ be a morphism of finite type k-schemes, $\dim X = m$, and $\dim Y = n$. Then, there is an open $U \subseteq Y$ such that for all $q \in U$, the fiber over q has pure dimension $m - n$ or is empty.*

Fiber dimension in general is discontinuous, but curiously, it obeys **upper semicontinuity** (we say that for all $\varepsilon > 0$, there's a $\delta > 0$ such that if $|x_0 - x| < \delta$, then $f(x) \le f(x_0) + \varepsilon$). The intuition is that the value can jump, but then the "upper part" is closed. This is exactly as in real analysis.

Proposition 4.16 (Vakil 11.4.2)**.** *Let $\pi : X \to Y$ be a morphism of finite type k-schemes.*

(1) The dimension of the fiber of π at a $p \in X$ (specifically, of the largest component of $\pi^{-1}(\pi(p))$ containing p) is upper semicontinuous on X.
(2) If in addition π is proper[6], then the dimension of the fiber above a $y \in Y$ is upper semicontinuous on Y.

Though it's surprising that upper semicontinuity exists, it appears in other places in algebraic geometry. It tells us that the dimension can increase when one takes limits. The dimension of the fiber is never smaller than what you think, but can be bigger, e.g. when we collapsed the z-axis onto the xy-plane in Example 4.14.

5. Regularity: 5/30/16

Today, Jay Hathaway spoke about sections 12.1–12.3, on regularity, to the sonorous sounds of high schoolers warming up for the Texas State Solo and Ensemble Festival.

First, we'll talk about the Zariski cotangent space. Recall that in differential geometry, a manifold is a locally ringed space (M, C^∞_M), with C^∞_M the sheaf of smooth functions. The cotangent space at an $x \in M$ is linear functionals on germs of smooth functions on x to first order: that is, if $\mathfrak{m}_x$ is the maximal ideal of the local ring $C^\infty_{M,x}$, then the cotangent space is $T^*_x M = \mathfrak{m}_x/\mathfrak{m}_x^2$ (the $\mathfrak{m}_x^2$ term contains all of the higher-order information). This motivates the algebraic definition of a cotangent space.

Definition 5.1. Let $(A, \mathfrak{m})$ be a local ring. Then, the **Zariski cotangent space** of A is the $(A/\mathfrak{m})$-vector space $\mathfrak{m}/\mathfrak{m}^2$. If X is a scheme, the **Zariski cotangent space** at a $p \in X$ is $\mathfrak{m}_p/\mathfrak{m}_p^2$, where $\mathfrak{m}_p$ is the maximal ideal of the local ring $\mathscr{O}_{X,p}$.

Just like in differential geometry, tangent vectors correspond to derivations.

Proposition 5.2 (Vakil ex. 12.1.A)**.** *Let X be a scheme and k be the residue field of $\mathscr{O}_{X,p}$ at a $p \in X$. Then, $(\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee \cong \mathrm{Der}_k(\mathscr{O}_{X,p}, \mathscr{O}_{X,p})$.*

Recall that $(\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee = \mathrm{Hom}_k(\mathfrak{m}_p/\mathfrak{m}_p^2, k)$; since this is the dual of the cotangent space, it's reasonable to call it the tangent space.

Partial proof. Let $\nabla : \mathscr{O}_{X,p} \to \mathscr{O}_{X,p}$ be a derivation, and write $f' = \nabla f$ for a germ $f \in \mathscr{O}_{X,p}$. The Leibniz rule tells us that $(fg)' = f'(p)g(p) + f(p)g'(p)$, so the map $f \mapsto f'(p)$ makes sense on $\mathfrak{m}_p$ (the functions that vanish at p) and vanishes on $\mathfrak{m}_p^2$ (functions vanishing to second order at p), so it defines a map $\phi : \mathfrak{m}_p/\mathfrak{m}_p^2 \to k$. ⊠

The other direction is more fiddly, and we'll see a better proof later in the notes. The idea is that we can write $\mathscr{O}_{X,p} = k \oplus \mathfrak{m}_p$ as a split square-zero extension (which is the tricky part, because it's not natural in any sense); then, given a morphism $\phi : \mathfrak{m}_p/\mathfrak{m}_p^2 \to k$ we define a derivation to be 0 on k and ϕ on $\mathfrak{m}_p$, more or less, and this satisfies the Leibniz rule. In a later chapter this is done in greater generality.

Suppose $\pi : X \to Y$ is a map of schemes, $p \in X$, and $q = \pi(p)$. Then, pullback gives us a map of stalks $\pi^\sharp : \mathscr{O}_{Y,q} \to \mathscr{O}_{X,p}$: since a map of schemes is a map of locally ringed spaces, this carries the maximal ideal $\mathfrak{n}_q \subset \mathscr{O}_{Y,q}$ to the maximal ideal $\mathfrak{m}_p \subset \mathscr{O}_{X,p}$. Thus, it descends to a pullback map on the cotangent spaces $\mathfrak{n}_q/\mathfrak{n}_q^2 \to \mathfrak{m}_p/\mathfrak{m}_p^2$. This works for general locally ringed spaces; if you do this for smooth manifolds, the dual of this map is the usual derivative Df of a smooth function f.

Proposition 5.3 (Vakil ex. 12.1.G)**.** *Let $X = \mathrm{Spec}\, k[x_1, \dots, x_n]/(f_1, \dots, f_n)$ for $f_1, \dots, f_n \in k[x_1, \dots, x_n]$. Then, $T^*_p X = \ker \mathrm{Jac}_p(f_1, \dots, f_r)$, which is the Jacobian of $f_1, \dots, f_r$ evaluated at p.*

This is a thing you can sit down and compute; much later in the notes, the sheaf of Kähler differentials can be employed to understand this more cleanly. The idea is that we take the ideal $(x_1, \dots, x_n)$, then mod out by all degree 2 monomials. After this, the f_i decompose into their first-order components.

The cotangent space is the beginning of our understanding of smoothness.

[6] Equivalently, it's a closed map, since we already have the other hypotheses.

Theorem 5.4. *Let $(A, \mathfrak{m})$ be a Noetherian local ring and $k = A/\mathfrak{m}$ be its residue field. Then,*

$$\dim A \leq \dim_k \mathfrak{m}/\mathfrak{m}^2. \tag{5.5}$$

The proof involves Nakayama's lemma.

Definition 5.6. If equality holds in (5.5), one says A is **regular**.

Definition 5.7. Let X be a locally Noetherian scheme.

- If $p \in X$ and $\mathscr{O}_{X,p}$ is a regular local ring, then X is **regular at** p.
- X is **regular** if it's regular at all $p \in X$.
- If X is not regular at p, it's called **singular at** p, and X is called **singular**.

6. An Algebraic Interlude: 6/1/16

Today, Tom reviewed §§7.2 and 7.3, discussing some commutative algebra that is necessary for understanding smoothness, and some finiteness conditions on morphisms.

Definition 6.1. Let $\varphi : B \to A$ be a ring homomorphism.

- An $a \in A$ is **integral** over B if there exists a *monic* polynomial $f \in B[x]$ such that $f(a) = 0$, i.e. there exist $b_0, \ldots, b_{n-1} \in B$ such that
$$a^n + \varphi(b_{n-1})a^{n-1} + \cdots + \varphi(b_0) = 0. \tag{6.2}$$
- A is **integral over** B if all $a \in A$ are integral over B. In this case, φ is called **integral**.
- If φ is integral and injective, φ is called an **integral extension**.

Integrality is a generalization of the algebraicity of a field extension.

Definition 6.3. Let $f : X \to Y$ be a morphism of schemes. Then, f is **integral** if for all affine opens $\operatorname{Spec} B \subset Y$ and affine opens $\operatorname{Spec} A \subset f^{-1}(\operatorname{Spec} B)$, the induced map on global sections $f^\sharp : B \to A$ is integral.

This is how we define almost all fancy properties of schemes: take some ring-theoretic property and require it to hold affine-locally.

Definition 6.4.

- A ring homomorphism of schemses $\varphi : B \to A$ is **finite** if it induces a finitely generated B-module structure on A. Often, one says that "A is a finite B-module."
- A morphism $f : X \to Y$ of schemes is **finite** if for all affine opens $\operatorname{Spec} B \subset Y$, the preimage $f^{-1}(\operatorname{Spec} B) \cong \operatorname{Spec} A$ is affine, and the induced map on global sections $f^\sharp : B \to A$ is finite.

Integral and finite morphisms of schemes have the nice properties we require of properties of morphisms.

Proposition 6.5 (Vakil ex. 7.2.A)**.** *Integrality and finiteness can be checked affine-locally.*

The proofs use the same trick that we always use to check this: reduce to the affine communication lemma. It doesn't come for free: one must check that the ring-theoretic statement is true for the ring A iff it's true for each A_{f_i}, where $(f_1, \ldots, f_i) = 1$. This is analogous to checking on an open cover. It's good to work this out, albeit not more than once.

Integrality plays well under quotients and localization; intuitively, integrality of morphisms of schemes is well-behaved locally.

Proposition 6.6 (Vakil ex. 7.2.B)**.** *Let $\varphi : B \to A$ be an integral morphism.*

(1) If $S \subset B$ is a multiplicative set, $S^{-1}\varphi : S^{-1}B \to S^{-1}A$ is integral.
(2) If $J \subseteq A$ is an ideal and $I = \varphi^{-1}(J)$, then $B/I \to A/J$ is integral.
(3) If $I' \subseteq B$ is an ideal, then $B/I' \to A/I'A$ is integral (here, $I'A$ is the ideal generated by $\varphi(I')$).

Moreover, (1) and (2) preserve the property that φ is an integral extension.

Surjective ring maps are tautologically integral, but we can do even better: they're finite, and we'll show finiteness implies integrality.

Partial proof. For the first part, suppose $a/s \in S^{-1}A$, so we know there exist b_i satisfying (6.2). When we multiply by s^n, this shows

$$\left(\frac{a}{s}\right)^n + \frac{b_{n-1}}{s}\left(\frac{a}{s}\right)^{n-1} + \cdots + \frac{b_0}{s^n} = 0,$$

so a/s is integral over $S^{-1}B$.

The second part follows from taking the integrality condition (6.2) mod J.

If φ is an integral extension, we just have to check injectivity. For part (1), this follows because localization is an exact functor: you can check that if $0/1 = \varphi(b)/\varphi(s)$ inside $S^{-1}A$, then there's a $t \in S$ such that $\varphi(t) = 0 = \varphi(tb)$, so $tb = 0$, and therefore $b/s = bt/st = 0$, so φ is injective. For (2), injectivity follows more or less by definition of the quotient. ⊠

Lemma 6.7. *Let $\varphi : B \to A$ be a ring homomorphism. Then, $a \in A$ is integral over B iff there's a subalgebra M of A containing a that is finitely generated as a B-module.*

Again, this is a property of algebraicity, and it invites an interesting question: given a monic polynomial satisfying a, and a monic polynomial satisfying a', how do we write down one satisfying $a + a'$? This is tricky, and there is one that exists, but the point of the lemma is that you need not do it directly.

Proof. In the forward direction, suppose a is integral over B. Then, $B[a] \subset A$ is a finitely generated B-module, because it's generated by $1, a, \ldots, a^{n-1}$ over B. Conversely, suppose $M = \langle m_1, \ldots, m_k \rangle_B$ is a finitely generated B-submodule of A containing a. That is, there are $\lambda_{ij} \in B$ such that

$$am_i = \sum_{j=1}^{k} \lambda_{ij} m_j.$$

If $\Lambda = (\lambda_{ij})$ is the matrix of these coefficients and $\vec{m} = (m_1, \ldots, m_k)^{\mathrm{T}}$, then this says that, as matrices over B, $(aI - \Lambda)\vec{m} = 0$. We'd like to invert this, but we're not over a field. Using the adjugate matrix, which does exist over rings, we have that $\det(aI - \Lambda)$ annihilates $\vec{m}$, and so since A' contains 1 and $(m_1, \ldots, m_k)$ generates M, $\det(aI - \Lambda) \cdot 1 = 0$. This is great, because $\det(aI - \Lambda)$ is a monic, degree-k polynomial with coefficients in $\varphi(B)$. ⊠

Extending Lemma 6.7, one can show that $a \in A$ is integral over B iff $B[a]$ is a finitely generated B-module.

Corollary 6.8 (Vakil cor. 7.2.2)**.** *If $\varphi : B \to A$ is a finite ring homomorphism, then it's integral.*

The converse is not true, e.g. the inclusion $\mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}$. Corollary 6.8 is a generalization of the field-theoretic statement that finite extensions are algebraic, but the converse is not true.

Proposition 6.9. *A composition of integral ring homomorphisms is integral.*

Proof. Let $\varphi : B \to A$ and $\psi : C \to B$ be integral ring homomorphisms. Let $a \in A$, so $a \in B[\varphi(b_1), \ldots, \varphi(b_n)] \subset A$. We can write each $\varphi(b_i)$ as a polynomial in finitely many $\varphi(\psi(\gamma_{ij}))$, and so a is generated over C by these finitely many λ_{ij}. ⊠

Proposition 6.10 (Vakil ex. 7.2.D)**.** *Let $\varphi : B \to A$ be a ring homomorphism. Then, the elements of A that are integral over B form a B-subalgebra $\overline{B} \subset A$, called the* ***integral closure*** *of B in A.*

If the ambient ring A is absent, an integral closure usually refers to the integral closure in the field of fractions.

The idea of the proof is that it reduces to checking that if a and a' are integral over B, then so are $a + a'$ and aa'. We want to look at $B[a + a']$ and $B[aa']$, which are subextensions of $B[a][a']$. We know $B[a][a']$ is integral over $B[a]$, and $B[a]$ is integral over B, so by Proposition 6.9, $B[a][a']$ is integral over A.

Using these smaller results, we can understand a bigger theorem, the lying over theorem.

Theorem 6.11 (Lying over, Vakil thm. 7.2.5)**.** *Let $\varphi : B \hookrightarrow A$ be an integral extension. Then, for any prime ideal $\mathfrak{p} \subset B$, there is a prime ideal $\mathfrak{q} \subset A$* ***lying over*** *$\mathfrak{p}$, i.e. $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$.*

What this also says is that if $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$ is an integral map of schemes, then φ is surjective as a map of sets. This can be useful.

We can extend this to a statement about chains of ideals.

Theorem 6.12 (Going up). *Let $\varphi : B \to A$ be an integral ring homomorphism, $n > m \geq 1$, $\mathfrak{q}_1 \subsetneqq \cdots \subsetneqq \mathfrak{q}_m$ be a chain of strictly increasing prime ideals of A, and $\mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_n$ be a chain of strictly increasing prime ideals of B. If each $\mathfrak{q}_i$ lies over $\mathfrak{p}_i$ for $1 \leq i \leq m$, then we can extend the chain: there exist $\mathfrak{q}_{m+1} \subsetneqq \cdots \subsetneqq \mathfrak{q}_n$ such that $\mathfrak{q}_i$ lies over $\mathfrak{p}_i$ for $1 \leq i \leq n$.*

Geometrically, this states that if $\varphi : \operatorname{Spec} A \to \operatorname{Spec} B$ is an integral morphism of schemes and I have a chain of irreducible subsets $Y_1 \supsetneqq Y_2 \supsetneqq \cdots \supsetneqq Y_m$ of $\operatorname{Spec} A$, a chain of irreducible subsets $X_1 \supsetneqq X_2 \supsetneqq \cdots \supsetneqq X_n$ of $\operatorname{Spec} B$, and $\varphi(Y_i) = X_i$, then we can find irreducible subsets mapping to $X_{m+1}, \ldots, X_n$ and preserving the chain relations.

The proof idea is to apply the lying over theorem many times. Interestingly, it tells you how to define the scheme-theoretic fiber, and therefore in some sense motivates the definition of the fiber product: the primes lying over $\mathfrak{p}$ are the fiber $\varphi^{-1}(\mathfrak{p})$ as a set: as a scheme, we take the fiber product with $\operatorname{Spec} k_{\mathfrak{p}}$, where $k_{\mathfrak{p}}$ is the residue field $k_{\mathfrak{p}} = B_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$; this is because the prime ideals lying over $\mathfrak{p}$ are the prime ideals lying over 0 in the residue field. Ring-theoretically, we get the pushout $A \otimes_B k_{\mathfrak{p}}$.

The point is, the setwise fiber has the same underlying set as the scheme-theoretic fiber, which admits the more abstract definition through a universal property. Equating these two viewpoints is nice to know. It also makes it easier to compute stalks of points in a fiber.

Now, let's talk about the various theorems called Nakayama's lemma. Like the cupcakes at the front of the class today, some are better than others.

Lemma 6.13 (Nakayama's lemma 1, Vakil 7.2.8). *Let A be a ring, $I \subseteq A$ be an ideal, and M be a finitely generated A-module such that $M = IM$. Then, there's an $a \in A$ such that $a = 1 \pmod I$ and $aM = 0$.*

Proof. Choose a generating set $m_1, \ldots, m_k$ for M; since $M = IM$, $M = \langle m_1, \ldots, m_k \rangle_I$. That is, there exist $\lambda_{ij} \in I$ for $1 \leq i, j \leq k$ such that

$$m_i = \sum_{j=1}^{k} \lambda_{ij} m_j.$$

If $\vec{m} = (m_1, \ldots, m_k)^{\mathrm{T}}$ and $\Lambda = (\lambda_{ij})$ as before, then $(1 - \Lambda)\vec{m} = 0$, and therefore we can choose $a = \det(1 - \Lambda)$, so $am_i = 0$ for each i, and $a = 1 \pmod I$ (since Λ is I-valued). ⊠

Recall that the **Jacobson radical** $\operatorname{Jac}(A)$ of a ring A is the intersection of its maximal ideals.

Lemma 6.14 (Nakayama's lemma 2, Vakil 7.2.9). *Let A be a ring, $I \subseteq A$ be an ideal contained in $\operatorname{Jac} A$, and M be a finitely generated A-module such that $M = IM$. Then, $M = 0$.*

Proof. Using Lemma 6.13, we have an $a \in A$ such that $a = 1 \pmod I$ (so $a = 1 + i$ for some $i \in I$) and $aM = 0$. For all maximal ideals $\mathfrak{m} \subset A$, $a \notin \mathfrak{m}$, because $a = 1 + i$ for some $i \in I \subset \mathfrak{m}$. Thus, a is a unit, so $M = aM = 0$. ⊠

This is slick, but doesn't show you why M must be finitely generated. A more explicit proof chooses (using Zorn's lemma) a maximal submodule $N \subsetneq M$ and $x \in M \setminus N$. Then, we can define a map $\varphi : A \twoheadrightarrow M/N$ sending $a \mapsto a \cdot [x]$. Hence, $A/\ker\varphi \cong M/N$ as A-modules (there's no good ring structure here), forcing $I \subseteq \ker\theta$. This implies $IM \subseteq N \subsetneq M$, but $IM = M$, so no such N exists, and therefore $M = 0$. So we don't need M to be finitely generated, which is pretty cool.

All the other versions of Nakayama's lemma follow from these two.

Lemma 6.15 (Nakayama's lemma 3). *Let A be a ring, $I \subset A$ be an ideal contained in $\operatorname{Jac} A$, M be an A-module, and $N \subset M$ be a submodule. If $N/IN \to M/IM$ is surjective, then $N = M$.*

These are useful for proving various submodules are the whole model, etc., which is useful for showing that xactness is a local condition, e.g. if M is an A-module, $M = 0$ iff $M_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p} \subset A$ iff $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m} \subset A$.

7. More Regularity and Smoothness: 6/2/16

Today's lecture was given by Jay and Danny.

Recall that in Proposition 5.2, we said that if $(A, \mathfrak{m})$ is a local ring with reside field k, then $\operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k) = \operatorname{Der}_k(A, k)$. We showed how to obtain a homomorphism given a derivation; let's go in the other direction.

Suppose $\phi : \mathfrak{m}/\mathfrak{m}^2 \to k$ is a homomorphism; then, we can write $A/\mathfrak{m}^2$ as a square-zero extension of k by $\mathfrak{m}/\mathfrak{m}^2$, i.e. as rings, $A/\mathfrak{m}^2 = k \oplus \mathfrak{m}/\mathfrak{m}^2$, sending $f \mapsto (f(p), f - f(p) \pmod{\mathfrak{m}^2})$. Now, we define a derivation ∇: if $\lambda \in k$ and $m \in \mathfrak{m}/\mathfrak{m}^2$, then let $\nabla(\lambda + m) = \phi(m)$. This obeys the Leibniz rule: since $(\mathfrak{m}/\mathfrak{m}^2)^2 = 0$ in the square-zero extension,

$$\nabla((\lambda_1 + m_1)(\lambda_2 + m_2)) = \nabla(\lambda_1\lambda_2 + \lambda_2 m_1 + \lambda_1 m_2) = \lambda_2\phi(m_1) + \lambda_1\phi(m_2),$$

and therefore ∇ is indeed a derivation.

A related result allows us to understand the tangent space to a scheme X as the first-order infinitesimal information at X.

Proposition 7.1. *Let X be a scheme, k be a field, and $p \in X$ be a k-valued point. Then,*

$$\mathrm{Hom}_{\mathsf{Sch}}(\operatorname{Spec} k[x]/(x^2), X) = T_pX.$$

Proof. Since the tangent space is locally defined, we may assume $X = \operatorname{Spec} A$, where A is a k-algebra, and p represents the maximal ideal $\mathfrak{m})_p \subset A$. A morphism $\pi : \operatorname{Spec} k[x]/(x^2) \to X$ therefore induces a ring map $\phi : A \to k[x]/(x^2)$ such that $\phi^{-1}(x) = \mathfrak{m}_p$; hence, it factors through the map $\widetilde{\phi} : A/\mathfrak{m}_p^2 \to k[x]/(x^2)$. Then, the desired tangent vector is the map $\mathfrak{m}_p/\mathfrak{m}_p^2 \to k[x]/(x^2) \to k$, where the first map is $\widetilde{\varphi}|_{\mathfrak{m}_p/\mathfrak{m}_p^2}$ and the second map takes the coefficient of x.

In the other direction, suppose we have a $\phi \in \mathrm{Hom}_k(\mathfrak{m}_p/\mathfrak{m}_p^2, k)$. Once again we can take a square-zero extension $A/\mathfrak{m}_p^2 \cong k \oplus \mathfrak{m}_p/\mathfrak{m}_p^2$, and so the desired morphism of schemes is Spec of the ring map $A \to A/\mathfrak{m}_p^2 = k \oplus \mathfrak{m}_p/\mathfrak{m}_p^2 \to k[x]/(x^2)$, where the latter map sends $\lambda + m \mapsto \lambda + \phi(m)x \pmod{\mathfrak{m}^2}$. One has to check these are inverses, but it follows because they were defined in the same way. ⊠

It would also be nice to know whether this bijection depends on the choice of $\operatorname{Spec} A \subset X$ containing p; it turns out not to, and also doesn't depend on the splitting, since A is a k-algebra (there's already a natural map $k \to A$), so it's suitably natural.

We can also now prove Theorem 5.4: if $(A, \mathfrak{m})$ is a Noetherian local ring with residue field, then the Krull dimension of A is at most $\dim_k \mathfrak{m}/\mathfrak{m}^2$: the dimension of the tangent space is an upper bound.

Proof of Theorem 5.4. Since A is Noetherian, $\mathfrak{m}$ is finitely generated, and therefore $\mathfrak{m}/\mathfrak{m}^2$ is a finite-dimensional k-vector space, say n-dimensional. Let $\{f_1, \dots, f_n\}$ be a basis of $\mathfrak{m}/\mathfrak{m}^2$. By one of the many versions of Nakayama's lemma (version 4 in Vakil's notes), this lifts to a generating set of $\mathfrak{m}$: there is a lift $\widetilde{f_i} \in \mathfrak{m}$ for each f_i such that $\mathfrak{m} = (\widetilde{f_1}, \dots, \widetilde{f_n})$. Krull's height theorem says that any ideal containing n elements has height at most n, so the height of $\mathfrak{m}$ is at most n.

Now, we'd like to talk about $\dim A$; since A is a local ring, any chain of its prime ideals is contained entirely in $\mathfrak{m}$; thus, the length of any chains of primes in A is bounded above by the height of $\mathfrak{m}$, i.e. $\dim A \le n \le \dim_k \mathfrak{m}/\mathfrak{m}^2$. (In fact, if a chain doesn't contain $\mathfrak{m}$, $\mathfrak{m}$ can be added to it, so $\dim A = n$.) ⊠

Recall that if A is a Noetherian local ring and the inequality in the above proof is an equality, A is called regular, and that a locally Noetherian scheme X is regular at a $p \in X$ if $\mathscr{O}_{X,p}$ is a regular ring.

The intuition for regularity is that at a singularity, there are "too many tangent directions," so the tangent space has too high of a dimension. For example, on the scheme that's the union of the x- and y-axes, the tangent space is one-dimensional everywhere except the origin, where there are two directions, and correspondingly a two-dimensional tangent space. Hence, this scheme is regular everywhere except the origin.

There's a criterion for regularity that will motivate the definition of smoothness.

Proposition 7.2. *Let p be a k-valued point of $X = \operatorname{Spec} k[x_1, \dots, x_n]/(f_1, \dots, f_r)$. If X is pure dimension d, then X is regular iff* $\operatorname{corank} \operatorname{Jac}(f_1, \dots, f_r)|_p = d$.

Proposition 7.3 (Vakil ex. 12.2.E)**.** *Let k be an algebraically closed field and $X = \operatorname{Spec} k[x_1, \dots, x_n]/(f)$ be a* ***hypersurface*** *(meaning it has codimension 1). Then, a closed point $p \in X$ is singular iff* $\operatorname{Jac} f = 0$.

That is, the singular points are cut out by $f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$. This is useful, because it means they form a closed subset.

Regularity is helpful, but there are a few drawbacks: for example, it's not obvious when the Jacobian criterion is sufficient; we know it's sufficient at k-valued points when k is algebraically closed, but that's somewhat restrictive. The fix is actually to define smoothness.

Definition 7.4. Let X be a finite-type k-scheme of pure dimension d. Then, X is **smooth (of dimension d)** if it can be covered by affine opens of the form $\operatorname{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ with $\operatorname{corank} \operatorname{Jac}(f_1, \ldots, f_r)|_p = d$ for all $p \in X$.

One thing that would be nice to know is whether this satisfies the affine communication lemma. We'll return to this much later, when we define the smoothness of a morphism and show it has nice properties. It will be hard to show that this is affine-local in general, since n and r aren't required to be fixed.

The interesting examples are all hard, and we will have to return to them later. However, we can do a few easier examples.

Example 7.5. Consider $\mathbb{A}_k^n = \operatorname{Spec} k[x_1, \ldots, x_n]$. Then, $\operatorname{Jac}(0) : k \to k^n$ always has corank n, and therefore $\mathbb{A}_k^n$ is smooth of dimension n, which is reassuring.

It's useful to compare regularity and smoothness.

Proposition 7.6 (Vakil ex. 12.2.I)**.** *Let k be an algebraically closed field and X be a finite-type k-scheme. Then, X is smooth iff it's regular at its closed points.*

Proof. Smooth definitely implies regular at closed points, by a previous exercise.

First, we show it for affines: suppose $X = \operatorname{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$. The locus L where $\operatorname{Jac}(f_1, \ldots, f_r)$ has corank d is a locally closed set, since bounding the rank above is a closed condition, but bounding it below is an open condition. That is, $L = U \cap F$, where U is open adn F is closed. If X is regular at all closed points, this contains all closed points, which are dense in X (since it's a finite type k-scheme: the closed points are those where the residue field is a finite extension of k, hence must be k). Thus, $F = X$, and $L = U$ is an open dense set. Thus, $L = X \setminus V(I)$ for an ideal $I \subseteq k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$, but L contains all maximal ideals, so I isn't contained in any maximal ideal, and therefore $I = A$, so $L = X$.

Since both regularity and smoothness can be checked locally, this suffices. ⊠

Here is another comparison between regularity and smoothness. Recall that a field k is **perfect** if $k = k^{\operatorname{char} k}$, including all characteristic 0 fields and finite fields, but excluding fields such as $\mathbb{F}_p(t)$.

Proposition 7.7. *Let k be a field.*

(1) Every smooth k-scheme is regular.
(2) If k is perfect, every regular, finite-type k-scheme is smooth.

Part 2 begins here, where Danny took over, and talked about §§12.4–12.6.

Discrete valuation rings are an excellent example of dimension 1 magic. We care about Noetherian local rings, because they're the stalks of pretty much every scheme we encounter. Using theorems like Krull's principal ideal theorem, we can induct as long as we understand the dimension 1 case, so let's think about that case.

We have a bunch of nice classes of rings that aren't *a priori* related to the Noetherian property, such as being a PID, being regular, being normal, and so on. In nice cases, we can relate these (which is the "dimension 1 magic" in question).

Theorem 7.8. *Let $(A, \mathfrak{m})$ be a 1-dimensional Noetherian local ring. Then, the following are equivalent.*

(1) $(A, \mathfrak{m})$ is regular.
(2) $\mathfrak{m}$ is principal.

Proof. To show (1) $\implies$ (2), suppose A is regular, so $\dim A = \dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$, so by Nakayama's lemma, a basis $\{f\}$ of $\mathfrak{m}/\mathfrak{m}^2$ lifts to a generator $\widetilde{f}$ of $\mathfrak{m}$, so $\mathfrak{m} = (\widetilde{f})$. Conversely, if $\mathfrak{m} = (r)$, then $\mathfrak{m}/\mathfrak{m}^2$ is a 1-dimensional vector space, as needed. ⊠

Proposition 7.9 (Vakil ex. 12.2.A)**.** *A zero-dimensional regular local ring is a field.*

Note that such a ring is trivially Noetherian.

Proof. Let A be such a ring and $\mathfrak{m}$ be its unique maximal ideal. By Nakayama's lemma, version 2 (Lemma 6.14), since $\mathfrak{m}/\mathfrak{m}^2 = 0$, then $\mathfrak{m} = \mathfrak{m}(\mathfrak{m})$, so $\mathfrak{m} = (0)$. Conversely, a field is zero-dimensional, and its maximal ideal is (0), so $(0)/(0)^2$ is zero-dimensional, so fields are regular. ⊠

Theorem 7.10 (Vakil thm. 12.2.13). *If $(A, \mathfrak{m})$ is a finite-dimensional regular local ring, then it's an integral domain.*

Proof. We induct on the dimension. The base case $n = 0$ follows from Proposition 7.9.

For the inductive step, suppose $n > 0$, and we know the result for things of dimension at most n. Let $f \in \mathfrak{m} \setminus \mathfrak{m}^2$, so $A/(f)$ is local and $\dim Z(A/(f)) = n - 1$ (where Z is the Zariski tangent space at $\mathfrak{m}$). Exercise 11.3.B informs us that since $(A, \mathfrak{m})$ is Noetherian, then for any $f \in \mathfrak{m}$, $\dim A/(f) \geq n - 1$, so by Theorem 5.4, $\dim A/(f) = n - 1$, so by the inductive assumption, $A/(f)$ is an integral domain.

We extend this to A: choose a minimal prime $\mathfrak{p}$ of A such that $\dim A/\mathfrak{p} = n$ (we're going to show $\mathfrak{p} = 0$): we can do this because $\dim A = n$, so there is a chain of primes of length n, and we can take $\mathfrak{p}$ to be the lowest prime on the chain. We can check $A/\mathfrak{p}$ is regular: its Zariski tangent space would need to be n-dimensional, and using Theorem 5.4 again, this is indeed the case.

The same argument with $A/\mathfrak{p}$ in place of A shows that $(A/\mathfrak{p})/(f) \cong A/(\mathfrak{p} + (f))$ is a regular local ring of dimension $n - 1$, and hence an integral domain. Thus, $A/(\mathfrak{p} + (f))$ is a quotient map between two maps of the same dimension; the kernel is a prime ideal $\mathfrak{q}$, and extends any chain of prime ideals from the codomain to one strictly longer — unless $\mathfrak{q} = (0)$, so this is an isomorphism: $(A/\mathfrak{p})/(f) \cong A/(\mathfrak{p} + (f))$. In particular, $\mathfrak{p} + (f) = (f)$, so $\mathfrak{p} \subsetneq (f)$, and therefore any $u \in \mathfrak{p}$ can be written as $u = fv$ for some $v \in A$. Since $\dim A?(\mathfrak{p} + (f)) < \dim A/\mathfrak{p}$, then $f \notin \mathfrak{p}$, and therefore $v \in \mathfrak{p}$. Thus, $\mathfrak{p} \subset (f)\mathfrak{p} \subset \mathfrak{p}$, so by Nakayama's lemma, $\mathfrak{p} = (0)$, so A is an integral domain. ⊠

The following statement is a corollary of the Artin-Rees lemma, which is in §12.9.

Proposition 7.11 (Vakil 12.5.2). *If $(A, \mathfrak{m})$ is a Noetherian local ring, then*

$$\bigcap_{i=1}^{\infty} \mathfrak{m}^i = 0.$$

This is actually a geometric statement: for example, it tells us that a holomorphic function that vanishes to all orders must be zero. Moreover, if one has a smooth, non-analytic function on a scheme, then its stalks must not be Noetherian.

Thus, we can extend Theorem 7.8. There are lots of nice algebraic properties that are equivalent.

Theorem 7.12. *The conditions in Theorem 7.8 are also equivalent to:*

(3) All ideals of $(A, \mathfrak{m})$ are of the form $\mathfrak{m}^i$, or (0).
(4) A is a PID.
(5) A is a discrete valuation ring.
(6) A is a UFD.
(7) A is an integrally closed integral domain.

These imply lots of things, e.g. characterizing the ideals of a DVR, and that a valuation on an integral domain is unique. One geometric application is to define zeros and poles of integral orders using the valuation $v : k^\times \to \mathbb{Z}$. That is, for a locally Noetherian scheme, we know how to define zeros and poles of various orders at any codimension 1 point!

8. Quasicoherent Sheaves: 6/6/16

Today Yuri will ramble quasicoherently about different types of sheaves, albeit without real proofs. However, we'll still hear which proofs are worth doing, and which aren't.

The setup for today is that over a scheme X, one can develop a theory of sheaves of modules, and one might want this to correspond to ordinary modules over a ring. We'll figure out which part of the theory carries over.

A starting insight is if X is a topological space, the category $\mathsf{Sh}(X)$ of sheaves of sets on X behaves a lot like the category of sets (which is a special case: $\mathsf{Set} = \mathsf{Sh}(\mathrm{pt})$), and so statements such as "module over a ring" can be directly translated into sheaf-theoretic logic. However, there are some bizarre-looking restrictions.

- The axiom of choice does not translate, because there exist surjective morphisms of sheaves $\mathscr{F} \twoheadrightarrow \mathscr{G}$ that do not admit sections (which correspond to choice functions in Set).

- The law of excluded middle does not hold: a statement about sheaves need not just be true or false: it can be true on some open sets and false on others. As such, any statement about sets that requires a proof by contradiction does not necessarily translate to sheaves.

However, many familiar constructions do not need these: the proof of existence of sheaf hom, tensor product, and direct sum, for example, goes through word-for-word, so these proofs aren't "interesting," as they're translations of proofs for modules that you already know.

Recall that a **presentation** of an R-module M is an exact sequence

$$R^{\oplus r} \longrightarrow R^{\oplus g} \longrightarrow M \longrightarrow 0,$$

where r denotes the relations and g denotes the generators. These always exist, since there is a "least efficient" presentation generated by all elements of M, with relations given by all of their relations.

Can we translate this to sheaves? That is, if X is a locally ringed space, does every sheaf $\mathscr{F}$ of R-modules on X admit a presentation

$$\mathscr{O}_X^{\oplus I} \longrightarrow \mathscr{O}_X^{\oplus J} \longrightarrow \mathscr{F} \longrightarrow 0,$$

at least locally?

Disappointingly, the answer is no.

Example 8.1. Consider a sheaf $\mathscr{F}$ of $\mathbb{Z}$-modules over $\mathbb{R}$ where $\mathscr{F}(U) = \mathbb{Z}$ if U doesn't contain the origin; if U contains the origin, we take $\mathscr{F}(U) = 0$. There is no local presentation of $\mathscr{F}$: it would have to at least surject onto the constant sheaf $\underline{\mathbb{Z}}$, but there is no way to surject both onto 0 (for a neighborhood of the origin) and onto $\mathbb{Z}$ over anything else. However, we can provide presentations for all stalks.

Having presentations, at least locally, is a good thing.

Definition 8.2. Let $(X, \mathscr{O}_X)$ be a locally ringed space and $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules on X.

- $\mathscr{F}$ is **quasicoherent** if it's locally presentable.
- $\mathscr{F}$ is **finite type** if it's locally finitely generated (i.e. there is a presentation where the generators form a finite-dimensional free $\mathscr{O}_X$-module).
- $\mathscr{F}$ is **finitely presented** if its presentation is (i.e. both the generators and relations are finite-dimensional).
- $\mathscr{F}$ is **coherent** if it's of finite type, and for any n, the kernel of the map $\mathscr{O}_X^n \to \mathscr{F}$ is finite type.
- $\mathscr{F}$ is **locally free** if, locally, there is an isomorphism $\mathscr{F} \cong \mathscr{O}_X^n$ of $\mathscr{O}_X$-modules.[7]

There are other, equivalent definitions over schemes or over Noetherian schemes. Quasicoherent sheaves will form an abelian category, as will coherent sheaves.

Locally free sheaves are analogous to vector bundles: if $\pi : E \to B$ is a vector bundle, we can take its sheaf $\mathscr{F}$ of sections. Since the vector bundle is locally trivializable, there's an open cover $\mathfrak{U}$ of B such that for each $U \in \mathfrak{U}$, $\pi|_U$ is isomorphic to the projection $U \times \mathbb{R}^n \to U$ (n may vary, but is constant on connected components). Hence, translating to sheaves, we recover the local freeness condition. In the other direction, we can take relative Spec: if $\mathscr{F}$ is locally free, $\operatorname{Spec}(\operatorname{Sym}\mathscr{F})$ will recover the vector bundle (there's a bit to check here).

Hence, as objects, locally free sheaves are the same as vector bundles. **Warning**: this is *not* an equivalence of categories! The morphisms are not the same: there are linear maps of locally free $\mathscr{O}_X$-modules that are not morphisms of vector bundles. Vector bundles are not an abelian category; there are issues with cokernels.

In any case, we know what vector bundles are for manifolds, which suggests that for schemes, we can define vector bundles to be locally free sheaves.

Proposition 8.3. *Let $\mathscr{E}$ be a locally free sheaf on X and $\mathscr{F}$ be an $\mathscr{O}_X$-module. Then, there is an isomorphim of $\mathscr{O}_X$-modules $\mathscr{E}^\vee \otimes_{\mathscr{O}_X} \mathscr{F} \cong \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F})$.*

Here, $\operatorname{Hom}_{\mathscr{O}_X}$ denotes sheaf hom.

Proof. We have an evaluation map $\operatorname{ev} : \mathscr{E} \otimes \mathscr{E}^\vee \to \mathscr{O}_X$, so composing $\operatorname{ev} \otimes 1 : \mathscr{E} \otimes \mathscr{E}^\vee \otimes \mathscr{F} \to \mathscr{O}_X \otimes \mathscr{F}$ with the isomorphism $\mathscr{O}_X \otimes \mathscr{F} \to \mathscr{F}$, we obtain a map $\mathscr{E} \otimes \mathscr{E}^\vee \otimes \mathscr{F} \to \mathscr{F}$; by the tensor-hom adjunction, this is the same data as a map $\mathscr{E}^\vee \otimes \mathscr{F} \to \operatorname{Hom}_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F})$.

[7] Sometimes people consider infinite-rank locally free sheaves, but today all of our locally free sheaves will be finite rank, akin to finite-dimensional vector spaces.

To check this map is an isomorphims, it suffices to check locally, and therefore assume $\mathscr{E} \cong \mathscr{O}_X^{\oplus n}$. The map becomes a map $\mathscr{O}_X^n \otimes \mathscr{F} \to \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X^n, \mathscr{F})$, which becomes an isomorphism $\mathscr{F}^n \to \mathscr{F}^n$. ⊠

Proposition 8.4 (Frobenius reciprocity)**.** *Suppose $f : X \to Y$ is a morphism of locally ringed spaces, $\mathscr{F}$ is an $\mathscr{O}_X$-module, and $\mathscr{E}$ is a locally free $\mathscr{O}_Y$-module. Then, there is an isomorphism $f_*\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{E} \to f_*(\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{E})$.*

Now, we'll actually talk about schemes, and give a nice characterization of quasicoherent sheaves over schemes: they're constructed from modules over a ring of functions.

Proposition 8.5. *Let X be a scheme and $\mathscr{F}$ be an $\mathscr{O}_X$-module. Then, $\mathscr{F}$ is quasicoherent iff for all affine opens $U = \operatorname{Spec} A \hookrightarrow X$, $\mathscr{F}|_U \cong \widetilde{M}$ for some $\Gamma(U, \mathscr{O}_X)$-module M.*

Here, $\widetilde{M}$ is the sheaf of $\mathscr{O}_X$-modules constructed by localization, in the same way that $\mathscr{O}_X$ was constructed: for a distinguished open $D(f) \subset \operatorname{Spec} A \hookrightarrow X$, we can choose $\widetilde{M}(D(f)) = M_f$.

Proving Proposition 8.5 takes some work, but is essentially a follow-your-nose argument. One important ingredient is that distinguished affine inclusions, i.e. inclusions of the form $\operatorname{Spec}(A_f) \hookrightarrow \operatorname{Spec} A \hookrightarrow X$, are cofinal in $\mathsf{Top}(X)$, the category of open subsets of X (meaning every open subset contains a distinguished affine open). This allows one to work only with these inclusions, and therefore reduce the proof to the affine case.

Lemma 8.6. *If $X = \operatorname{Spec} A$ is an affine scheme, then quasicoherent sheaves on X are the same as A-modules.*

In this case, given a quasicoherent sheaf $\mathscr{F}$, let $M = \Gamma(\operatorname{Spec} A, \mathscr{F})$. Quasicoherence implies $\Gamma(\operatorname{Spec} A_f, \mathscr{F}) \cong \Gamma(\operatorname{Spec} A, \mathscr{F})_f$, which means $\widetilde{M}(U) \cong \mathscr{F}(U)$ for all distinguished affine opens $U \subset \operatorname{Spec} A$, which implies it for all open subsets. We can show this locally, using the presentation of $\mathscr{F}$. Most of the proofs interspersed throughout the text are fairly formal, e.g. verifying definitions. Many of these hold for all locally ringed spaces and aren't so interesting. But the exercises near the end, which are specifically about schemes (or specific kinds, e.g. over number fields), are definitely worth your time.

Remark. The fact that $\Gamma(\operatorname{Spec} A_f, \mathscr{F}) \cong \Gamma(\operatorname{Spec} A, \mathscr{F})_f$ is true for more than affine schemes: if X is QCQS (quasicoherent and quasiseparated) and $f \in \Gamma(X, \mathscr{O}_X)$, we can define $X_f = \{p \in X \mid f(p) \notin \mathfrak{m}_{X,p}\}$ as a subscheme; then, there is a natural isomorphism $\Gamma(X_f, \mathscr{O}_X) \cong \Gamma(X, \mathscr{O}_X)_f$.

Quasicoherent sheaves are locally determined by modules; what about coherent sheaves?

Proposition 8.7. *Let X be a Noetherian scheme. Then, an $\mathscr{O}_X$-module is a coherent sheaf iff it's locally isomorphic to $\widetilde{M}$ for $\Gamma(U, \mathscr{O}_X)$-modules M that are finite type.*

As locally free sheaves are akin to vector bundles, we can define line bundles.

Definition 8.8. A **line bundle** (or **invertible sheaf**[8]) is a locally free sheaf of rank 1.

By multiplicativity of dimension, if $\mathscr{E}$ and $\mathscr{F}$ are line bundles, so are $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{F}$ and $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F})$ are line bundles too.

Focally free sheaves form a monoid under tensor product, and so line bundles are exactly the invertible elements. That is, they form a group $\mathrm{Pic}(X)$, called the **Picard group** of X.

Remark. Later, when we know what cohomology is (there are many definitions: derived functors are the fancy version, but in many cases it relates to how one assembles cocycles), we will see an isomorphism $H^1(X, \mathscr{O}_X) = \mathrm{Pic}(X)$. This is because a line bundle on X is defined by trivial data on an open cover of X, along with the data of how to glue them together, and this is the same cocycle condition that defines a cohomology class in H^1. Thinking through this for vector bundles (e.g. take $X = S^1$) may be helpful.

One can define $\mathscr{O}_X^*$ to be the subsheaf of $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{O}_X)$ of isomorphisms (or the automorphism sheaf of $\mathscr{O}_X$), which corresponds to $\mathrm{GL}(1, \mathbb{R})$ in the differentiable category. The cocycle condition is that the transition functions have to be valued in $\mathscr{O}_X^*$, which is what Čech cohomology gives you.

Since $\mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^\times$, then it's homotopy equivalent to $\mathbb{F}_2 \cong \mathrm{O}_1$. Hence, if X is a compact CW complex, $H^1(X; \mathbb{F}_2)$ recovers isomorphism classes of line bundles on X, and since $\mathbb{C}^\times$ deformation retracts onto its unit circle $S^1 \cong \mathrm{U}_1$, then $H^1(X; S^1) \cong H^2(X; \mathbb{Z})$ classifies isomorphism classes of line bundles on X.[9]

[8] The name "invertible sheaf" comes from the fact that these are exactly the sheaves that are invertible under $\otimes_{\mathscr{O}_X}$. The idea is that $\mathscr{E}^\vee \otimes \mathscr{E} \cong \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E})$, and the latter sheaf has a global section given by the identity map, hence is trivial.

[9] Since $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$, then $H^2(X; \mathbb{Z}) = [X, \mathbb{CP}^\infty]$, and $\Omega\mathbb{CP}^\infty \simeq S^1$.

9. Coherent Sheaves: 6/9/16

Today's talk was given by Yan Zhou, on the second part of Chapter 13.

Recall that if X is a scheme, an $\mathscr{O}_X$-module $\mathscr{F}$ is quasicoherent if for all affine opens $\operatorname{Spec} A \subset X$, $\mathscr{F}|_{\operatorname{Spec} A} \cong \widetilde{M}$, where $\widetilde{M}$ is the $\mathscr{O}_X$-module associated to a $\Gamma(\operatorname{Spec} A, \mathscr{O}_X)$-module M.

There are a bunch of finiteness conditions one can put on quasicoherent sheaves (or the modules locally defining them).

- One could ask that all such M are finitely generated, i.e. there's a surjection $A^{\oplus n} \to M \to 0$.
- One could require M to be finitely presented, meaning there's an exact sequence
$$A^{\oplus n} \longrightarrow A^{\oplus m} \longrightarrow M \longrightarrow 0.$$
- A module M is **coherent** if it's finitely generated and if for every map $A^{\oplus n} \to M$, the kernel is finitely generated.

In particular, if A is Noetherian, the kernel K of such a surjection is a submodule of A^n, which is finitely generated, and therefore K must also be finitely generated. Hence, *if A is Noetherian, all three of these finiteness conditions are the same.*

Recall that a closed embedding defines a sheaf of ideals: if $i : Y \hookrightarrow X$ is a closed embedding (closed subscheme), it defines the sheaf of ideals $\mathscr{I}_{X/Y}$ that fits into the short exact sequence
$$0 \longrightarrow \mathscr{I}_{X/Y} \longrightarrow \mathscr{O}_X \longrightarrow i_*\mathscr{O}_Y \longrightarrow 0,$$
so the kernel of $\mathscr{O}_X \to i_*\mathscr{O}_Y$. The converse isn't necessarily true: a sheaf of ideals might not define a closed embedding. In this case, quasicoherence rescues us.

Proposition 9.1. *There is a bijection between the isomorphism classes of quasicoherent sheaves of ideals on a scheme X and the closed subschemes of X.*

That is, if $\mathscr{I}$ is a quasicoherent sheaf of ideals, then it determines a closed subscheme $Y = \operatorname{supp} \mathscr{I}$, with structure sheaf $\mathscr{O}_X/\mathscr{I}$. Affine-locally, we can explicate this: if $\operatorname{Spec} A \hookrightarrow X$ is an affine open, then $Y = \operatorname{Spec}(A/\mathscr{I}(\operatorname{Spec} A))$; quasicoherence is what guarantees that these glue together to define a scheme.

Now, we'll talk about a bunch of exercises. These all use Nakayama's lemma in the case of local rings, which is actually a different statement than Lemmas 6.13, 6.14, and 6.15 that we already discussed.

Lemma 9.2 (Nakayama's lemma (local rings))**.** *Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module. Then a basis of $M/\mathfrak{m}M$ lifts to a minimal set of generators for M.*

Before we discuss a geometric consequence of Nakayama's lemma, recall that if $\mathscr{F}$ is a sheaf on a scheme X, its fiber at a $p \in X$ is $\mathscr{F}|_p = \mathscr{F}_p \otimes_{\mathscr{O}_{X,p}} k(p)$, where $k(p)$ denotes the residue field at p.

Finite-rank vector bundles are analogous to locally free coherent sheaves, and so we should expect that for locally free coherent sheaves, the rank of the fiber, as a $k(p)$-vector space, should be locally constant. On coherent sheaves more generally, the rank may jump, but it will still be relatively well-behaved.

Lemma 9.3 (Geometric Nakayama's lemma, Vakil ex. 13.7.E)**.** *Let X be a scheme, $U \subset X$ be open, $p \in U$, and $\mathscr{F}$ be a finite-type quasicoherent sheaf on X. If $a_1, \dots, a_n \in \mathscr{F}(U)$ are such that their images $\overline{a}_1, \dots, \overline{a}_n \in \mathscr{F}_p$ form a basis for $\mathscr{F}_p/\mathfrak{m}_p\mathscr{F}_p$, then there exists an affine open $\operatorname{Spec} A \subseteq U$ containing p such that*

(1) $a_1|_{\operatorname{Spec} A}, \dots, a_n|_{\operatorname{Spec} A}$ generate $\mathscr{F}(\operatorname{Spec} A)$, and
(2) for any $q \in \operatorname{Spec} A$, $a_1|_q, \dots, a_n|_q$ generate $\mathscr{F}_q$.

The idea is that if the fiber is finitely generated, then it generates the sheaf nearby.

Proof. This proposition's name suggests that we should use Nakayama's lemma, version 9.2. This tells us that $\overline{a}_1, \dots, \overline{a}_n$ lift to a minimal set of generators for the stalk $\mathscr{F}_p$. Hence, since $\mathscr{F}$ is finite-type quasicoherent, there is some affine open $\operatorname{Spec} A' \subset U$ containing p such that $\mathscr{F}|_{\operatorname{Spec} A'} \cong \widetilde{M}$, where M is an A'-module.

Let $b_1, \dots, b_k$ be a set of generators for M. At p, each $b_i|_p \in (a_1|_p, \dots, a_n|_p)$, so for each i, there's an open neighborhood on which this is true as functions, not just as germs. Since there are finitely many, their intersection is still an open neighborhood of p, and hence contains a distinguished affine open $\operatorname{Spec} A = \operatorname{Spec} A'_f \subset \operatorname{Spec} A'$ containing p, and therefore $b_i|_{\operatorname{Spec} A} \in (a_1|_{\operatorname{Spec} A}, \dots, a_n|_{\operatorname{Spec} A})$ for each i; since $\mathscr{F}$ is quasicoherent, $b_1|_{\operatorname{Spec} A}, \dots, b_k|_{\operatorname{Spec} A}$ generate $\mathscr{F}(\operatorname{Spec} A)$. ⊠

Definition 9.4. If $\mathscr{F}$ is a finite-type quasicoherent sheaf on a scheme X, its **rank** at a $p \in X$ is the dimension of its fiber: $\varphi(p) = \dim_{k(p)}(\mathscr{F}_p \otimes_{\mathscr{O}_{X,p}} k(p))$.

The rank φ is upper semicontinuous, meaning at the set $\{p \mid \varphi(p) > n\}$ is closed, ultimately following from Lemma 9.3. In particular, if X is irreducible, we can look at the generic point η: if $\varphi(\eta) = n$, then the rank at any point in X is at least n, and there is a dense open set where the rank is exactly n.

Proposition 9.5 (Vakil ex. 13.7.F)**.** *Let $\mathscr{F}$ be a coherent sheaf on a scheme X, and suppose that for some $p \in X$, $\mathscr{F}_p$ is a free $\mathscr{O}_{X,p}$-module. Then, $\mathscr{F}$ is locally free on an open neighborhood of p.*

The takeaway is that being locally free is a stalk-local property: $\mathscr{F}$ is locally free iff for all $p \in X$, $\mathscr{F}_p$ is a free $\mathscr{O}_{X,p}$-module.

Proof. We once again use Lemma 9.3. We may assume $\mathscr{F}|_p \neq 0$, because if it is zero, then geometric Nakayama's lemma implies it's zero in a neighborhood, which is locally free, if silly.

If it's nonzero, there's an open neighborhood U of p and a finite set of sections $a_1, \ldots, a_n \in \mathscr{F}(U)$ such that $a_1|_p, \ldots, a_n|_p$ are a basis for $\mathscr{F}|_p$. Hence, there is an open neighborhood $Y \subset U$ such that for all $q \in Y$, $a_1|_q, \ldots, a_n|_q$ generate $\mathscr{F}|_q$. Since $\mathscr{F}$ is coherent, there is a surjection $\phi : (\mathscr{O}_X|_Y)^{\oplus n} \to \mathscr{F}|_Y \to 0$ that is an isomorphism at p (since $\mathscr{F}|_p$ is free), and $\ker(\phi)$ is coherent.

One can show that the support of a coherent sheaf is closed, and $p \notin \operatorname{supp}(\ker \varphi)$, as $\phi|_p$ is an isomorphism. Thus, $V = Y \setminus \operatorname{supp}(\ker \phi)$ is an open neighborhood of p on which ϕ is an isomorphism, so $\mathscr{F}|_V$ is a free $\mathscr{O}_X$-module. ⊠

We won't prove the next proposition, but Vakil pretty much walks you through it. It's also an exercise in Hartshorne, albeit with no hint.

Proposition 9.6. *Let X be a reduced scheme and $\mathscr{F}$ be a finite-type quasicoherent sheaf on X. If the rank of $\mathscr{F}$ is constant, then $\mathscr{F}$ is locally free.*

When proving this, you will once again use Lemma 9.3 to produce a proof that looks similar to the one for Proposition 9.5: you will concoct a surjection $(\mathscr{O}_X|_{\operatorname{Spec} A})^{\oplus n} \to \mathscr{F}|_{\operatorname{Spec} A} \to 0$ for some $\operatorname{Spec} A \subset X$, and then show that it's an isomorphism.

10. Line Bundles: 6/13/16

These are Arun's lecture notes on line bundles and divisors, corresponding to sections 14.1 and 14.2 in Vakil's notes. I'm planning on talking about the following topics:

- A few nice examples of line bundles on $\mathbb{P}^n$.
- Weil divisors and their relation to invertible sheaves.
- Using the class group to compute the Picard group, and if time, some actual examples.

Throughout this lecture, X will be a normal, reduced, Noetherian scheme that's regular in codimension 1. `#sorrynotsorry`

Line bundles on $\mathbb{P}^n$. The first part of this lecture will be an extended example, of nice classes of line bundles on projective spaces. Throughout this section, let A be a ring.

Example 10.1. First, we'll define a line bundle $\mathscr{O}(1) = \mathscr{O}_{\mathbb{P}^1_A}(1)$ on $\mathbb{P}^1_A = \operatorname{Proj} A[x_0, x_1]$. Recall that $\mathbb{P}^1_A$ is covered by two affine subsets $U_0 = D(x_0) = \operatorname{Spec} A[x_{1/0}]$ and $U_1 = D(x_1) = \operatorname{Spec} A[x_{0/1}]$; $\mathscr{O}(1)$ is trivial on those subsets, so it's completely specified by the two transition functions. Over U_0, a section of $\mathscr{O}(1)$ is an element of $A[x_{1/0}]$, and similarly for U_1.

We define $\mathscr{O}(1)$ to be the line bundle whose transition functions are: from U_0 to U_1, multiply by $x_{0/1} = x_{1/0}^{-1}$, and from U_1 to U_0, multiply by $x_{1/0} = x_{0/1}^{-1}$. These satisfy the cocycle condition, so we obtain a line bundle $\mathscr{O}(1)$.

Suppose $s \in \Gamma(\mathbb{P}^1_A, \mathscr{O}(1))$; then, s is the data of polynomials $f \in A[x_{1/0}]$ and $g \in A[x_{0/1}]$ such that $f(1/x_{0/1})x_{0/1} = g(x_{0/1})$. This forces f to be linear: $f(x_{1/0}) = ax_{1/0} + b$, and therefore $g(x_{0/1}) = a + bx_{0/1}$. Thus, $\dim \Gamma(\mathbb{P}^1_A, \mathscr{O}(1)) = 2$. Since $\dim \Gamma(\mathbb{P}^1_A, \mathscr{O}_{\mathbb{P}^1_A}) = 1$, then $\mathscr{O}(1)$ is a nontrivial line bundle: it's not isomorphic to the structure sheaf. Notice also that if we homogenize, $x_{1/0} = x_1/x_0$ and so $ax_{1/0} + b$ is naturally identified with $ax_1 + bx_0$. Thus, the global sections of $\mathscr{O}(1)$ are naturally identified with the degree-1 homogeneous polynomials in $A[x_0, x_1]$.

Example 10.2. In the same way, we can define $\mathscr{O}(n)$ on $\mathbb{P}^1_A$, where the transition functions are instead multiplication by $x_{0/1}^n = x_{1/0}^{-n}$ and vice versa. If $n \geq 0$, a section $s \in \Gamma(\mathbb{P}^1_A, \mathscr{O}(n))$ is identified with a degree-n polynomial in $x_{1/0}$ on U_0, or a homogeneous degree-n polynomial in $A[x_0, x_1]$. Thus, $\Gamma(\mathbb{P}^1_A, \mathscr{O}(n)) = n+1$. However, if $n < 0$, a global section would determine polynomials $f \in A[x_{1/0}]$ and $g \in A[x_{0/1}]$ such that $f(1/x_{0/1})x_{0/1}^n = g(x_{0/1})$, so we're forced to conclude $f, g = 0$. Thus, if $n < 0$, $\dim \Gamma(\mathbb{P}^1_A, \mathscr{O}(n)) = 0$.

Under this identification, the tensor product of line bundles turns into polynomial multiplication, so $\mathscr{O}(m) \otimes \mathscr{O}(n) = \mathscr{O}(m+n)$. Additionally, $\mathscr{O}(0)$ is the structure sheaf. This implies $\mathscr{O}(-n) = \mathscr{O}(n)^\vee$, since $\mathscr{O}(-n) \otimes \mathscr{O}(n) = \mathscr{O}(0) = \mathscr{O}_{\mathbb{P}^1_A}$. Hence, if $m \neq n$, $\mathscr{O}(m) \not\cong \mathscr{O}(n)$: if at least one of m or n is nonnegative, this is clear because their global sections have different dimensions, and if otherwise, then the global sections of $\mathscr{O}(m)^\vee$ and $\mathscr{O}(n)^\vee$ have different dimensions. That is, the map $n \mapsto \mathscr{O}(n)$ defines an injection $\mathbb{Z} \hookrightarrow \operatorname{Pic}(\mathbb{P}^1_A)$.

Example 10.3. In the same way, we can define $\mathscr{O}(n) = \mathscr{O}_{\mathbb{P}^m_A}(n)$ on $\mathbb{P}^m_A$. Here, we have $n+1$ affine opens $U_i = \operatorname{Spec} A[x_{0/i}, \dots, x_{m/i}]/(x_{i/i} - 1)$. We let $\mathscr{O}(n)$ be trivial on these affines, with the transition function from U_i to U_j being multiplication by $x_{i/j}^n = x_{j/i}^{-n}$. Thus, these also satisfy the cocycle condition, so define a line bundle over $\mathbb{P}^n_A$.

If $n \geq 0$, a global section restricts on an affine to a polynomial of degree at most n, and therefore after homogenizing, a global section is defined by a homogeneous, degree-n polynomial in $A[x_0, \dots, x_m]$, and vice versa. Hence, $\dim \Gamma(\mathbb{P}^m_A, \mathscr{O}(n)) = \binom{m+n}{m}$.

Once again, $\mathscr{O}(\ell) \otimes \mathscr{O}(n) = \mathscr{O}(\ell + n)$, so by the same line of reasoning as before, $n \mapsto \mathscr{O}(n)$ defines an injection $\mathbb{Z} \hookrightarrow \operatorname{Pic}(\mathbb{P}^m_A)$.

It turns out that over a field k, these are the only line bundles over $\mathbb{P}^n_k$. In order to prove this, we introduce the formalism of Weil divisors and their imperfect dictionary to line bundles.

Weil divisors.

Definition 10.4. The **group of Weil divisors** of a scheme X, denoted $\operatorname{Weil} X$, is the free abelian group on the set of codimension-1 irreducible closed subsets of X. Thus, a Weil divisor D is a formal linear combination

$$D = \sum_{Y \subset X \text{ codim. } 1} n_Y [Y], \tag{10.5}$$

where $n_Y \in \mathbb{Z}$ and all but finitely many n_Y are zero.

- If $Y \subset X$ is an irreducible closed subset, $[Y]$ is called an **irreducible** divisor.
- If D is as in (10.5) and $n_Y \geq 0$ for all Y, then D is called **effective**. We define a partial ordering on $\operatorname{Weil} X$ in which $D_1 \leq D_2$ iff $D_2 - D_1$ is effective.
- The **support** of a Weil divisor (10.5) is the set $\bigcup_{n_Y \neq 0} Y$.
- If $U \subset X$ is an open subset, we have a **restriction map** $\operatorname{Weil} X \to \operatorname{Weil} U$ by defining $[Y] \mapsto [Y \cap U]$ and extending $\mathbb{Z}$-linearly.

For example, if X is a curve, the Weil divisors are linear combinations of closed points.

Definition 10.6. Let $\mathscr{F}$ be a sheaf on X; then, a **rational section** s of $\mathscr{F}$ is a section of $\mathscr{F}|_U$, where U is an open, dense subset of X. Two rational sections are equal if they agree on a dense open subset. I'll write the space of rational sections of $\mathscr{F}$ over an open set V as $K(V, \mathscr{F})$.

In particular, on a variety over a field k, a rational function (i.e. to $\mathbb{A}^1_k$) is the same as a rational section of $\mathscr{O}_X$. This is analogous to the generalization from meromorphic functions to meromorphic 1-forms in the theory of Riemann surfaces.

Let $\mathscr{L}$ be a line bundle on X and s be a rational section of $\mathscr{L}$ that does not vanish on any irreducible component of X. If $Y \subset X$ is a codimension-1 irreducible component of X and η_Y is its generic point, then $\mathscr{O}_{X,\eta_Y}$ is a discrete valuation ring, and a trivialization determines an isomorphism $\mathscr{O}_{X,\eta_Y} \cong \mathscr{L}_{\eta_Y}$. Thus, $s|_Y$ has a valuation $\operatorname{val}_Y(s)$, which is independent of the choice of trivialization because any two trivializations will differ by an invertible germ. As such, s determines a Weil divisor

$$\operatorname{div}(s) = \sum_Y \operatorname{val}_Y(s)[Y],$$

called its **divisor of zeros and poles**. If $Q = \{(\mathscr{L}, s)\}/\cong$ denotes the set of isomorphism classes of line bundles and rational sections, Q is an abelian group under tensor product, and div is a group homomorphism $\operatorname{div} : Q \to \operatorname{Weil} X$. We're going to use this homomorphism to calculate the Picard group.

Lemma 10.7 (Vakil ex. 13.1.K)**.** *A rational section with no poles is regular.*

Proposition 10.8 (Vakil prop. 14.2.1)**.** div *is injective.*

Proof. Suppose $\operatorname{div}(\mathscr{L}, s) = 0$, so s has no poles by Lemma 10.7. Since $\mathscr{L}$ is an $\mathscr{O}_X$-module, acting on s defines a morphism $\times s : \mathscr{O}_X \to \mathscr{L}$. We'll show this is an isomorphism, so $(\mathscr{O}_X, 1) \cong (\mathscr{L}, s)$. In fact, it suffices to show $\times s$ is an isomorphism on an open cover $\mathfrak{U}$ of X that trivializes $\mathscr{L}$.

Let $U \in \mathfrak{U}$, so that there is an isomorphism $i : \mathscr{L}|_U \to \mathscr{O}_X|_U$, and let $s' = i(s)$. Then, multiplication by s' defines a map $\times s' = i \circ \times s : \mathscr{O}_X|_U \to \mathscr{O}_X|_U$. Since s' has neither zeros nor poles, it's a regular section and $1/s'$ is a regular section, so $\times s'$ is invertible, and hence an isomorphism; thus, s is also an isomorphism on U. ⊠

The next construction will be a kind of inverse.

Definition 10.9. If D is a Weil divisor, define a sheaf $\mathscr{O}_X(D)$ whose sections on an open $U \subset X$ are the rational functions t on U whose zeros and poles are constrained by D, i.e. $\operatorname{div}|_U t + D|_U \geq 0$, along with the zero section. If U is contained in an irreducible component of X, then

$$\Gamma(U, \mathscr{O}_X(D)) = \{t \in K(X)^\times : \operatorname{div}|_U t + D|_U \geq 0\} \cup \{0\}.$$

If $\mathscr{L}$ is a line bundle, define $\mathscr{L}(D) = \mathscr{L} \otimes \mathscr{O}_X(D)$.

$\mathscr{L}(D)$ can be interpreted as rational sections of $\mathscr{L}$ with zeros and poles constrained by D; by algebraic Hartogs' lemma, it's isomorphic to $\mathscr{L}$ away from $\operatorname{supp} D$.

Lemma 10.10 (Vakil ex. 14.2.C)**.** *$\mathscr{O}_X(D)$ and $\mathscr{L}(D)$ are quasicoherent sheaves.*

Using the distinguished affine criterion for quasicoherence, this follows because $\mathscr{O}_X$ and $\mathscr{L}$ are quasicoherent. In fact, in pleasant circumstances, we can do better than quasicoherence.

Proposition 10.11 (Vakil ex. 14.2.E.)**.** *Let $\mathscr{L}$ be an invertible sheaf and $s \in K(X, \mathscr{L})^\times$. Then, there is an isomorphism $\mathscr{O}_X(\operatorname{div} s) \cong \mathscr{L}$ such that if $\sigma : K(X) \to K(X, \mathscr{L})$ is the induced map on rational sections, $\sigma(1) = s$.*

Example 10.12 (Vakil ex. 14.2.F)**.** For example, when $X = \mathbb{P}^m_A$ and $\mathscr{L} = \mathscr{O}(n)$, then a degree-$n$ homogeneous polynomial in $A[x_0, \ldots, x_m]$ defines a rational section s of $\mathscr{L} = \mathscr{O}(n)$. Therefore $\mathscr{O}_{\mathbb{P}^n_A}(\operatorname{div} s) \cong \mathscr{O}(m)$ by Proposition 10.11.

Definition 10.13.

- If D is a divisor such that $D = \operatorname{div} f$ for a rational *function* f, D is called a **principal divisor**. The principal divisors form a subgroup $\operatorname{Prin} X \subseteq \operatorname{Weil} X$.
- If D is a divisor that restricts to principal divisors on an open cover of X, then D is called **locally principal**. Locally principal divisors form a subgroup $\operatorname{LocPrin} X \subseteq \operatorname{Weil} X$.
- The **class group** $\operatorname{Cl} X = \operatorname{Weil} X / \operatorname{Prin} X$.

Notice that if $D = \operatorname{div} f$ is principal, Proposition 10.11 tells us $\mathscr{O}_X(D) \cong \mathscr{O}_X$, since f is a rational function, i.e. a rational section of $\mathscr{O}_X$. Thus, if D is locally principal, $\mathscr{O}_X(D)$ is locally isomorphic to $\mathscr{O}_X$, and therefore is a line bundle.

Proposition 10.14 (Vakil ex. 14.2.G)**.** *The converse is true: if $\mathscr{O}_X(D)$ is invertible, then* $\operatorname{div}(\sigma(1)) = D$, *and D is locally principal.*

In particular, $\operatorname{LocPrin} X$ is the image of div, so we have a commutative diagram

$$\begin{array}{ccccc}
Q & \underset{\operatorname{div},\ \sim}{\overset{D \mapsto (\mathscr{O}_X(D), \sigma(1))}{\rightleftarrows}} & \operatorname{LocPrin} X & \hookrightarrow & \operatorname{Weil} X \\
\downarrow & & \downarrow {\scriptstyle / \operatorname{Prin} X} & & \downarrow {\scriptstyle / \operatorname{Prin} X} \\
\operatorname{Pic} X & \xleftarrow{D \mapsto \mathscr{O}_X(D)} & \operatorname{LocPrin} X / \operatorname{Prin} X & \hookrightarrow & \operatorname{Cl} X.
\end{array} \tag{10.15}$$

In particular, the map $D \mapsto \mathscr{O}_X(D)$ along the bottom left is surjective.

Proposition 10.16. *This map is an isomorphism.*

Computing Picard groups. By Proposition 4.10, if A is a UFD, all codimension-1 prime ideals are principal, so all Weil divisors on $\operatorname{Spec} A$ are principal. Thus, $\operatorname{Cl}\operatorname{Spec} A = 0$, and therefore $\operatorname{Pic}\operatorname{Spec} A = 0$. For example, $k[x_1, \dots, x_n]$ is a UFD, so $\operatorname{Pic}(\mathbb{A}^n_k) = 0$.

Geometrically, this makes sense: $\mathbb{A}^n$ is akin to the complex manifold $\mathbb{C}^n$, which is contractible, and so "shouldn't have nontrivial line bundles." It's also true that $\mathbb{A}^n_k$ has no nontrivial vector bundles, but this is the much harder Quillen-Suslin theorem.

Another tool for computing Picard groups is excising subsets of schemes. Removing a subset of codimension greater than 1 doesn't affect the class group (though it may affect the Picard group), and removing a subset of codimension 1 affects it in a controlled way.

If Z is an irreducible, codimension-1 subset of X, then the following is a short exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1 \mapsto [Z]} \operatorname{Weil} X \longrightarrow \operatorname{Weil}(X \setminus Z) \longrightarrow 0,$$

and when we quotient by $\operatorname{Prin} X$, we obtain the **excision exact sequence for class groups**, which is merely right exact:

$$\mathbb{Z} \longrightarrow \operatorname{Cl} X \longrightarrow \operatorname{Cl} X \setminus Z \longrightarrow 0. \tag{10.17}$$

For example, open subschemes of $\mathbb{A}^n$ have trivial class groups and therefore trivial Picard groups.

Example 10.18. Suppose $X = \mathbb{P}^n_k$; then, the hyperplane $Z = V(x_0)$ is a codimension-1 closed, irreducible subset, and therefore (10.17) specializes to

$$\mathbb{Z} \longrightarrow \operatorname{Cl}\mathbb{P}^n_k \longrightarrow \operatorname{Cl}\mathbb{A}^n_k = 0 \longrightarrow 0,$$

so $\mathbb{Z} \twoheadrightarrow \operatorname{Cl}\mathbb{P}^n_k$. However, the line bundles we saw at the beginning defined an injection $\mathbb{Z} \hookrightarrow \operatorname{Pic}\mathbb{P}^n_k \hookrightarrow \operatorname{Cl}\mathbb{P}^n_k$, so we're forced to conclude $\operatorname{Pic}\mathbb{P}^n_k \cong \mathbb{Z}$, generated by $\mathscr{O}(1)$. Using this, we can define the **degree** of a line bundle on $\mathbb{P}^n_k$ to be $\deg \mathscr{O}(d) = d$.

We can generalize this to understand factorial schemes more generally.

Proposition 10.19 (Vakil ex. 14.2.I)**.** *If X is factorial and $D \in \operatorname{Weil} X$, then $\mathscr{O}_X(D)$ is an invertible sheaf.*

Corollary 10.20 (Vakil prop. 14.2.10)**.** *Suppose X is factorial. Then, the map $\operatorname{Pic} X \hookrightarrow \operatorname{Cl} X$ is an isomorphism.*

We know for every Weil divisor D, $\mathscr{O}_X(D)$ is invertible, so the map $\operatorname{LocPrin} X \hookrightarrow \operatorname{Weil} X$ is an isomorphism, and this remains true when we quotient by $\operatorname{Prin} X$.

Example 10.21 (Vakil ex. 14.2.K)**.** Let $Y \subset \mathbb{P}^n_k$ be a hypersurface cut out by an irreducible degree-d polynomial $f \in k[x_0, \dots, x_n]$, and $X = \mathbb{P}^n_k \setminus Y$. Thus, the (irreducible) divisor $[Y] = \operatorname{div} f$, so by Example 10.12, $\mathscr{O}_{\mathbb{P}^n_k}([Y]) = \mathscr{O}(\deg f) = \mathscr{O}(d)$, which in the isomorphism $\mathbb{Z} \cong \operatorname{Cl}\mathbb{P}^n_k$ is identified with $d \in \mathbb{Z}$. Hence, (10.17) for X specializes to

$$\mathbb{Z} \xrightarrow{\cdot d} \mathbb{Z} \longrightarrow \operatorname{Cl} X \longrightarrow 0,$$

so $\operatorname{Cl}(X) \cong \mathbb{Z}/d$. Since X is factorial, then by Corollary 10.20, $\operatorname{Pic}(X) \cong \mathbb{Z}/d$.

11. Effective Cartier Divisors and Closed Subschemes: 6/16/16

Today Tom talked about a mishmash of topics broadly relating to closed subschemes and their relationship to ideal sheaves, allowing us to tie together lots of ideas we've seen before to things we've learned about quasicoherent sheaves. This allows us to understand when we can move quasicoherent sheaves along maps of schemes, and when we can move closed subschemes along maps of schemes. This in particular is useful in moduli problems: the **Hilbert scheme** of a given scheme is the moduli space of its closed subschemes, and so identifying closed subschemes with a nice class of sheaves allows us to reduce the problem to a sheafy question, which is how the Hilbert scheme (and many more general constructions) are made.

We'll start with some material from Chapter 8, which might be review.

Definition 11.1. A morphism $\iota : X \to Y$ of schemes is a **closed embedding** if it's affine and for all affine opens $\operatorname{Spec} B \subset Y$, so that $\iota^{-1}(\operatorname{Spec} B) \cong \operatorname{Spec} A$, then the induced ring map $B \to A$ is surjective.

In this case, $A \cong B/I$, and $(\operatorname{Spec} B) \cap \operatorname{Im}(\iota) \cong V(I)$.

Definition 11.2 (Vakil ex. 8.1.D)**.** The condition of being a closed embedding is affine-local.

This is not hard to prove.

Proposition 11.3. *If $\iota : X \to Y$ is a closed embedding, then the induced morphism of schemes $\iota^\sharp : \mathscr{O}_Y \to \iota_* \mathscr{O}_X$ is surjective.*

This is fiddly to prove, since surjectivity can't always be checked on global sections; in particular, $(\iota_*\mathscr{O}_X)_p = 0$ for all p not in the image. In any case, we can take the kernel $\mathscr{I}_{X/Y} = \ker(\iota^\sharp)$, which fits into the exact sequence

$$0 \longrightarrow \mathscr{I}_{X/Y} \longrightarrow \mathscr{O}_Y \longrightarrow \iota_*\mathscr{O}_X \longrightarrow 0.$$

Over any open set $U \subset Y$, $\mathscr{I}_{X/Y}(U)$ is an ideal of $\mathscr{O}_Y(U)$, and therefore $\mathscr{I}_{X/Y}$ is called a **sheaf of ideals**. More than just any sheaf of ideals, it also has the following property.

Proposition 11.4 (Vakil ex. 8.1.G)**.** *Let $\operatorname{Spec} B \subset Y$ be an affine open and $f \in B$. In the diagram*

$$\begin{array}{ccc} \mathscr{I}_{X/Y}(\operatorname{Spec} B) & \longrightarrow & \mathscr{I}_{X/Y}(\operatorname{Spec} B_f) \\ & \searrow \qquad \nearrow_{\alpha} & \\ & \mathscr{I}_{X/Y}(\operatorname{Spec} B)_f, & \end{array}$$

the map α (induced by property of localization) is an isomorphism.

This is actually a sufficient condition, though this is harder to prove.

Proposition 11.5 (Vakil prop. 8.1.H)**.** *Given the data of an ideal $I(B) \subset B$ for all affine opens $\operatorname{Spec} B \subset Y$ such that for all $f \in B$, $I(B_f) = I(B)_f$, then there exists a unique closed subscheme $\iota : X \hookrightarrow Y$ such that $\mathscr{I}_{X/Y}(\operatorname{Spec} B) = I(B)$ for all affine opens $\operatorname{Spec} B \subset Y$.*

This has a curious consequence. We know that over an affine scheme, a sheaf is uniquely determined by its behavior on distinguished opens, and all such opens are determined by localizing at various global sections. Thus, a sheaf of ideals on an affine scheme is uniquely determined by its global sections, and on a general scheme, a sheaf of ideals can be determined from an affine cover.

Rephrasing Proposition 11.5, it says there is a bijection between the closed subschemes of Y and the sheaves of ideals with the localization property.

This also allows us to define intersections and unions of closed subschemes as schemes by using the sheaf of ideals. *A priori*, one needs to do a lot of gluing to ensure things work out, but since local data of the ideal sheaf given by a closed subscheme is determined by global data, we actually have a lot less work to do.

Definition 11.6. Let $X, Z \hookrightarrow Y$ be closed subschemes.

- Their **scheme-theoretic intersection** $X \cap Z$ is the closed subscheme corresponding to the ideal sheaf $\mathscr{I}_{X/Y} + \mathscr{I}_{Z/Y}$.
- Their **scheme-theoretic union** $X \cup Z$ is the closed subscheme corresponding to the ideal sheaf $\mathscr{I}_{X/Y} \cap \mathscr{I}_{Z/Y}$.[10]

One must check that these ideal sheaves have the localization property, but this is quick to check, and automatically ensures that on every affine open, we have the usual intersection and union of closed subsets.

The reason we're talking about this now is that quasicoherence allows us to clarify this relationship. Recall that one of our criteria for quasicoherence looks quite similar to what we've been talking about today.

Proposition 11.7 (Vakil ex. 13.3.D)**.** *Let X be a scheme and $\mathscr{F}$ be a sheaf on X. Then, $\mathscr{F}$ is quasicoherent iff for all open affines $\operatorname{Spec} A \subset X$ and $f \in A$, the map $\varphi : \Gamma(\operatorname{Spec} A, \mathscr{F})_f \to \Gamma(\operatorname{Spec}(A_f), \mathscr{F})$ induced by*

[10] We use the intersection rather than the product so that $X = X \cup X$: otherwise, inside $k[x]$, $V(x) \cup V(x)$ would be $V(x^2)$, which is nonreduced. This is interesting and sometimes useful, but not what we're looking for today.

localization in the diagram

$$\begin{array}{ccc} \Gamma(\operatorname{Spec} A, \mathscr{F}) & \longrightarrow & \Gamma(\operatorname{Spec} A_f, \mathscr{F}) \\ & \searrow \qquad \nearrow_{\varphi} & \\ & \Gamma(\operatorname{Spec} A)_f, & \end{array}$$

and this isomorphism is canonical.

The canonicity of this isomorphism is important for gluing things together.

The punchline is that *there is a bijection between closed subschemes and quasicoherent sheaves of ideals.* This is exciting and useful because we can do things with quasicoherent sheaves.

If X is in addition quasicompact and quasiseparated, we can upgrade Proposition 11.7 by just checking for global sections! The key is, of course, that in this case X is a finite union of affine opens, such that all intersections of these affine opens are themselves affine. This is necessary because localization commutes with finite products, but not infinite ones — an example of the category-theoretic fact that limits commute with limits, and colimits commute with colimits, but colimits only commute with finite limits (and vice versa).

Effective Cartier divisors. Within the bijection between closed subschemes and quasicoherent sheaves of ideals, one can ask whether particularly nice kinds of closed subschemes correspond to particularly nice sheaves of ideals. One such class is effective Cartier divisors.

Definition 11.8. A closed embedding $i : X \hookrightarrow Y$ is **locally principal** if there is an affine open cover $\mathfrak{U}$ of X such that on each $U_i = \operatorname{Spec}(A_i) \in \mathfrak{U}$, X is "cut out by a single equation," i.e. $\pi^{-1}(U_i) \cong \operatorname{Spec}(A_i/(a_i))$ for some $a_i \in A_i$.

That is, locally X is cut out by a principal ideal. Note that this is *not* an affine-local condition: local principality with respect to one open cover does not imply local principality for all affine covers, as principality depends on some sort of choice.

Definition 11.9. With notation as in Definition 11.8, $X \hookrightarrow Y$ is an **effective Cartier divisor** if the a_i are not zero divisors.

Proposition 11.10 (Vakil ex. 8.4.A)**.** *If $t \in A$ is not a zero divisor, then it's not a zero divisor in $A_\mathfrak{p}$ for all primes $\mathfrak{p} \subset A$.*

The converse is *not* true: this is not stalk-local.

Effective Cartier divisors are a nice kind of closed subscheme, so they will correspond to a particularly nice kind of sheaf of ideals. Locally, the sheaf of ideals gives us an exact sequence of A-modules

$$0 \longrightarrow (a) \longrightarrow A \longrightarrow A/(a) \longrightarrow 0;$$

since a isn't a zero divisor, $A \to (a)$ is injective, and since A has 1, it's also surjective. Thus, this short exact sequence is really the one induced by multiplication by a:

$$0 \longrightarrow A \xrightarrow{\cdot a} A \longrightarrow A/(a) \longrightarrow 0.$$

Thus, $\mathscr{I}_{X/Y}(\operatorname{Spec} A_i) \cong A_i$ for all $\operatorname{Spec} A_i \in \mathfrak{U}$; since in addition we know $\mathscr{I}_{X/Y}$ is quasicoherent, then this says $\mathscr{I}_{X/Y}$ is an invertible sheaf: $\mathscr{I}_{X/Y}(\operatorname{Spec} A_i) \cong \mathscr{O}_{\operatorname{Spec} A_i}$.

So the sought-for correspondence would be between effective Cartier divisors and invertible sheaves of ideals (invertible implies quasicoherent); we hope to make this a bijection. If $D \hookrightarrow X$ is an effective Cartier divisor, let $\mathscr{O}_X(D) = \mathscr{I}_{X/Y}^\vee$, or $\mathscr{I}_{D/X} = \mathscr{O}_X(-D)$.[11]

Proposition 11.11 (Vakil ex. 14.3.C)**.** *Let $\mathscr{L}$ be an invertible sheaf on X and $s \in \mathscr{L}(X)$ be a section that is "locally not a zero divisor" (i.e. there's an affine cover of X on which $\mathscr{L}$ is trivial and s restricts to non-zerodivisors[12]). Then, $D = V(s)$ is an effective Cartier divisor and $\mathscr{O}_X(D) = \mathscr{L}$.*

[11] There's a very different way to define Cartier divisors and effective Cartier divisors in terms of a sheaf of total fractions; this streamlines the connection and nomenclature between Weil and Cartier divisors. Unlike Weil divisors, though, Cartier divisors can be defined on any scheme, and on suitably nice schemes, there's an isomorphism between the groups of Cartier divisors and Weil divisors.

[12] Asking for s to not be a zerodivisor stalk-locally is not equivalent!

Partial proof. Well, once you know what it means for s to locally not be a zero divisor, let $\mathfrak{U}$ be a trivializing affine open cover for $\mathscr{L}$ such that $s|_{U_i}$ is not a zero divisor on $U_i = \operatorname{Spec} A_i$. Then, on U_i, $D \cong \operatorname{Spec}(A_i/(a_i))$, so D is an effective Cartier divisor. ⊠

This is almost all of the data of the bijection we wanted. One important point is why an invertible sheaf is the same as an invertible sheaf with a section, and the ideal is that we have a map $\mathscr{O}_X \to \mathscr{O}_X(D)$, and the image of 1 defines a canonical section s_D for us, and given a section s_D for $\mathscr{L}$, one can define a map $\mathscr{O}_X \to \mathscr{L}$ by sending $1 \mapsto s_D$; since $\mathscr{L}$ is quasicoherent, this data, which is about global sections, localizes to define a map on every distinguished affine open.

12. Quasicoherent Sheaves on Projective Schemes: 6/20/16

Today, Danny spoke about the first part of chapter 15, which coveres quasicoherent sheaves on projective A-schemes:

- §1 is about how quasicoherent sheaves come from graded modules.
- §2 is about invertible and twisted quasicoherent sheaves.
- §3 is the Hilbert syzygy theorem.
- §4 is about globally generated sheaves, etc.

The first part of this chapter is a lot of tiny details and short exercises; hopefully we can tackle Hilbert's syzygy theorem, which is a bigger trophy.

Quasicoherent sheaves and graded modules. Recall that a quasicoherent sheaf is one that's locally obtained from a module over every affine open, so a module M over a ring A determines a quasicoherent sheaf $\widetilde{M}$ on $\operatorname{Spec} A$. Correspondingly, if we're handed a graded module over a graded ring $S_\bullet$, we would like to turn it into a quasicoherent sheaf over $\operatorname{Proj} S$.

In today's lecture, all graded rings will be $\mathbb{Z}_{\geq 0}$-graded, i.e. there will be no nontrivial terms in negative degree. Given such a graded ring $S_\bullet$, $\operatorname{Proj} S_\bullet$ is a projective A-scheme, where $A = S_0$; the points of $\operatorname{Proj} S_\bullet$ are in bijection with the homogeneous prime ideals in $S_\bullet$ that don't contain the irrelevant ideal S_+ (consisting of all positively graded elements). We will often assume S_+ is finitely generated, and sometimes assume it's generated in degree 1 — when this arises, it is an extremely important condition.

The distinguished opens of $\operatorname{Proj} S_\bullet$ are as follows: if $f \in S_\bullet$ is a positive-degree homogeneous polynomial, $D(f) \cong \operatorname{Spec}((S_\bullet)_f)_0$: we localize $S_\bullet$ at f, and then take the terms of degree 0. For the sake of notation, we will let $S_{(f)}$ denote $((S_\bullet)_f)_0$, and we will do the same thing for graded modules.

One's instinct is, given a graded module $M_\bullet$ over the graded ring $S_\bullet$, that we should get a quasicoherent sheaf $\widetilde{M_\bullet}$ by making the construction $\widetilde{M_\bullet}(D(f)) = M_{(f)}$. That is, over a distinguished open, which is affine, we take the usual sheaf-from-module construction.

Proposition 12.1. *This $\widetilde{M_\bullet}$ defines a quasicoherent sheaf on* $\operatorname{Proj} S_\bullet$.

Proof. Recall that way back in chapter 2, Vakil discusses how to obtain a sheaf from a sheaf on the collection of distinguished affine opens; however, it might not have the same sections over a distinguished open, unless it satisfies the cocycle condition, so let's check that.

Let $D(f_1)$, $D(f_2)$, and $D(f_3)$ be three distinguished opens in $\operatorname{Proj} S_\bullet$. Restriction defines a map $M_{(f_1)} \to M_{(f_1 f_2)}$, which is *not* localization. This means that for the cocycle condition, one has to check that the maps $\varphi_{ij} : M_i|_{U_i \cap U_j} \to M_i|_{U_i \cap U_j}$ are the identity, which is in fact the case. ⊠

Next question: what are the stalks of this sheaf?

Proposition 12.2. *If $p \in \operatorname{Proj} S_\bullet$ is the point corresponding to the homogeneous prime ideal $\mathfrak{p} \subset S_\bullet$, then* $(\widetilde{M_\bullet})_p \cong M_{(\mathfrak{p})} = ((M_\bullet)_\mathfrak{p})_0$.

This will be helpful to have around.

$M_\bullet \mapsto \widetilde{M_\bullet}$ defines a functor $\sim$: $\mathsf{GrMod}_{S_\bullet} \to \mathsf{QC}(\operatorname{Proj} S_\bullet)$. This is functorial because if $f : M_\bullet \to N_\bullet$ is a morphism, it determines maps $M_{(f)} \to N_{(f)}$ for each $f \in S_\bullet$, and the diagram

$$\begin{array}{ccc} M_{(f)} & \longrightarrow & N_{(f)} \\ \downarrow & & \downarrow \\ M_{(fg)} & \longrightarrow & N_{(fg)} \end{array}$$

commutes, as needed.

Proposition 12.3. $\sim$ *is an exact functor.*

Proof. On stalks, $\sim$ is just localization, which is exact; then, exactness of a sequence of sheaves may be checked on stalks. ⊠

However, $\sim$ is not an equivalence of categories: for example, this construction only depends on the "asymptotic data" of $M_\bullet$. What this means precisely is that if $M_\bullet$ and $N_\bullet$ agree for all but finitely mnay terms, then $\widetilde{M_\bullet} \cong \widetilde{N_\bullet}$; this is because localizing involves taking products, which raises the degree arbitrarily as one takes the limit to obtain stalks (which determine isomorphism data). This allows one to write down examples of $M_\bullet$ and $N_\bullet$ that aren't isomorphic, but such that their associated sheaves are, and this is means we don't have an equivalence of categories.

Another useful fact is that we can define a map from M_0 to global sections. If $x \in M_0$, then over the distinguished open $D(f)$, we let $x \mapsto x/1 \in D(f)$; these agree on overlaps (they're the same function, after all), so these stitch together into a canonical map $M_0 \to \Gamma(\operatorname{Proj} S_\bullet, \widetilde{M_\bullet})$. In general, though, this is not an isomorphism, e.g. $S_\bullet = k[x]$, and $M_\bullet = k[x]/(x^2)$. Then, $M_0 = k$, but $\Gamma(\mathbb{P}^0_k, \widetilde{M_\bullet}) = k[x]/(x^2)$.

Invertible sheaves on projective A-schemes. Let's return to the world of graded modules that may have negative degree. These admit degree shifts.

Definition 12.4. Let $S_\bullet$ be a graded ring and $M_\bullet$ be a graded $S_\bullet$-module. Then, for any $n \in \mathbb{Z}$, $M(n)_\bullet$ (sometimes also written $M_\bullet[n]$), the **degree shift** by n of $M_\bullet$, is the graded module whose m^{th} homogeneous terms are $M(n)_m = M_{m+n}$.

That is, we've literally shifted the grading by n. We can take $\widetilde{M(n)_\bullet}$; its distinguished sections are

$$\Gamma(D(f), \widetilde{M(n)_\bullet}) = ((M_\bullet)_f)_n,$$

which we'll call $M(n)_{(f)}$.

Example 12.5. For $S_\bullet = A[x_0, \dots, x_n]$, so that $\operatorname{Proj} S_\bullet = \mathbb{P}^n_A$, $\widetilde{S(n)_\bullet} = \mathscr{O}(n)$ from Example 10.2.

This motivates the following definition.

Definition 12.6. Let $X = \operatorname{Proj} S_\bullet$ be a projective scheme. Then, we let $\mathscr{O}_X(n) = \widetilde{S(n)_\bullet}$.

Note that in general, this is not locally isomorphic to the structure sheaf $\widetilde{S_\bullet}$. There is a canonical $\mathscr{O}_X$-linear map $\mathscr{O}_X(n) \otimes_{\mathscr{O}_X} \mathscr{O}_X(m) \to \mathscr{O}_X(m+n)$, but this is in general not an isomorphism; over the distinguished open $D(f)$, it is the map $S(n)_{(f)} \otimes S(m)_{(f)} \to S(m+n)_{(f)}$ sending $(x/f^n) \otimes (y/f^m) \mapsto xy/f^{m+n}$.[13]

We can also use this to twist sheaves.

Definition 12.7. Let $\mathscr{F}$ be a quasicoherent sheaf on $X = \operatorname{Proj} S_\bullet$; then, define $\mathscr{F}(n) = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n)$.

Proposition 12.8. *If $S_\bullet$ is generated in degree 1, then $\mathscr{O}_X(n)$ is invertible.*

Proof. Let $D(f)$ be a distinguished open of X and $d = \deg f$. Then, for any $n \in \mathbb{Z}$, multiplication by f^n defines a map $\mathscr{O}_X(D(f)) \to \mathscr{O}(nd)(D(f))$ (sending $x \mapsto f^n x$), and $y \mapsto y/f^n$ defines an inverse map. Thus, $\mathscr{O}(n) \cong \mathscr{O}(nd)$ over $D(f)$. In particular, if $\deg f > 1$, then over $D(f)$, $\mathscr{O}_X \not\cong \mathscr{O}_X(1)$, so if $S_\bullet$ is generated in degree 1 by $\{f_i\}_{i \in I}$, where each f_i is a homogeneous, degree-1 polynomial, $\mathfrak{U} = \{D(f_i)\}_{i \in I}$ is a cover of X by distinguished affine opens such that on each $D(f_i) \in \mathfrak{U}$, $\mathscr{O}_X \cong \mathscr{O}_X(n)$ for all $n \in \mathbb{Z}$; thus, $\mathscr{O}_X(n)$ is invertible. ⊠

[13]This arises from a more general canonical map $\widetilde{M_\bullet} \otimes \widetilde{N_\bullet} \to \sim(M_\bullet \otimes N_\bullet)$, which is not in general an isomorphism, and can be quite badly behaved.

One can strengthen this: it's necessarily true that for some d, the terms of $S_\bullet$ of degrees at least d are generated by those of degree d; in this case, the same argument shows that $\mathscr{O}_X(nd)$ is invertible for each n, but in general $\mathscr{O}_X \not\cong \mathscr{O}_X(1)$.

The Hilbert syzygy theorem. Let $S_\bullet = k[x_0, \dots, x_n]$, where k is a field. Then, if $M_\bullet$ is a graded module, one might want to study the sizes of different-degree pieces of the module, i.e. the **Hilbert function** $H_M(d) = \operatorname{rank} M_d$. This has a quite geometric interpretation: if $M_\bullet = S_\bullet/I_\bullet$ encodes the information of a projective subvariety, then H_M encodes lots of important information about this subvariety, including asymptotic information about its dimension.

For example, if $M_\bullet = S_\bullet$, then $H_S(d) = \binom{n+d}{d}$: the free case is the nicest case. If the module isn't free, then we can try to construct a free resolution; in fact, we want it to be a graded resolution, where each map is degree zero.

For niceness, assume M is finitely generated, in particular generated by finitely many homogeneous elements $m_1 \in M_{a_1}, \dots, m_k \in M_{a_k}$. Thus we obtain a surjective map

$$\underbrace{\bigoplus_{i=1}^{k} S(-a_i)}_{F_0} \xrightarrow{g_0} M_\bullet \longrightarrow 0$$

sending $(0, \dots, 0, 1, 0, \dots, 0)$ (with the 1 in the j^{th} entry) to m_j. Then, $\ker(g_0)$ is another finitely generated graded $S_\bullet$-module, so we can make the same construction to obtain a surjection $F_1 \to \ker(g_0)$ from a free graded module with finitely generated kernel. Iterating this, we obtain a resolution for $M_\bullet$

$$\cdots \longrightarrow F_m \xrightarrow{g_m} F_{m-1} \xrightarrow{g_{m-1}} \cdots \longrightarrow F_0 \xrightarrow{g_0} M \longrightarrow 0, \tag{12.9}$$

where each term is a graded free module of finite rank, and each map preserves the grading.

The point of the Hilbert syzygy theorem is that in nice conditions, this will be finite. Recall that a finite long exact sequence has a nice formula for its Euler characteristic:

$$\sum (-1)^k \dim V_k = 0.$$

This will allow us to determine the Hilbert function in terms of the really nice Hilbert functions of f.g. free graded modules.

Theorem 12.10 (Hilbert syzygy theorem)**.** *If $S_\bullet = k[x_0, \dots, x_n]$ and $M_\bullet$ is a finitely generated graded $S_\bullet$-module, then there exists a graded free resolution of finite rank, where the morphisms preserve degree, and of length at most $n+1$.*

It might not be exactly (12.9), but the point is there's one that looks a lot like it.

The idea is to first show this for $M_\bullet = k$, by forming the **Koszul complex** of graded $S_\bullet$-modules

$$0 \longrightarrow \mathcal{K}_n \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{K}_0 \xrightarrow{d} k \longrightarrow 0, \tag{12.11}$$

where $\mathcal{K}_p$ has basis $\{e_{i_1,\dots,i_p} \mid 0 \le i_j \le n\}$ and the differential is determined by

$$d(e_{i_1,\dots,i_p}) = \sum_{j=1}^{p} (-1)^i x_j e_{i_1,\dots,\widehat{\imath}_j,\dots,i_p}.$$

Here, $\widehat{\imath}_j$ denotes the absence of the index i_j (all the other indices are intact). The last map sends $p(x)e_0 \mapsto p(0)$. To make this graded, the degree of $e_{i_1,\dots,i_p}$ has to be set correctly.

The next step of the proof is to take a minimal nice resolution of $M_\bullet$, i.e. one with the smallest number of generators that has the properties we've been looking for. Minimality has the great property that all the maps save $F_0 \twoheadrightarrow M$ will become 0 after applying $- \otimes_{S_\bullet} k$. This is because any basis element e of a term F_j in a minimal resolution is mapped to a combination of terms with strictly positive degree; otherwise, there's a relation amongst the generators of F_{j-1}, so $F_\bullet$ isn't minimal.

Since this destroys all the maps after applying $- \otimes_{S_\bullet} k$, it makes it really easy to compute Tor: if $i \ge 1$, $\operatorname{Tor}_i^{S_\bullet}(M_\bullet, k) = H^i(F_\bullet) = F_i \otimes_{S_\bullet} k$. But since Tor is symmetric, this is also $\operatorname{Tor}_i^{S_\bullet}(k, M_\bullet)$, which is obtained from a projective resolution of k, e.g. the Koszul complex (12.11) by tensoring with M, and so this has to be 0 for $i > n+1$.

Here's another reason to care.

Theorem 12.12. *Let $S_\bullet$ be a $\mathbb{Z}$-graded ring finitely generated in degree 1 and $\mathscr{F}$ be a quasicoherent sheaf on* $\operatorname{Proj} S_\bullet$*. Then, there's a surjection*

$$\bigoplus_{i=1}^N \mathscr{O}_X(-n_i) \longrightarrow \mathscr{F} \longrightarrow 0.$$

In particular, if $\mathscr{F}$ is coherent and $S_\bullet$ is a finitely generated k-algebra, then $\Gamma(\operatorname{Proj} S_\bullet, \mathscr{F})$ is finite-dimensional.

13. Finite Type Quasicoherent Sheaves on Projective Schemes: 6/23/16

Today's talk was given by Richard.

The goal of §15.3 is to prove the following theorem.

Theorem 13.1. *If $S_\bullet$ is a graded ring and $\mathscr{F}$ is a finite type quasicoherent sheaf on $X = \operatorname{Proj} S_\bullet$, then $\mathscr{F}$ admits a presentation of the form*

$$\bigoplus_{i=1}^N \mathscr{O}_X(n_i) \longrightarrow \mathscr{F} \longrightarrow 0.$$

This will be proven by invoking a theorem of Serre that's proven in the next chapter. It's also an excuse to make a few useful definitions.

Definition 13.2. Let X be a scheme and $\mathscr{F}$ be an $\mathscr{O}_X$-module.

- $\mathscr{F}$ is **globally generated** if it admits a surjection from a direct sum of free sheaves $\mathscr{O}_X^{\oplus I} \twoheadrightarrow \mathscr{F}$.
- If $\mathscr{F}$ admits a surjection from a finite direct sum of free sheaves, it's **finitely globally generated**.
- If $p \in X$, $\mathscr{F}$ is **globally generated at** p if there's a surjection $\mathscr{O}_{X,p}^{\oplus I} \twoheadrightarrow \mathscr{F}_p$.

Since surjectivity on stalks is equivalent to surjectivity as a map of sheaves, $\mathscr{F}$ is globally generated iff it's globally generated at all points.

Example 13.3. On an affine scheme, quasicoherent sheaves are globally generated; finite type sheaves are finitely globally generated.

Proposition 13.4 (Vakil ex. 15.3.B)**.** *If $\mathscr{F}$ and $\mathscr{G}$ are globally generated, then so is $\mathscr{F} \otimes \mathscr{G}$.*

Proof. Let $p \in X$ be arbitrary, so that there exist surjections $\mathscr{O}_{X,p}^{\oplus I} \twoheadrightarrow \mathscr{F}_p$ and $\mathscr{O}_{X,p}^{\oplus J} \twoheadrightarrow \mathscr{G}_p$. Since the tensor product is right exact, then we obtain a surjective map $\mathscr{O}_{X,p}^{\oplus IJ} \twoheadrightarrow \mathscr{F}_p \otimes \mathscr{G}_p$. ⊠

Proposition 13.5 (Vakil ex. 15.3.E)**.** *Let $\mathscr{L}$ be an invertible sheaf. Then, $\mathscr{L}$ is globally generated iff for every $p \in X$, there's a section s_p of $\mathscr{L}$ that doesn't vanish at p.*

Proof. In the reverse direction, we can define the map $\mathscr{O}_{X,p} \twoheadrightarrow \mathscr{L}_p$ by sending the generator to s_p, which is surjective, so $\mathscr{L}$ is globally generated at every point.

In the forward direction, the surjection $\mathscr{O}_{X,p}^{\oplus I} \twoheadrightarrow \mathscr{L}_p$ defines a section that maps to the generator, which necessarily doesn't vanish at p. ⊠

Definition 13.6. Let $\mathscr{L}$ be an invertible sheaf on a scheme X.

- The **base points of** $\mathscr{L}$ are the points of X where all global sections vanish. This defines a subscheme of X, called the **base point locus**.
- The complement of the base point locus is the **base point-free locus**.

Propositions 13.4 and 13.5 tell us that $\mathscr{L}$ is generated by global sections iff it's base point-free, and that the tensor product of base point-free sheaves is base point-free.

Peeking ahead to chapter 16, we'll be able to use this to characterize maps of $X \to \mathbb{P}^n$.

Theorem 13.7 (Vakil thm. 16.4.1)**.** *Given a scheme X, there is a bijection*

$$\operatorname{Hom}_{\mathsf{Sch}}(X, \mathbb{P}^n_{\mathbb{Z}}) \longleftrightarrow \{(\mathscr{L}, s_0, \dots, s_n) \mid \mathscr{L} \in \operatorname{Pic}(X), s_i \in \Gamma(X, \mathscr{L}) \text{have no common zeros}\}/\text{isomorphism}.$$

There is also a version over k, using $\mathbb{P}^n_k$.

This should be thought of as an analogue of the following proposition.

Proposition 13.8 (Vakil ex. 6.3.N)**.** *Let A be a ring and X be an A-scheme. Then,* $\mathrm{Hom}_{\mathsf{Sch}/B}(X, \mathbb{P}^n_B)$ *is in bijection with the set of n-tuples of functions on X that have no common zeros.*

Proof of Theorem 13.7. First, suppose we have the data $(\mathscr{L}, s_0, \dots, s_n)$, with $\mathscr{L}$ a line bundle and $s_0, \dots, s_n$ global sections. Let $\mathfrak{U}$ be a trivializing open cover for $\mathscr{L}$; for each $U_j \in \mathfrak{U}$, we'll define a map $U_j \to \mathbb{P}^n_{\mathbb{Z}}$, then stitch all of these maps together.

Let $g_{i,j} = s_i|_{U_j}$; then, let $\phi_j : U_j \to \mathbb{P}^n$ send $p \mapsto [g_{0,j}(p) : \dots : g_{n,j}(p)]$. These ϕ_j agree with the transition functions: if f_{ij} is the transition function for $\mathscr{L}$ from $U_i \to U_j$, then $g_{k,i} = f_{ij} g_{k,j}$ for $1 \le k \le n$, so $\phi_i = \phi_j$ on $U_i \cap U_j$, meaning these define a map $X \to \mathbb{P}^n_{\mathbb{Z}}$.

In the other direction, given a map $\pi : X \to \mathbb{P} - \mathbb{Z}^n$, $\mathscr{L}$ will be the pullback of the anticanonical bundle $\mathscr{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1)$, with $s_0, \dots, s_n$ the pullback of a basis of its global sections. ⊠

To prove Theorem 13.1, we'll need to invoke a theorem of Serre (which definitely means that it's hard).

Theorem 13.9 (Serre's theorem A)**.** *Let $S_\bullet$ be a graded ring generated in degree 1 and finitely generated over S_0, and let $\mathscr{F}$ be a finite type quasicoherent sheaf on $\mathrm{Proj}\, S_\bullet$. Then, there exists an n_0 such that for all $n \ge n_0$, $\mathscr{F}(n) = \mathscr{F} \otimes_{\mathscr{O}_{\mathrm{Proj}\, S_\bullet}} \mathscr{O}_{\mathrm{Proj}\, S_\bullet}$ is finitely globally generated.*

Using this, the proof of Theorem 13.1 isn't particularly hard: we can twist and untwist.

Proof of Theorem 13.1. Theorem 13.9 tells us there's an n such that $\mathscr{F}(n)$ is finitely globally generated, so there exist m global sections $s_1, \dots, s_m$ of $\mathscr{F}(n)$ such that the map

$$(f_1, \dots, f_m) \longmapsto (s_1 f_1 + \dots + s_m f_m)$$

defines a surjection $\mathscr{O}_X^{\oplus m} \twoheadrightarrow \mathscr{F}(n)$. Since $\mathscr{O}_X(-n)$ is locally free and tensor product is right exact, applying $- \otimes_{\mathscr{O}_X} \mathscr{O}_X(-n)$ turns this into a surjection

$$\bigoplus_{i=1}^{m} \mathscr{O}_X(-n) \twoheadrightarrow \mathscr{F}(n) \otimes_{\mathscr{O}_X} \mathscr{O}_X(-n) = \mathscr{F}.$$ ⊠

Let $S_\bullet$ be a finitely generated graded algebra generated in degree 1, so that $\mathscr{O}_X(n)$ is invertible for all n, and let $X = \mathrm{Proj}\, S_\bullet$. Obtaining a quasicoherent sheaf from a graded module is functorial (and we called this functor $\sim$), but not an equivalence of categories: there's no map in the other direction. However, there will be a functor $\Gamma_\bullet$ landing in the subcategory of saturated graded modules, which talk to graded modules via a free-forgetful adjunction. We're going to show that $(\sim, \Gamma_\bullet)$ are adjoints and define an equivalence of categories between $\mathsf{SatGrMod}_{S_\bullet}$ and QCoh_X.

Recall that

$$\dim_k \Gamma(\mathbb{P}^m_k, \mathscr{O}_{\mathbb{P}^m_k}(n)) = \binom{m+n}{n}.$$

Therefore, if $X = \mathrm{Proj}\, S_\bullet$ is a closed subscheme of $\mathbb{P}^N_A$, we can do something similar with the following definition.

Definition 13.10. Given a quasicoherent sheaf $\mathscr{F}$, let $M_n = \Gamma(X, \widetilde{M}(n)_\bullet)$. We will define $\Gamma_\bullet$ to have n^{th} graded part $\Gamma_n(\mathscr{F}) = \Gamma(\mathrm{Proj}\, S_\bullet, \mathscr{F}(n))$.

saturated In particular, you can check that if $M_\bullet$ is any graded module, one can define its **saturation** to be $\Gamma_\bullet(\widetilde{M_\bullet})$, and a graded module to be **saturated** if it's its own inverse. Then, it's possible, but harder to show, that there is a natural isomorphism $\widetilde{\Gamma_\bullet(\mathscr{F})} \to \mathscr{F}$, which is the bulk of the hard work of the equivalence of categories.

14. Pushforwards and Pullbacks of Quasicoherent Sheaves: 6/27/16

"Don't put this in the notes, but..."

Today, Jay talked about the first part of Chapter 16, which is about pushforwards and pullbacks of quasicoherent sheaves.

We would like these to be adjoints, and moreover for the pushforward-pullback adjunction to be locally modeled on a ring-theoretic adjunction: if $f : A \to B$ is a map of rings, it makes A-modules into B-modules, which is a functor we'll call $(\cdot)_B$. Then, there is an adjunction

$$-\underset{B}{\otimes} A : \mathsf{Mod}_B \leftrightarrows \mathsf{Mod}_A : (\cdot)_B$$

between A-modules and B-modules.

We bring this to the land of sheaves.

Proposition 14.1. *Let $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes and M be an A-module. Then, the pushforward $\pi_*\widetilde{M} = \widetilde{M_B}$.*

Proof. We check on distinguished opens. If $g \in B$,

$$\begin{aligned}\pi_*\widetilde{M}(D(g)) &= \widetilde{M}(\pi^{-1}(D(g)))\\ &= \widetilde{M}(D(\pi^\sharp(g)))\\ &= M_{\pi^\sharp(g)} = (M_B)_g.\end{aligned}$$

⊠

Though this is not particularly profound, it has some good consequences.

Corollary 14.2. *If π is a morphism of affine schemes, π_* sends quasicoherent sheaves to quasicoherent sheaves.*

Since quasicoherence is affine-local, we can actually conclude something stronger.

Corollary 14.3. *If π is an affine morphism, π_* sends quasicoherent sheaves to quasicoherent sheaves.*

Corollary 14.4. *If π is an affine morphism, π_* is exact.*

Proof sketch. Locally, π_* looks like the functor $(\cdot)_B : \mathsf{Mod}_A \to \mathsf{Mod}_B$, which is exact. ⊠

There's another useful criterion from an earlier chapter, which is a little easier to prove using Proposition 14.1.

Proposition 14.5. *Let $\pi : X \to Y$ be a QCQS morphism and $\mathscr{F}$ be a quasicoherent $\mathscr{O}_X$-module. Then, $\pi_*\mathscr{F}$ is a quasicoherent $\mathscr{O}_Y$-module.*

Proof. Since quasicoherence is an affine-local condition, we may without loss of generality assume Y is affine, and therefore X is a QCQS scheme. Thus, there is a finite affine open cover $\mathfrak{U} = \{U_1, \dots, U_n\}$ of X such that each intersection

$$U_i \cap U_j = \bigcup_{k=1}^{n_{ij}} U_{ijk}$$

is a finite union of affine open sets. Let $\pi_i = \pi|_{U_i}$, $\mathscr{F}_i = \mathscr{F}|_{U_i}$, $\pi_{ijk} = \pi|_{U_{ijk}}$, and $\mathscr{F}_{ijk} = \mathscr{F}|_{U_{ijk}}$. In particular, each π_i and π_{ijk} is a map of affine schemes.

Consider the exact sequence

$$0 \longrightarrow \pi_*\mathscr{F} \xrightarrow{\text{res}} \bigoplus_i (\pi_i)_*\mathscr{F}_i \xrightarrow{\text{diff.}} \bigoplus_{i,j,k} (\pi_{ijk})_*\mathscr{F}_{ijk}, \tag{14.6}$$

where the first map is restriction and the second takes the difference on open subsets. Since π_i and π_{ijk} are maps of affine schemes, then by Corollary 14.3, each $(\pi_i)_*\mathscr{F}_i$ is quasicoherent, and same for $(\pi_{ijk})_*\mathscr{F}_{ijk}$. Since the category of quasicoherent sheaves is abelian, then it is closed under finite direct sums, so the latter two terms in (14.6) are quasicoherent, and therefore $\pi_*\mathscr{F}$ must be as well. ⊠

Remark. If you try to replace "quasicoherent" with "coherent," everything goes wrong without additional hypotheses. Consider the map $f : \mathbb{A}^1_k \to \operatorname{Spec} k$ induced from the inclusion $k \hookrightarrow k[t]$. Since $k[t]$ isn't finitely generated over k, then $f_*\mathscr{O}_{\mathbb{A}^1_k}$ isn't of finite type over $\mathscr{O}_{\operatorname{Spec} k}$.

Proposition 14.7 (Vakil ex. 16.2.C)**.** *Let X and Y be Noetherian schemes and $\pi : X \to Y$ be a finite morphism. If $\mathscr{F}$ is a coherent sheaf on X, then $\pi_*\mathscr{F}$ is a coherent sheaf on Y.*

Proof. We'll first reduce to the case $\mathscr{F} = \mathscr{O}_X$. Let $f : B \to A$ be a map of rings such that f induces the structure of a *finite* B-module on A, and let M be a finitely generated A-module. Then, M_B is a finitely generated B-module: if $\{e_1, \dots, e_\ell\}$ generate M as an A-module and $\{g_1, \dots, g_k\}$ generate A as a B-module, then as a B-module, M is generated by $\{g_ie_j : 1 \le i \le k, 1 \le j \le \ell\}$. Hence, if $\pi_*\mathscr{O}_X$ is coherent, then for any coherent sheaf $\mathscr{F}$ on X, $\pi_*\mathscr{F}$ is locally finitely generated over $\pi_*\mathscr{O}_X$, which is locally finitely generated over $\mathscr{O}_Y$, and therefore $\pi_*\mathscr{F}$ is locally finitely generated over $\mathscr{O}_Y$.

Now we show $\pi_*\mathscr{O}_Y$ is coherent. Let $\mathfrak{U} = \{\operatorname{Spec} A_i\}_{i\in I}$ be an affine open cover of Y; since π is finite and therefore affine, $\pi^{-1}(\operatorname{Spec} A_i) = \operatorname{Spec} B_i$ for some rings B_i, and $\pi^\sharp : A_i \to B_i$ induces a finite A_i-module structure on B_i. Thus, $\pi_*\mathscr{O}_X|_{\operatorname{Spec} B_i}$ is a finite type $\mathscr{O}_Y|_{\operatorname{Spec} A_i}$-module; since both X and Y are Noetherian, finite type is the same as finitely presented (since finitely generated and finitely presented are the same notion over Noetherian rings). It suffices to check that this behaves well with localization, but *a priori* $\pi_*\mathscr{O}_X$ and $\mathscr{O}_Y$ are both quasicoherent, so we're set. ⊠

That's all for pushforwards; let's talk about pullbacks. These are harder, because there's no natural $\mathscr{O}$-module structure on pullbacks.

Remark. Let $\pi : X \to Y$ be a map of schemes.

- You might think we could take the inverse image functor π^{-1}, but if $\mathscr{F}$ is an $\mathscr{O}_Y$-module, $\pi^{-1}\mathscr{F}$ might not be an $\mathscr{O}_X$-module.
- Pullback and pushforward will form an adjunction $\pi^* \vdash \pi_*$. Recall that we want to locally model this with the adjunction $\operatorname{Hom}_A(M \otimes_B A, N) \cong \operatorname{Hom}_B(M, N_B)$ between A-modules and B-modules; we're going to promote this into an adjunction of $\mathscr{O}$-modules, establishing a natural isomorphism

$$\operatorname{Hom}_{\mathscr{O}_X}(\pi^*\mathscr{F}, \mathscr{G}) \xrightarrow{\cong} \operatorname{Hom}_{\mathscr{O}_Y}(\mathscr{F}, \pi_*\mathscr{G}). \tag{14.8}$$

By abstract nonsense, adjoint functors are unique, so this determines π^* completely, if it exists. To solve this, we will construct it.

Definition 14.9. Let $\pi : X \to Y$ be a map of schemes, so there's an induced map of structure sheaves $\pi^\sharp : \pi^{-1}\mathscr{O}_Y \to \mathscr{O}_X$. If $\mathscr{F}$ is an $\mathscr{O}_Y$-module, then $\pi^{-1}\mathscr{F}$ is a $\pi^{-1}\mathscr{O}_Y$-module, so we define the **pullback sheaf** $\pi^*\mathscr{F} = \pi^{-1}\mathscr{F} \otimes_{\pi^{-1}\mathscr{O}_Y} \mathscr{O}_X$.

This has the important property that

$$\pi^*\mathscr{O}_Y = \pi^{-1}\mathscr{O}_Y \underset{\pi^{-1}\mathscr{O}_Y}{\otimes} \mathscr{O}_X = \mathscr{O}_X, \tag{14.10}$$

i.e. pullback sends structure sheaves to structure sheaves.

Proposition 14.11. *This construction satisfies the universal property* (14.8).

Proof. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module and $\mathscr{G}$ be an $\mathscr{O}_Y$-module. By a sequence of abstract nonsense,

$$\begin{aligned}\operatorname{Hom}_{\mathscr{O}_X}(\pi^*\mathscr{G}, \mathscr{F}) &\cong \operatorname{Hom}_{\mathscr{O}_X}(\pi^{-1}\mathscr{G} \underset{\pi^{-1}\mathscr{O}_Y}{\otimes} \mathscr{O}_X, \mathscr{F})\\ &\cong \operatorname{Hom}_{\pi^{-1}\mathscr{O}_Y}(\pi^{-1}\mathscr{G}, \mathscr{F})\\ &\cong \operatorname{Hom}_{\mathscr{O}_Y}(\mathscr{G}, \pi_*\mathscr{F}),\end{aligned}$$

and since each of these is a natural isomorphism, then so is their composition. ⊠

This establishes the adjunction we were looking for.

Proposition 14.12 (Vakil ex 16.3.D)**.** *The pullback of a quasicoherent sheaf is quasicoherent.*

Proof. If $\pi : X \to Y$ is a morphism of schemes, then $\pi^* : \mathsf{Mod}_{\mathscr{O}_Y} \to \mathsf{Mod}_{\mathscr{O}_X}$ is a left adjoint functor between abelian categories, so it commutes with colimits and is right exact. If $\mathscr{G}$ is a quasicoherent sheaf on Y, then it locally admits a presentation

$$\bigoplus_I \mathscr{O}_Y \longrightarrow \bigoplus_J \mathscr{O}_Y \longrightarrow \mathscr{G} \longrightarrow 0,$$

so applying π^* and (14.10), we obtain an exact sequence

$$\bigoplus_I \mathscr{O}_X \longrightarrow \bigoplus_J \mathscr{O}_X \longrightarrow \pi^*\mathscr{G} \longrightarrow 0,$$

showing $\pi^*\mathscr{G}$ is quasicoherent. ⊠

Pushforward doesn't necessarily preserve quasicoherence, because it's a right adjoint, so doesn't commute with arbitrary colimits. However, it does commute with finite colimits, which is why the finiteness hypothesis is necessary in Proposition 14.1.

Corollary 14.13. *Let $\pi : X \to Y$ be a QCQS morphism. Then, it defines an adjoint pair $\pi^* : \mathsf{QCoh}_Y \leftrightarrows \mathsf{QCoh}_X : \pi_*$.*

Proposition 14.14. *Let $\pi : X \to Y$ be a morphism.*

1. *$\pi^*\mathscr{O}_Y \cong \mathscr{O}_X$.*
2. *The pullback π^* of a finite type $\mathscr{O}_Y$-module is a finite type $\mathscr{O}_X$-module.*
3. *The pullback π^* of a locally free $\mathscr{O}_Y$-module is a locally free $\mathscr{O}_X$-module, and a trivialization pulls back to a trivialization.*
4. *If $\rho : Y \to Z$ is another morphism of schemes, there is a natural isomorphism $\pi^* \circ \rho^* \cong (\rho \circ \pi)^*$.*
5. *Pullback is right exact and commutes with tensor products.*

(Note that (1) is just (14.10).)

15. Relative $\mathscr{S}pec$ and $\mathscr{P}roj$: 7/11/16

These are Arun's lecture notes on relative $\mathscr{S}pec$ and $\mathscr{P}roj$, and using the latter to define projective morphisms, corresponding to §§17.1–17.3 in Vakil's notes. I'm planning to talk about:

- Construction of the relative Spec $\mathscr{S}pec\,\mathscr{B}$ of a quasicoherent sheaf $\mathscr{B}$ of $\mathscr{O}_X$-algebras, and some of its properties; in particular, all affine morphisms arise from $\mathscr{S}pec$.
- Construction of the relative Proj $\mathscr{P}roj\,\mathscr{S}_\bullet$ of a quasicoherent sheaf $\mathscr{S}_\bullet$ of nice quasicoherent graded sheaves of $\mathscr{O}_X$-algebras.

Relative Spec. Rings are objects, and Spec produces geometric objects from algebraic ones. We'd like to produce a relative analogue, which produces geometric morphisms from algebraic morphisms. A morphism $B \to A$ of rings induces a B-algebra structure on A, and taking Spec makes $\operatorname{Spec} A$ into a scheme over $\operatorname{Spec} B$. We will generalize this, using what Vakil calls "the high-falutin' language of representable functors."

Throughout this section, X is a scheme and $\mathscr{B}$ is a quasicoherent sheaf of $\mathscr{O}_X$-algebras.

Proposition 15.1. *Let $F_{\mathscr{B}} : \mathsf{Sch}_X^{\mathsf{op}} \to \mathsf{Set}$ denote the functor sending an X-scheme $\mu : W \to X$ to*

$$\operatorname{Hom}_{\mathsf{Alg}_{\mathscr{O}_X}}(\mathscr{B}, \mu_*\mathscr{O}_W).$$

*Then, F is representable; we call the representing object $\beta : \mathscr{S}pec\,\mathscr{B} \to X$ the **relative Spec** of $\mathscr{B}$.*

Recall that in order to show that a functor $F : \mathsf{Sch}_X^{\mathsf{op}} \to \mathsf{Set}$ is representable, we have to show two things.

1. F must be a **Zariski sheaf**, meaning that for every X-scheme W, then the assignment $U \mapsto F(U)$ for all open $U \subseteq W$ defines a sheaf of sets on W. In other words: does the data of what F does locally uniquely determine what it does globally?
2. F must be covered by **open subfunctors** $\{F_i\}$. F_i is an open subfunctor of F if for all X-schemes W, if $h_W = \operatorname{Hom}_{\mathsf{Sch}_X}(-, W)$ and $\phi : h_W \to F$ is a morphism (natural transformation), then $F_i \times_F h_W$ is representable and represented by an open subscheme U_i of W. A collection of open subfunctors is a cover if $\{U_i = F_i \times_F h_W\}$ is an open cover of W, for all W.

Lemma 15.2. *$F_{\mathscr{B}}$ is a Zariski sheaf.*

Proof. For any X-scheme W, $F_{\mathscr{B}}(W) = \operatorname{Hom}_{\mathsf{Alg}_{\mathscr{O}_X}}(\mathscr{B}, \mu_*\mathscr{O}_W)$. Morphisms of sheaves define a sheaf: given compatible local data, there is exactly one way to glue it into a globally defined morphism. ⊠

Let $\mathfrak{U}$ be an affine open cover of X, and let $U_i = \operatorname{Spec} A_i \in \mathfrak{U}$ be arbitrary. Let $B_i = \mathscr{B}(U_i)$, and define $F_{\mathscr{B},i} : \mathsf{Sch}_X^{\mathsf{op}} \to \mathsf{Set}$ to send

$$W \longmapsto \operatorname{Hom}_{\mathsf{Alg}_{A_i}}(B_i, \Gamma(U_i, \mu_*\mathscr{O}_W)).$$

Proposition 15.3 (Vakil ex. 17.1.A)**.** *If $X = \operatorname{Spec} A$ is affine and $\mathscr{B} = \widetilde{B}$, where B is an A-algebra, then $F_{\mathscr{B}}$ is represented by $\operatorname{Spec} B \to \operatorname{Spec} A$.*

Proof. Since $\mathscr{B}$ is quasicoherent, $\operatorname{Hom}_{\mathsf{Alg}_{\mathscr{O}_X}}(\mathscr{B}, \mu_*\mathscr{O}_W) = \operatorname{Hom}_{\mathsf{Alg}_A}(B, \Gamma(W, \mathscr{O}_W)) = \operatorname{Hom}_{\mathsf{Sch}_X}(W, \operatorname{Spec} B)$, so $\operatorname{Spec} B$ represents $F_{\mathscr{B}}$. ⊠

Proposition 15.4 (Vakil prop. 17.1.3)**.** *If $\beta : \mathscr{S}pec\,\mathscr{B} \to X$ represents $F_{\mathscr{B}}$ and $U \hookrightarrow X$ is an open embedding, then $\beta|_U : \mathscr{S}pec\,\mathscr{B} \times_X U \to U$ represents $F|_{\mathscr{B}|_U}$ over U.*

Proof. Let $V = \mathscr{S}pec\,\mathscr{B} \times_X U$, and let W be a U-scheme, so the structure map $\mu : W \to X$ making it an X-scheme factors through U. We have a collection of natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Sch}_U}(W, V \to U) &\cong \mathrm{Hom}_{\mathsf{Sch}_X}(\mu : W \to X, \mathscr{S}pec\,\mathscr{B}) \\ &\cong \mathrm{Hom}_{\mathsf{Alg}_{\mathscr{O}_X}}(\mathscr{B}, \mu_*\mathscr{O}_W) \\ &\cong \mathrm{Hom}_{\mathsf{Alg}_{\mathscr{O}_W}}(\mu^*\mathscr{B}, \mathscr{O}_W) \\ &\cong \mathrm{Hom}_{\mathsf{Alg}_{\mathscr{O}_W}}((\mu|_U)^*\mathscr{B}, \mathscr{O}_W) \\ &\cong \mathrm{Hom}_{\mathsf{Alg}_{\mathscr{O}_X}}(\mathscr{B}|_U, (\mu|_U)_*\mathscr{O}_W). \end{aligned}$$ ⊠

Corollary 15.5. *$F_{\mathscr{B},i}$ is represented by $U_i \times_X \operatorname{Spec} B$.*

Proposition 15.6 (Vakil ex. 17.1.B)**.** *Each F_i is an open subfunctor of F, and $\mathfrak{V} = \{F_i : U_i \in \mathfrak{U}\}$ covers F.*

Proof. Let $\phi : h_W \to F$ be a natural transformation (where $s : W \to X$ is an X-scheme), and let $U_i \hookrightarrow X$ be open. Thus, both of the following squares are pullback squares in $\mathsf{Fun}(\mathsf{Sch}^{\mathsf{op}}, \mathsf{Set})$, which means the four corners form a pullback square too.

$$\begin{array}{ccc} h_W \times_F F_i & \longrightarrow & h_W \\ \downarrow & \lrcorner & \downarrow \phi \\ F_i & \longrightarrow & F \\ \downarrow & \lrcorner & \downarrow \\ h_{U_i} & \hookrightarrow & h_X \end{array}$$

Taking just the outer corners, everything except possibly the upper left corner is representable, so the pullback square is a diagram in Sch:

$$\begin{array}{ccc} ?? & \longrightarrow & W \\ \downarrow & \lrcorner & \downarrow \\ U & \hookrightarrow & X. \end{array}$$

Hence, we know the pullback exists, so $h_W \times_F F_i$ is representable, and since pullbacks preserve open embeddings, the representing scheme W_i is an open subscheme of W. Hence, each $F_{\mathscr{B},i}$ is an open subfunctor of $F_{\mathscr{B}}$. If $p \in W$, then there's some $U_i \subset X$ that's an open neighborhood of $s(p)$, and W_i contains p (as a subscheme of W), so $\mathfrak{V}$ covers F. ⊠

Lemma 15.2 and Proposition 15.6 together imply Proposition 15.1: we've shown that $\mathscr{S}pec\,\mathscr{B}$ exists, and we have a local model for it. In particular, from this local model we know:

Corollary 15.7. *$\beta : \mathscr{S}pec\,\mathscr{B} \to X$ is an affine morphism.*

Impressively, the converse is true.

Proposition 15.8 (Vakil ex. 17.1.D)**.** *If $\mu : Z \to X$ is an affine morphism, there is a natural isomorphism of X-schemes $Z \cong \mathscr{S}pec\,\mu_*\mathscr{O}_Z$.*

Proof. Let $U = \operatorname{Spec} A \subset X$ be an affine open subset; since μ is affine, $\mu^{-1}(U) = \operatorname{Spec} B \subset Z$ for some A-algebra B. Since $\Gamma(U, \mu_*\mathscr{O}_Z) = \Gamma(\mu^{-1}(U), \mathscr{O}_Z) = B$, and B is quasicoherent, then this trickles down: $\mu_*\mathscr{O}_Z|_U = \widetilde{B}$. By Proposition 15.3, $\operatorname{Spec} B \to \operatorname{Spec} A$, i.e. $\mu^{-1}(U) \to U$, satisfies the universal property for $\mathscr{S}pec\,\mu_*\mathscr{O}_Z|_U$, so there's a natural isomorphism $\mu^{-1}(U) \to \mathscr{S}pec\,\mu_*\mathscr{O}_Z|_U$ over U. By Proposition 15.4, as U varies, these isomorphisms agree on intersections, and so glue together to define a natural isomorphism $Z \to \mathscr{S}pec\,\mu_*\mathscr{O}_Z$. ⊠

The assignment

$$W \longmapsto \mathrm{Hom}_{\mathsf{Alg}_{\mathscr{O}_X}}(\mathscr{B}, \mu_*\mathscr{O}_W)$$

is functorial in $\mathscr{B}$, and therefore $\mathscr{S}pec : \mathsf{QCAlg}_{\mathscr{O}_X}^{\mathsf{op}} \to \mathsf{Sch}_X$ is a contravariant functor.

Proposition 15.9 (Vakil ex. 17.1.E)**.** *There is an equivalence of categories between* $\mathrm{QCoh}(\mathscr{S}pec\,\mathscr{B})$ *and the category of quasicoherent $\mathscr{B}$-modules on X.*

Proof sketch. Let $\beta : \mathscr{S}pec\,\mathscr{B} \to X$ denote the structure morphism and $\mathscr{F} \in \mathrm{QCoh}(\mathscr{S}pec\,\mathscr{B})$. The equivalence is realized by β_*: for any open $U \subset X$, $\beta^{-1}(U) \cong \operatorname{Spec} \mathscr{B}(U)$, so since $\mathscr{F}$ is an $\mathscr{O}_{\mathscr{S}pec\,\mathscr{B}}$-module, $\mathscr{B}(U)$ acts on $\mathscr{F}(\operatorname{Spec} \mathscr{B}(U))$, and these actions are compatible with restriction, so $\beta_*\mathscr{F}$ is a quasicoherent $\mathscr{B}$-module over X.

The functor in the other direction is β^*: given a quasicoherent $\mathscr{B}$-module $\mathscr{G}$ and an open $U \subset X$, we recover $\beta^*\mathscr{G}(\operatorname{Spec} U) = \mathscr{G}(U)$ as $\mathscr{B}(U)$-modules and extend it to the other opens of $\mathscr{S}pec\,\mathscr{B}$. ⊠

This is useful when X is simple and $\mathscr{S}pec\,\mathscr{B}$ is complicated.

$\mathscr{S}pec$ behaves well under base change:

Proposition 15.10 (Vakil ex. 17.1.F)**.** *Let $\mu : Z \to X$ be a map of schemes and $\mathscr{B}$ be a quasicoherent sheaf of algebras on X. Then, there is a natural isomorphism $Z \times_X \operatorname{Spec} \mathscr{B} \xrightarrow{\sim} \operatorname{Spec} \mu^*\mathscr{B}$.*

Definition 15.11. Let $\mathscr{F}$ be a finite-rank locally free sheaf on a scheme X. Then, its **total space** to be $\mathscr{S}pec(\operatorname{Sym}^\bullet \mathscr{F}^\vee)$.

The intuition is that we want to add an extra dimension, which corresponds to taking polynomials in one variable. The coordinate-free way description of this is $\operatorname{Sym}^\bullet(-^\vee)$.

Proposition 15.12 (Vakil ex. 17.1.G)**.** *The total space is a rank-n vector bundle, i.e. for every $p \in X$, there's an open neighborhood $U \subset X$ of p such that $\mathscr{S}pec(\operatorname{Sym}^\bullet \mathscr{F}^\vee|_U) \cong \mathbb{A}^n_U$ as U-schemes.*

Proof. Let $\mathfrak{U}$ be an affine open cover of X that trivializes $\mathscr{F}$. For each $U_i \in \mathfrak{U}$, write $U_i = \operatorname{Spec} A_i$. Since U_i is affine,

$$\mathscr{S}pec(\operatorname{Sym}^\bullet \mathscr{F}^\vee|_{U_i}) = \operatorname{Spec}(\operatorname{Sym}^\bullet(\mathscr{F}(U_i)^\vee))$$

Since $\mathscr{F}$ is trivial on U,

$$= \operatorname{Spec}(\operatorname{Sym}^\bullet((\mathscr{O}_X(U_i)^{\oplus n})^\vee)) = \operatorname{Spec}(\operatorname{Sym}^\bullet(A_i[x_1, \dots, x_n])).$$

The symmetric algebra of a polynomial module is just the polynomial algebra: take formal expressions in n variables that commute. Hence,

$$= \operatorname{Spec}(A_i[x_1, \dots, x_n]) = \mathbb{A}^n_{U_i}.$$

⊠

In particular, $\mathbb{A}^n_X = \mathscr{S}pec(\mathscr{O}^n_X)$, as it should.

The dual construction $\mathscr{S}pec(\operatorname{Sym}^\bullet \mathscr{F})$ is called the **abelian cone** associated to $\mathscr{F}$.

Relative Proj. Parallel to the construction of $\mathscr{S}pec$, we define $\mathscr{P}roj$, a relative generalization of the Proj construction, and analyze some of its properties. We assume some hypotheses on the quasicoherent sheaves of algebras that we apply $\mathscr{P}roj$ to; in some cases they can be weakened.

Definition 15.13. Let X be a scheme. Then, a **nicely graded $\mathscr{O}_X$-algebra $\mathscr{S}_\bullet$** is a quasicoherent sheaf of $\mathbb{Z}^{\geq 0}$-graded algebras over $\mathscr{O}_X$ satisfying two additional hypotheses:

(1) $\mathscr{S}_\bullet$ is generated in degree 1, i.e. the natural map $\operatorname{Sym}^\bullet_{\mathscr{O}_X} \mathscr{S}_1 \to \mathscr{S}_\bullet$ is surjective; and
(2) $\mathscr{S}_1$ is finite type.

Condition (1) is affine-local. Vakil calls nicely graded $\mathscr{O}_X$-algebras "quasicoherent sheaves of $\mathbb{Z}^{\geq 0}$-graded algebras finitely generated in degree 1," which is more of a mouthful; our notation is more concise, but nonstandard.

For the rest of this talk, $\mathscr{S}_\bullet$ will be a nicely graded $\mathscr{O}_X$-algebra, as will anything else we ever apply $\mathscr{P}roj$ to, unless otherwise specified.

Unlike $\mathscr{S}pec$, the universal property for $\mathscr{P}roj$ is much messier, so we'll define it through a more explicit construction. We start with a general method for constructing schemes over a base.

Proposition 15.14 (Vakil ex. 17.2.B)**.** *Let X be a scheme, and suppose we have the following data.*

(i) For every affine open $U \subset X$, a morphism $\pi_U : Z_U \to U$.
(ii) For each open inclusion $V \hookrightarrow U$ of affine opens, an open embedding $\rho^U_V : Z_V \hookrightarrow Z_U$.

Assume this data satisfies:

(a) For each open inclusion $V \hookrightarrow U$ of affine opens, ρ^U_V induces an isomorphism $Z_V \xrightarrow{\sim} \pi_U^{-1}(V)$ of schemes over V.

(b) For a nested inclusion $W \hookrightarrow V \hookrightarrow U$, $\rho_W^U = \rho_V^U \circ \rho_W^V$.

Then, there exists a unique X-scheme $\pi : Z \to X$ along with isomorphisms $i_U : \pi^{-1}(U) \to Z_U$ for all affine opens U such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(V) & \hookrightarrow & \pi^{-1}(U) \\ \downarrow{\scriptstyle i_V} & & \downarrow{\scriptstyle i_U} \\ Z_V & \xrightarrow{\rho_V^U} & Z_U. \end{array}$$

Corollary 15.15 (Vakil ex. 17.2.3)**.** *Consider the data of, over every affine open $U = \operatorname{Spec} A \subset X$, the U-scheme $\operatorname{Proj}_A \mathscr{S}_\bullet(U)$. These satisfy the conditions of Proposition 15.14, and therefore define an X-scheme $\mathscr{P}roj_X \mathscr{S}_\bullet \to X$, called the* ***relative Proj*** *of $\mathscr{S}_\bullet$.*

Proof. We haven't specified all of the data needed to construct $\mathscr{P}roj_X \mathscr{S}_\bullet$; we also need open embeddings arising from inclusions of affine opens $V \hookrightarrow U$ of X. This is local on the target, so for an $f \in \mathscr{S}_\bullet(U)$, we can work in $D(f) \cong \operatorname{Spec}((\mathscr{S}_\bullet(U)_f)_0)$. The sheaf restriction map induces an open inclusion $\operatorname{Spec}((\mathscr{S}_\bullet(V)_{f|_V})_0) \hookrightarrow \operatorname{Spec}((\mathscr{S}_\bullet(U)_f)_0)$; we need the induced (restriction) map on structure sheaves to be an isomorphism, but this is "tautologically" true: we defined this map to restrict sections, so of course it restricts sections. Thus, we've defined open embeddings $D(f|_U) \hookrightarrow D(f)$ for all $f \in \mathscr{S}_\bullet(V)$, and these embeddings patch together to the desired open embedding.

Now, we must check that the data satisfies conditions (a) and (b). The latter is true because the restriction maps for $\mathscr{S}_\bullet$ have that property. Now we just need to check that in the diagram

$$\begin{array}{ccccc} \operatorname{Proj} \mathscr{S}_\bullet(V) & \xrightarrow{\alpha} & \pi_U^{-1}(V) & \hookrightarrow & \operatorname{Proj} \mathscr{S}_\bullet(U) \\ \downarrow{\scriptstyle \pi_V} & & \downarrow{\scriptstyle \pi_U} & \swarrow{\scriptstyle \pi_U} & \\ V & \hookrightarrow & U, & & \end{array}$$

α is an isomorphism. On the level of sets, this is true because points (homogeneous prime ideals) in $\operatorname{Proj} \mathscr{S}_\bullet(V)$ are exactly those that were pulled back from U, and then the isomorphism on structure sheaves exists for the same reason as before: we're asking for this map to induce an isomorphism onto a restriction, but restriction is how we defined the map. ⊠

Definition 15.16 (Vakil ex. 17.2.D)**.** We construct an invertible sheaf $\mathscr{O}_{\mathscr{P}roj_X \mathscr{S}_\bullet}(1)$ on $\mathscr{P}roj_X \mathscr{S}_\bullet$. If $U = \operatorname{Spec} A \subset X$ is an affine open, then we will define $\mathscr{O}(1)$ over $\operatorname{Proj}_A \mathscr{S}_\bullet(U)$ to be $\mathscr{O}_{\operatorname{Proj} \mathscr{S}_\bullet(U)}(1)$. Since $\mathscr{S}_\bullet$ is a sheaf and $\mathscr{O}(1)$ is just obtained from degree-shifts of $\mathscr{S}_\bullet$, then this behaves sheafily over subsets of the form $\operatorname{Proj}_A \mathscr{S}_\bullet(\operatorname{Spec} A)$, in that compatible sections glue uniquely. By quasicoherence, this suffices to compatibly define it on all affine open subsets of $\mathscr{P}roj_X \mathscr{S}_\bullet$, since we have defined it on an open cover, and hence on all opens.

Definition 15.17. Let $\mathscr{F}$ be a finite type quasicoherent sheaf on X. Then, $\mathbb{P}\mathscr{F} = \mathscr{P}roj(\operatorname{Sym}^\bullet \mathscr{F})$ is called its **projectivization**. If $\mathscr{F}$ is locally free of rank $n+1$, $\mathbb{P}\mathscr{F}$ is a **projective bundle** (or **$\mathbb{P}^n$-bundle**).

In particular, $\mathbb{P}_X^n = \mathbb{P}(\mathscr{O}_X^{\oplus n+1})$, as it should.

Example 15.18 (Vakil exm. 17.2.4)**.** Suppose C is a regular curve and $\mathscr{F}$ is locally free of rank 2 over C. Then, $\mathbb{P}\mathscr{F}$ is called a **ruled surface** over C; it's ruled by copies of $\mathbb{P}^1$. If $C \cong \mathbb{P}^1$, then $\mathbb{P}\mathscr{F}$ is called a **Hirzebruch surface**.

In the next chapter, we'll prove all vector bundles on $\mathbb{P}^1$ split as a direct sum of line bundles; this implies every Hirzebruch surface is of the form $\mathbb{P}(\mathscr{O}(m) \oplus \mathscr{O}(n))$, which turns out to only depend on the difference $n - m$, and is denoted $\mathbb{F}_{n-m}$.

Proposition 15.19 (Vakil ex. 17.2.H)**.** *If $\mathscr{S}_\bullet$ is nicely graded, there is a canonical closed embedding*

$$\begin{array}{ccc} \mathscr{P}roj\, \mathscr{S}_\bullet & \xhookrightarrow{i} & \mathbb{P}\mathscr{S}_1 \\ & {\scriptstyle \beta}\searrow \quad \swarrow & \\ & X, & \end{array}$$

as well as an isomorphism $\mathscr{O}_{\mathscr{P}roj\,\mathscr{S}_\bullet}(1) \cong i^*\mathscr{O}_{\mathbb{P}\mathscr{S}_1}(1)$.

Proposition 15.20 (Vakil ex. 17.2.I)**.** *Let $\mathscr{F}$ be a locally free sheaf of rank $n+1$ on X; then, there is a bijection between the set of sections $s : X \to \mathbb{P}\mathscr{F}$ and the set of surjections $\mathscr{F} \twoheadrightarrow \mathscr{L}$ where $\mathscr{L}$ is invertible.*

16. Nice Results About Curves: 7/14/16

Today, Yan spoke about the second half of chapter 17, on applications to curves and projective morphisms.

Definition 16.1. A morphism $\pi : X \to Y$ of schemes is **projective** if there exists an isomorphism $X \xrightarrow{\sim} \operatorname{Proj} \mathscr{S}_\bullet$ of Y-schemes for some quasicoherent sheaf of $\mathscr{O}_Y$-algebras $\mathscr{S}_\bullet$, i.e. there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\sim} & \operatorname{Proj} \mathscr{S}_\bullet \\ & {}_{\pi}\searrow \quad \swarrow & \\ & Y. & \end{array}$$

Here are some nice properties of projective morphisms.

- The composition of projective morphisms is projective.
- Finite morphisms are projective. Recall that a morphism $\pi : X \to Y$ is **finite** if locally, it looks like $\operatorname{Spec} B \to \operatorname{Spec} A$, where the induced map $A \to B$ on rings makes B a finite A-module. Normalization is one of the most important examples of finite morphisms, which follows because a map into an integral closure is a finite morphism.

Now, let's talk about applications to curves. First, a theorem we'll use many times.

Theorem 16.2 (Curve-to-projective extension theorem)**.** *Suppose C is a pure dimension 1 Noetherian scheme over a scheme S and $p \in C$ is a closed point. If Y is a projective S-scheme, then an S-morphism $\mathbb{C} \setminus p \to Y$ extends to all of C.*

This extension need not be unique.

Classification theorems in algebraic geometry tend to involve birational equivalence, so here's a nice result on birationality and curves: every integral curve has a birational model (i.e. is birational to something) that is regular and projective. More precisely:

Proposition 16.3. *Let C be an integral curve of finite type over a field k. Then, there exists a regular projective curve C' birational to C.*

In order to prove this, we'll need the normalization.

Lemma 16.4 (Normalization)**.** *Let X be an integral scheme, $K(X)$ be its field of functions, and $L/K(X)$ be a finite extension. Then, there exists a scheme $\widetilde{X}$, called a* ***normalization*** *of X, such that $K(\widetilde{X}) \cong L$.*

Proof of Proposition 16.3. We first reduce to the case where C is affine. Since C is a curve, its function field $K(C)$ is an extension of k of transcendence degree 1. By the Noether normalization lemma (Lemma 3.12), there exists an $x \in K(C) \setminus k$ such that $K(C)/k(x)$ is a finite extension (so x contains all of the transcendence, so to speak).

We can identify $\operatorname{Spec} k[x] \cong \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$ as everything but the point at infinity; then, since $K(C)$ is a finite extension of $k(x) = K(\mathbb{P}^1_k)$, so let C' be a normalization of $\mathbb{P}^1_k$. Then, $C' \to \mathbb{P}^1_k$ is finite. To see why C' is regular, Theorem 7.12 proves that, since we're in dimension 1, regularity is equivalent to integrality, and C' is projective because it has a finite map to $\mathbb{P}^1_k$. ⊠

Proposition 16.5. *All regular, proper curves over a field k are projective.*

The proof will use the valuative criterion for properness; valuations are surprisingly crucial for understanding regular, projective curves. As such, we briefly digress to valuative criteria for separatedness and properness, which was in §12.7.

Proposition 16.6 (Valuative criteria for separatedness and properness)**.** *Consider a commutative diagram*

$$\begin{array}{ccc} \operatorname{Spec} K(A) & \longrightarrow & X \\ {\scriptstyle\pi'}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \operatorname{Spec} A & \xrightarrow{\ \rho\ } & Y, \end{array}$$

where A is a DVR, $\pi : X \to Y$ is finite type, π' is an open embedding, and Y is locally Noetherian.

(1) If π is separated, there is at most one way to lift ρ to the dotted arrow.
(2) If π is proper, there is exactly one way to lift ρ to the dotted arrow.

This has immediate applications to separated curves.

Proposition 16.7. *Let X be an integral, separated, Noetherian curve, so that if $p \in X$ is a closed point, then the inclusion $\mathscr{O}_{X,p} \subset K(X)$ induces a DVR structure on $\mathscr{O}_X$. If $p, q \in X$ are two distinct such points, then their local rings have distinct induced valuations.*

Proof. Let p and q be two points whose local rings have the same induced valuation structure, and in particular are both isomorphic to some ring R. Then, the fraction fields of $\mathscr{O}_{X,p}$ and $\mathscr{O}_{X,q}$ are $K(X)$; if η denotes the generic point of X, so we can write down a diagram

$$\begin{array}{ccc} \operatorname{Spec} K(X) & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ \operatorname{Spec} R & \longrightarrow & \operatorname{Spec} \mathbb{Z}. \end{array}$$

Since R is a DVR, $\operatorname{Spec} R$ has two points, a generic point ξ and a closed point θ. Thus, there are two ways to lift $\operatorname{Spec} R \to \operatorname{Spec} \mathbb{Z}$ to X: one maps $\xi \mapsto \eta$ and $\theta \mapsto p$, and the other sends $\xi \mapsto \eta$ and $\theta \to q$, which contradicts Proposition 16.6. ⊠

This provides an important perspective on curves: we can think of them algebraically, in terms of their function fields and discrete valuation rings: there's a bijection between valuation structures induced from $K(X)$ and closed points of X.

Proposition 16.8. *Let C be an integral, proper curve over a field k. Then, for any discrete valuation ring $R \subset K(C)$, there is a closed point $p \in X$ such that $\mathscr{O}_{X,p}$ is* ***dominated by*** *R, i.e. there is an inclusion $\mathscr{O}_{X,p} \hookrightarrow R$ that is a homomorphism of local rings: it sends the maximal ideal to the maximal ideal, and induces an inclusion of residue fields.*

Proof sketch of Proposition 16.5. Let C be a regular proper curve; we may assume C is irreducible, so there's a regular projective C' that is birational to C, and we obtain an open embedding $C \hookrightarrow C'$: every closed point in C corresponds to a unique DVR in $K(C)$, and therefore to one in $K(C')$, and therefore to a unique closed point in C'. Since C is proper, then this map is actually surjective, so this map is an isomorphism. ⊠

Another nice result on curves is a chain of equivalences of categories; the equivalence of the last two is the most important, and can be made more general (e.g. replace curves with integral k-varieties, and replace transcendence degree 1 with finite transcendence degree).

Proposition 16.9. *Let k be a field; then, the following categories are equivalent.*

(1) The category of irreducible, regular, projective curves over k and surjective k-morphisms.
(2) The category of irreducible, regular, projective curves over k and dominant k-morphisms.
(3) The category of irreducible, regular, projective curves over k and dominant rational maps over k.
(4) The category of integral curves of finite type over k and dominant rational maps over k.
(5) The opposite category of the category of finitely generated field extensions of k with transcendence degree 1 and k-morphisms.

The proof idea is to go from (4) to (1), because inclusion defines functors (1) to (2) to (3) to (4), and the last case can be dealt with by an earlier theorem.

This implies, for example, that if C is a quasiprojective reduced curve, then it's birational to a unique regular birational curve, which follows from Theorem 16.2: we already proved that integral curves have projective regular birational models, and uniqueness follows because any two birational models have isomorphic function fields, so by Proposition 16.9, they must be isomorphic.

Here's another implication.

Proposition 16.10. *Let $\pi : C \to C'$ be a dominant map between projective curves over a field k; then, π is a finite morphism.*

Proof. Since π is dominant and C' and C are projective, Proposition (16.9) produces a map $K(C') \hookrightarrow K(C)$; since both have the same transcendence degree over k, this must be a finite extension, and so we can form the normalization C'' of C' in $K(C)$; this means $K(C'') \cong K(C)$, and so C'' and C are birational, and hence isomorphic. Since C'' is normal, and hence regular, then it's projective, and the normalization map $C \cong C'' \to C'$ is a finite morphism. ⊠

We can get a stronger results with an extra condition, albeit one that's unintuitive.

Proposition 16.11 (Vakil prop. 17.4.5)**.** *Let $\pi : C \to C'$ be a finite morphism of projective schemes, where C has no embedded points and C' is regular. Then, $\pi_*\mathscr{O}_C$ is a finite-rank locally free sheaf.*

In this case, $\operatorname{rank} \pi_*\mathscr{O}_C$ is called the **degree** of π.

A more general, and less weird, condition, is that every associated point of C is mapped to the generic point of some irreducible component of C'. Later, when we talk about flat morphisms, we'll be able to show that a morphism of curves is flat iff every associated point is mapped to a generic point of an irreducible component, and so Proposition 16.11 will be true when π is a flat morphism.

On Riemann surfaces, which are nice curves over $\mathbb{C}$, we already knew this from maps being branched covers with constant degree.

17. Čech Cohomology: 7/18/16

Today, Yuri talked about Čech cohomology of quasicoherent sheaves. Throughout today's lecture, all schemes are quasicompact and separated, and all sheaves are quasicoherent; these hypotheses can be generalized somewhat if you want.

Definition 17.1. If X is a locally ringed space, the i^{th} **cohomology** $H^i(X,-) : \mathrm{QCoh}(X) \to \mathsf{Ab}$ is the i^{th} derived functor of the global sections functor $\Gamma : \mathrm{QCoh}(X) \to \mathsf{Ab}$.

This definition makes some formal properties easy to prove:

- $H^0 = \Gamma$.
- Suppose
$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$
is a short exact sequence of sheaves on X. Then, we have a long exact sequence in cohomology:
$$\cdots \longrightarrow H^i(X,\mathscr{F}') \longrightarrow H^i(X,\mathscr{F}) \longrightarrow H^i(X,\mathscr{F}'') \xrightarrow{\delta} H^{i+1}(X,\mathscr{F}') \longrightarrow \cdots$$
- Cohomology has nice pushforward and pullback properties: if $\pi : X \to Y$ is a map of schemes, $\mathscr{F}$ is a quasicoherent sheaf on X, and $\mathscr{G}$ is one on Y, then π induces natural maps $H^i(Y,\pi_*\mathscr{F}) \to H^i(X,\mathscr{F})$ and $H^i(Y,\mathscr{G}) \to H^i(X,\pi^*\mathscr{G})$; if π is affine, the first is an isomorphism.

However, it's not so easy to actually compute cohomology of anything using this definition; we'll need a model.

Example 17.2. If $Y = \mathbb{P}^n$ and $X \hookrightarrow \mathbb{P}^n$ is a closed embedding, then for a sheaf $\mathscr{F}$ on X, the last point implies $H^i(X,\mathscr{F}) = H^i(\mathbb{P}^n,\pi_*\mathscr{F})$, which allows us to reduce questions about projective schemes to the case over $\mathbb{P}^n$.

Example 17.3. Some of this stuff shows up in GAGA, a collection of results relating complex varieties to complex manifolds. Let X be a smooth complex variety, which defines a complex manifold X^h that's locally cut out of $\mathbb{C}^n$ by the same equations that cut X out of $\mathbb{A}^n_{\mathbb{C}}$.[14] Let $\mathscr{H}$ denote the sheaf of holomorphic functions on X^h, so if $\mathscr{F}$ is an **algebraic sheaf** ($\mathscr{O}_X$-module on X); since regular functions are (locally) polynomials, which are holomorphic, then the identity map defines a morphism of locally ringed spaces

[14] This can be done in more generality: if X is smooth, you get an analytic space of some sort, a manifold with singularities.

$\pi : (X^h, \mathscr{H}) \to (X, \mathscr{O}_X)$; the pullback on functions is inclusion $\pi^{-1}\mathscr{O}_X \hookrightarrow \mathscr{H}$. This map defines a morphism in cohomology

$$H^i(X, \mathscr{F}) \longrightarrow H^i(X^h, \pi^*\mathscr{F}). \tag{17.4}$$

The remarkable theorem is that this data is an equivalence.

Theorem 17.5 (GAGA). *If X is projective, then $\pi^* : \mathsf{Coh}(\mathscr{O}_X) \to \mathsf{Coh}(\mathscr{H})$ is an equivalence of categories and (17.4) is an isomorphism.*

Another useful theorem, which is considerably easier to prove, is a vanishing criterion.

Theorem 17.6 (Vanishing). *Suppose X can be covered by $n+1$ affine opens; then, for all $i > n$ and all $\mathscr{F}$, $H^i(X, \mathscr{F}) = 0$.*

In particular, cohomology is trivial on affine schemes. Impressively, the converse is true (though harder).

In general, given an open cover $\mathfrak{U}$ of X, the complicated part of the cohomology of X could come from the complicated parts of the cohomology over the sets in $\mathfrak{U}$, or the ways in which they fit together. In general, one uses the **Čech-Leray spectral sequence**

$$E_2^{pq} = \check{H}^p(\mathfrak{U}, \mathcal{H}^q(-, \mathscr{F})) \implies H^{p+q}(X, \mathscr{F}),$$

but if the cohomology of each $U_\alpha \in \mathfrak{U}$ is trivial, then all that's left is the data on how they fit together, which is entirely combinatorial.[15]

In topology, a sufficient condition for this is that $\mathfrak{U}$ is a **good cover**, meanin that each $U_\alpha \in \mathfrak{U}$ is contractible, as are all finite intersections of opens in $\mathfrak{U}$. But really, all we need is for cohomology to vanish, which is easier to generalize: we know that cohomology is trivial on affine opens, and there are lots of those.

Definition 17.7. Let $\mathfrak{U} = \{U_i \mid i \in I\}$ be an affine open cover of X, and write $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$, and so forth, where $i, j, k \in I$. The **Čech complex** is the complex of abelian groups

$$\prod_i \mathscr{F}(U_i) \xrightarrow{\partial} \prod_{i,j \text{ distinct}} \mathscr{F}(U_{ij}) \xrightarrow{\partial} \prod_{i,j,k \text{ distinct}} \mathscr{F}(U_{ijk}) \longrightarrow \cdots \tag{17.8}$$

where the differential is defined by an alternating sum. The cohomology of this complex, denoted $\check{H}^\bullet(\mathfrak{U}, \mathscr{F})$ is called the **Čech cohomology** of X **relative to the cover** $\mathfrak{U}$.[16] Then, we define the **Čech cohomology** of X to be

$$\check{H}^\bullet(X, \mathscr{F}) = \varprojlim_{\text{all covers } \mathfrak{U}} \check{H}^\bullet(\mathfrak{U}, \mathscr{F}).$$

There are a few notions of what it means to take a limit over all covers $\mathfrak{U}$ of X.

- A cover $\mathfrak{U}$ is a subset of the set of open subsets of X such that every point in X is in some set in the cover.
- A cover $\mathfrak{U}$ is a collection of open immersions $U_i \hookrightarrow X$ such that the induced map $\coprod_i U_i \to X$ is surjective.

The first is a set; the second is more categorical. It's also easier to generalize the second definition: replacing open immersions with étale or flat maps leads to notions like étale covers, etc.

The thing we'd like to do with this is find a cover $\mathfrak{U}$ whose cohomology is the Čech cohomology of X, so that we don't need to take a colimit. This is one reason we restrict to good covers in the differentiable case.

Proposition 17.9. *If X is a quasicompact, separated space and $\mathfrak{U}$ is a finite cover of X, then $\check{H}^\bullet(\mathfrak{U}, \mathscr{F}) = H^\bullet(X, \mathscr{F})$.*

More generally (from FAC), we have the following result.

Theorem 17.10. *Let X be a space and $\mathfrak{U}$ a cover for it. Suppose there is a family of covers $\{\mathfrak{W}^\alpha\}$ such that*

(1) any cover $\mathfrak{W}$ can be refined by some $\mathfrak{W}^\alpha$, and

(2) for all α and all $U \in \mathfrak{U}$, $\check{H}^\bullet(\mathfrak{W}^\alpha|_U, \mathscr{F}) = 0$.

Then, $\check{H}^\bullet(\mathfrak{U}, \mathscr{F}) = H^\bullet(X, \mathscr{F})$.

[15] The Mayer-Vietoris sequence is a very simplified version of the Čech-Leray spectral sequence.

[16] There's a way to define this using a simplicial abelian group.

Over quasicompact, separated schemes, we use the set of finite affine covers.

Proposition 17.9 immediately implies Theorem 17.6: there's no way to choose $n+1$ distinct sets in the cover, so the complex (17.8) dies at degree $n+1$.

Another nice vanishing result is that cohomology vanishes above the dimension of X.

Theorem 17.11 (Dimensional vanishing)**.** *Let k be a field and X be a projective k-scheme. Then, if $i > \dim X$, $H^i(X, \mathscr{F}) = 0$ for all QC sheaves $\mathscr{F}$.*

The proof embeds X into a high-dimensional projective space, and then covers it with at most n hyperplanes, using the fact that sufficiently many hyperplanes in projective space have to run into each other.

Theorem 17.12 (Künneth formula)**.** *Let k be a field, X and Y be k-schemes, and $\mathscr{F}$ be a QC sheaf on X and $\mathscr{G}$ be a QC sheaf on Y. Then, there is a natural isomorphism*

$$H^\bullet(X \times_k Y, \mathscr{F} \boxtimes \mathscr{G}) \cong H^\bullet(X, \mathscr{F}) \otimes H^\bullet(X, \mathscr{G}).$$

Here,

- *if π_1 and π_2 are the projections of $X \times_k Y$ onto its first and second components, respectively, then $\mathscr{F} \boxtimes \mathscr{G} = \pi_1^* \mathscr{F} \otimes \pi_2^* \mathscr{G}$, and*
- *the tensor product on the right-hand side is the tensor product of graded rings.*

Even for ordinary topological spaces, one needs k to be a field, or there will be higher Ext terms preventing the formula from being exactly correct.

One calculation (whose proof is omitted) yields many useful consequences.

Theorem 17.13. *Let A be a Noetherian ring.*

(1) If $m \geq 0$, $H^0(\mathbb{P}^n_A, \mathscr{O}(m))$ is a free A-module of rank $\binom{n+m}{m}$.
(2) If $m \leq -(n+1)$, $H^n(\mathbb{P}^n_A, \mathscr{O}(m))$ is free A-module of rank $\binom{-m-1}{-n-m-1}$.
(3) If otherwise, $H^i(\mathbb{P}^n_A, \mathscr{O}(m)) = 0$.

Corollary 17.14. *Let X be a projective A-scheme, where A is a Noetherian ring.*

(1) If $\mathscr{F}$ is a coherent sheaf on X, then $H^i(X, \mathscr{F})$ is a coherent (i.e. finitely generated) A-module.
(2) For all $N \gg 0$ and $i > 0$, $H^i(X, \mathscr{F}(N)) = 0$.

Proof. As mentioned in Example 17.2, we may without loss of generality assume $X = \mathbb{P}^n_A$. Since $\mathscr{F}$ is coherent, there's a surjection $s : \mathscr{O}_X(m)^{\oplus p} \twoheadrightarrow \mathscr{F}$; let $\mathscr{G} = \ker(s)$. The short exact sequence

$$0 \longrightarrow \mathscr{G} \longrightarrow \mathscr{O}(m)^{\oplus p} \longrightarrow \mathscr{F} \longrightarrow 0, \tag{17.15}$$

which induces a long exact sequence in cohomology that stops:

$$\cdots \longrightarrow H^n(\mathbb{P}^n, \mathscr{G}) \longrightarrow H^n(\mathbb{P}^n, \mathscr{O}(m))^{\oplus p} \longrightarrow H^n(\mathbb{P}^n, \mathscr{F}) \longrightarrow 0.$$

By Theorem 17.13, $H^n(\mathbb{P}^n, \mathscr{O}(m))^{\oplus p}$ is coherent, so $H^n(\mathbb{P}^n, \mathscr{F})$ must be too.

For the second part, twist (17.15) by applying $- \otimes \mathscr{O}(N)$; this is right exact, so we obtain

$$\mathscr{G}(N) \longrightarrow \mathscr{O}(m+N)^{\oplus p} \longrightarrow \mathscr{F}(N) \longrightarrow 0.$$

Since $\mathscr{O}(m+N)$ has trivial cohomology when N is sufficiently large, so must $\mathscr{F}(N)$, by the same surjection from the same long exact sequence. ⊠

If X is a projective k-scheme, sometimes its cohomology groups are unpleasant to calculate, or don't behave as well as you'd like. For example, we saw that the cohomology of $\mathscr{O}(m)$ over $\mathbb{P}^n_A$ is a cut-off polynomial, and we might like it to actually be a polynomial.

Definition 17.16. If X is a projective k-scheme and $\mathscr{F}$ is a coherent sheaf on X, the **Euler characteristic** is

$$\chi(X, \mathscr{F}) = \sum (-1)^i \dim_k H^i(X, \mathscr{F}).$$

By dimensional vanishing, this is an integer. There's also this nice fact:

Proposition 17.17. *Given a short exact sequence of coherent sheaves*

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0,$$

the Euler characteristic is additive: $\chi(X, \mathscr{G}) = \chi(X, \mathscr{F}) + \chi(X, \mathscr{H})$.

Today, Danny continued talking about chapter 18, starting at §18.4, on the Riemann-Roch theorem for projective curves over a field.

Definition 18.1. Let X be a projective scheme over a field k and $\mathscr{F}$ be a coherent sheaf on X. Thus, each $\check{H}^i(X, \mathscr{F})$ is finite-dimensional, and only finitely many are nonzero, so it makes sense to define the **Euler characteristic**

$$\chi(X, \mathscr{F}) = \sum_i (-1)^i \dim_k \check{H}^i(X, \mathscr{F}),$$

which is always finite.

Proposition 18.2 (Vakil ex. 18.4.A)**.** *If*

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \cdots \longrightarrow \mathscr{F}_n \longrightarrow 0$$

is an exact sequence of coherent sheaves, then

$$\sum_{i=1}^n (-1)^i \chi(X, \mathscr{F}_i) = 0.$$

To prove this, start with a short exact sequence, where it follows from the long exact sequence in cohomology.

Theorem 18.3 (Riemann-Roch, Vakil thm. 18.4.1)**.** *Let C be a regular projective curve over a field k and D be a divisor on C. Then,*

$$\chi(C, \mathscr{O}_C(D)) = \deg D + \chi(C, \mathscr{O}_C).$$

Proof. Let p be a closed point on C, so we have a closed embedding $\iota : p \to C$, to which we associate a short exact sequence of sheaves

$$0 \longrightarrow \mathscr{I}_{p/C} \longrightarrow \mathscr{O}_X \longrightarrow \iota_* \mathscr{O}_p \longrightarrow 0. \tag{18.4}$$

Since C is regular, then the stalk $\mathscr{O}_p$ is a DVR, so $\mathscr{I}_{p/C}$ is the ideal of stalks that vanish, hence to at least degree 1, and therefore $\mathscr{I}_{p/C} = \mathscr{O}_C(-p)$.

Now, we induct: if $\deg D = 0$, this is trivially true. In the general case, apply $- \otimes_{\mathscr{O}_C} \mathscr{O}_C(D)$, which is exact because $\mathscr{O}_C(D)$ is a line bundle. This turns (18.4) into

$$0 \longrightarrow \mathscr{O}_C(D-p) \longrightarrow \mathscr{O}_C(D) \longrightarrow (\iota_* \mathscr{O}_p) \otimes_{\mathscr{O}_C} \mathscr{O}_C(D) \longrightarrow 0.$$

Tensoring a skyscraper with a line bundle doesn't change the skyscraper: it would tensor by the stalk of the line bundle, but this is trivial. Thus the sequence simplifies to

$$0 \longrightarrow \mathscr{O}_C(D-p) \longrightarrow \mathscr{O}_C(D) \longrightarrow \iota_* \mathscr{O}_p \longrightarrow 0.$$

Applying Proposition 18.2,

$$\chi(C, \mathscr{O}_C(D)) = \chi(C, \mathscr{O}_C(D-p)) + \chi(C, \mathscr{O}_p).$$

What's the cohomology of a skyscraper sheaf? We can choose an affine cover $\mathfrak{U}$ of C such that p lies in only one set in $\mathfrak{U}$; then, the Čech complex is 0 everywhere except the first index, so $H^0(C, \mathscr{O}_p) = \Gamma(C, \mathscr{O}_p) = \kappa(p)$, which has dimension $\deg p$,[17] and $H^1(C, \mathscr{O}_p) = 0$ is killed by the 0-cochains. Dimensional vanishing guarantees that higher cohomology vanishes. Thus,

$$\chi(C, \mathscr{O}_C(D)) = \chi(C, \mathscr{O}_C(D-p)) + \deg p,$$

and since $\deg(D-p) < \deg D$, the formula follows by induction. ⊠

Definition 18.5. Let $\mathscr{L}$ be line bundle on a regular projective curve C over k. Then, we define its **degree** to be

$$\deg_C \mathscr{L} = \chi(C, \mathscr{L}) - \chi(C, \mathscr{O}_C).$$

If s is a rational section of $\mathscr{L}$ and $D = \operatorname{div} s = \sum v_p(s)[p]$, then by Theorem 18.3, $\deg_C \mathscr{L} = \deg D$.

This shows, for instance, that a rational function on a regular projective curve over k has the same number of poles as zeros (apply this with $\mathscr{L} = \mathscr{O}_C$). Similarly, one can show that $\deg(\mathscr{L}_1 \otimes \mathscr{L}_2) = \deg(\mathscr{L}_1) + \deg(\mathscr{L}_2)$.

[17]Recall that the **degree** of a closed point over a k-scheme is the degree of its residue field over k, as a field extension.

Proposition 18.6 (Vakil ex. 18.4.K)**.** *Let C be as before and $\mathscr{L}$ be an ample line bundle on C. Then, $\deg \mathscr{L} > 0$.*

Proof. We know that $\mathscr{L}^{\otimes n}$ is very ample for some large n, so if $\deg(\mathscr{L}^{\otimes n})$ is positive, so is $\deg \mathscr{L}$. Since $\mathscr{L}^{\otimes n}$ is generated by global sections, then in particular it has at least one section s, and s must vanish somewhere, since $\mathscr{L}^{\oplus n}$ is not the structure sheaf. So it has at least one zero and no poles, so $\deg(\operatorname{div} s) > 0$, and thus $\deg \mathscr{L} > 0$. ⊠

Theorem 18.7 (Serre duality for smooth projective varieties)**.** *Let k be a field and X be a smooth, projective, geometrically irreducible k-variety of dimension n.*[18] *Then, there exists a **dualizing sheaf** ω_X, which is an invertible sheaf such that if $\mathscr{L}$ is any invertible sheaf, then*[19]

$$h^i(X, \mathscr{L}) = h^{n-i}(X, \omega_X \otimes \mathscr{L}^\vee).$$

This allows us to restate Theorem 18.3: if $\mathscr{L} = \mathscr{O}_C(D)$, it's equivalent to

$$h^0(\mathscr{O}_C(D)) - h^1(\mathscr{O}_C(D)) = \deg D + h^0(\mathscr{O}_C) - h^1(\mathscr{O}_C)$$
$$h^0(\mathscr{L}) - h^1(\mathscr{L}) = \deg \mathscr{L} + 1 - h^1(\mathscr{O}_C)$$
$$h^0(\mathscr{L}) - h^0(\omega_X \otimes \mathscr{L}^\vee) = \deg \mathscr{L} + 1 - h^0(\omega_X).$$

Plugging in $\mathscr{L} = \omega_X$ shows that $\deg \omega_X = 2g - 2$, where $g = h^0(\omega_X)$. For this reason, $h^0(\omega_X)$ is sometimes called the **genus** of X.

Now, we'll sketch the classification of vector bundles on $\mathbb{P}^1_k$. In Vakil's words, this is called Grothendieck's theorem, because Grothendieck doesn't have enough theorems named after him.

Theorem 18.8 (Grothendieck's theorem)**.** *Let $\mathscr{E}$ be a rank r vector bundle on $\mathbb{P}^1_k$. Then, there is a unique nondecreasing sequence of integers $a_1 \le a_2 \le \cdots \le a_r$ such that*

$$\mathscr{E} \cong \mathscr{O}(a_1) \oplus \mathscr{O}(a_2) \oplus \cdots \oplus \mathscr{O}(a_r). \tag{18.9}$$

Proof sketch. For uniqueness, one studies maps $\mathscr{O}(m) \to \mathscr{E}$ for all $m \in \mathbb{Z}$; assuming $\mathscr{E}$ is of the form in (18.9), the set of such maps forms a k-vector space. We can realize these as certain maps of graded modules, and then uniquely read off the dimensions of these spaces of maps from the identification (18.9).

For existence, we'll induct on r. We've already classified line bundles on $\mathbb{P}^1_k$ in Example 10.18, which takes care of $r = 1$.

Proposition 18.10 (Vakil ex. 18.5.D)**.** *For $m \ll 0$, $\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{O}(m), \mathscr{E}) \ne 0$, and for $m \gg 0$, $\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{O}(m), \mathscr{E}) = 0$.*

Proof. By applying $- \otimes_{\mathscr{O}_X} \mathscr{O}(m)$, we have an isomorphism of sheaf hom

$$\mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{O}(m), \mathscr{E}) \cong \mathscr{H}\!om_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{E}(-m)).$$

Taking global sections, $\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{O}(m), \mathscr{E}) = H^0(\mathbb{P}^1_k, \mathscr{E}(-m))$. By Serre vanishing, there exists some a_r such that $\operatorname{Hom}(\mathscr{O}(a_r), \mathscr{E})$ is nonzero, but $\operatorname{Hom}(\mathscr{O}(m), \mathscr{E}) = 0$ if $m > a_r$. ⊠

Now, choose some nonzero morphism $\varphi : \mathscr{O}(a_r) \to \mathscr{E}$.

Claim. φ is an injection.

This involves showing that $\ker(\varphi)$ is locally free (note: this is not true in general), and therefore must be all of $\mathscr{O}(a_r)$ or trivial.

The next step is to show that $\mathscr{F} = \operatorname{coker}(\varphi)$ is locally free. It has rank $r - 1$, so by induction we can decompose it à la (18.9), yielding a short exact sequence

$$0 \longrightarrow \mathscr{O}(a_r) \longrightarrow \mathscr{E} \longrightarrow \mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_{r-1}) \longrightarrow 0.$$

We want to show $a_r \ge a_{r-1}$. First, tensor with $\mathscr{O}(-a_r - 1)$, which is an exact functor, and then take the long exact sequence in cohomology:

$$\cdots \longrightarrow H^0(\mathscr{E}(-a_r - 1)) \longrightarrow H^0(\mathscr{F}(-a_r - 1)) \longrightarrow H^1(\mathscr{O}(-1)) \longrightarrow \cdots$$

[18] Scheme with more than five hypotheses: bingo!

[19] Recall that $h^i(X, \mathscr{F}) = \dim_k H^i(X, \mathscr{F})$.

We chose a_r such that $H^0(\mathscr{E}(-a_r-1))=0$, and we have already calculated $H^1(\mathscr{O}(-1))=0$, so $\mathscr{F}(-a_r-1)$ has no global sections and hence a_r is greater than any of $a_1,\dots,a_{r-1}$.

Turning this into a splitting becomes a problem in linear algebra: the matrix of transition functions is in $\mathrm{GL}_r(k((x)))$ (valued in a ring of Laurent series), and playing with the diagonal finishes the proof. ⊠

Here's a criterion that arises from Chapter 16; it's useful for a result in this section.

Proposition 18.11 (Vakil ex. 16.6.G)**.** *If $\pi : X \to Y$ is a finite morphism of schemes and $\mathscr{L}$ is an ample line bundle on Y, then $\pi^*\mathscr{L}$ is ample on X.*

We've already had lots of criteria for ampleness; here's another using cohomology, at least on Noetherian schemes.

Theorem 18.12 (Serre's criterion for ampleness, Vakil thm. 18.7.1)**.** *Suppose A is a Noetherian ring, X is a proper A-scheme, and $\mathscr{L}$ is an invertible sheaf on X. Then, the following are equivalent.*

(1) $\mathscr{L}$ is ample.
(2) For any coherent sheaf $\mathscr{F}$ on X, there is an n_0 such that for $n \geq n_0$, $H^i(X, \mathscr{F}\otimes\mathscr{L}^{\otimes n})=0$ for all $i>0$.

This can be used to show that a line bundle is ample iff its restriction to the reduced subscheme is ample, or that it's ample iff its restriction to all irreducible components of X is ample. These are not true for very ample line bundles, which suggests why ampleness is better.

Proof sketch: (2) $\implies$ (1). One characterization of ampleness is that for any coherent sheaf $\mathscr{F}$, there's an $n \gg 0$ such that $\mathscr{F}\otimes\mathscr{L}^{\otimes n}$ is globally generated.

First, the strategy is to show this for the structure sheaf (which is coherent over itself, because X is Noetherian). This means we need to show that $\mathscr{L}^{\otimes n}$ is globally generated. It suffices to show that for any closed point p, there's a neighborhood $U_p \subset X$ containing p such that $\mathscr{L}^{\otimes n}$ is globally generated at all $q \in U_p$. We can do this because all quasicompact schemes have closed points, and moreover every closed subset contain a closed point, so these U_p cover X.

If $i : p \hookrightarrow X$ is a closed point, then we have a sheaf of ideals $\mathscr{I}_{p/C}$, and $\mathscr{I}_{p/C}\otimes\mathscr{L}^{\otimes p}$ is coherent (since coherent sheaves form an abelian category). For $n \geq n_0$, $H^i(X, \mathscr{I}_{p/C}\otimes\mathscr{L}^{\otimes n})=0$ for $i>0$, by assumption. Thus, if we take the cohomology long exact sequence associated to the short exact sequence

$$0 \longrightarrow \mathscr{I}_{p/C}\otimes\mathscr{L}^{\otimes n} \longrightarrow \mathscr{L}^{\otimes n} \longrightarrow \mathscr{L}^{\otimes n}|_p \longrightarrow 0$$

tells us that we have a surjection $\Gamma(\mathscr{L}^{\otimes n}) \twoheadrightarrow \Gamma(\mathscr{L}^{\otimes n}|_p)$, i.e. the map to its stalks is surjective, which is exactly the content that $\mathscr{L}^{\otimes n}$ is globally generated at p. By geometric Nakayama's lemma, this is true for all q in a neighborhood of p.

The same argument applies to $n, n+1, \dots, 2n-1$, producing neighborhoods $V_0,\dots,V_{n-1}$, so now for any $m \geq n_0$, it's a tensor product of some list of these $\mathscr{L}^{\otimes n},\dots,\mathscr{L}^{\otimes(2n-1)}$, and therefore is globally generated on $V_0\cap\cdots\cap V_{n-1}$. ⊠

19. Curves: 7/25/16

"This is because $1+1=2\dots$ it's nice to finally have an application of that."

Today, Tom talked about the first five sections of Chapter 19, on applications to curves. Most of it is cobbling together things that we already know, but there are two hard theorems.

Throughout this chapter, all curves are projective, geometrically integral, and geometrically regular (hence geometrically irreducible and equidimensional). And, of course, 1-dimensional.

Definition 19.1. Let k be a field; then, for an adjective A, a k-scheme X has property "geometrically A" if $X \times_{\operatorname{Spec} k} \operatorname{Spec}\overline{K}$ has property A for any algebraically closed field extension $\overline{K}$ of k.[20]

The idea is to apply this to properties that aren't preserved under base change, e.g. **geometrically reduced**, **geometrically irreducible**, **geometrically integral**, and **geometrically regular**.

[20] It's equivalent to define this using base change just to $\operatorname{Spec}\overline{k}$, or to $\operatorname{Spec} K$ for all field extensions K/K.

Facts about projective regular curves. Let C be a projective, regular curve over a field k. Here are some properties of C, from tautological to difficult.

Tier 1. C is Noetherian, proper, and regular.

Tier 2. All stalks are integral domains. C is factorial, which requires dimension 1 (ultimately coming from the miracle of dimension 1 DVRs).

Tier 3. C is a finite disjoint union of its irreducible components, and each irreducible component is integral. (If two irreducible components were in the same connected component, then the stalk at their intersection would not be an integral domain).

Tier 4. C is normal (i.e. admits an affine cover by schemes that are Spec of an integral domain; this is not stalk-local).

Why do we care?

- Normality is the setting in which locally principal Weil divisors are in bijection with the data $(\mathscr{L}, s)$ of a line bundle and a rational section.
- Factoriality means that every divisor is locally principal.
- If $f : C \to C'$ is a finite morphism of Noetherian curves (with no other assumptions), then it is surjective. In our case, a projective morphism with finite fibers is finite, and most of our nonconstant morphisms will have finite fibers.

If you like worrying about the details, these facts are good to keep in mind. If you don't, well, we will be using these facts, so it's good to review them.

Proposition 19.2 (Serre duality for curves)**.** *Let ω_C be the canonical bundle and $\mathscr{F}$ be a coherent sheaf on C. Then, $h^i(C, \mathscr{F}) = h^{1-i}(C, \omega_C \otimes \mathscr{F}^\vee)$ for $i = 0, 1$.*

This is especially nice because, over a curve, the only nonvanishing dimensions would be 0 and 1, and since 0 reduces to global sections, we can calculate cohomology only in terms of global sections of various sheaves.

Proposition 19.3 (Riemann-Roch for curves)**.** *Let $\mathscr{L}$ be a line bundle on C. Then,*

$$h^0(C, \mathscr{L}) - h^0(C, \omega_C \otimes \mathscr{L}^\vee) = \deg \mathscr{L} - g + 1, \tag{19.4}$$

where $g = H^1(C, \mathscr{O}_C)$ is the arithmetic genus.[21]

There are various nice ways to define the canonical bundle in terms of, say, the cotangent bundle, but the point is that it has nice properties and always exists. In particular, we can use Proposition 19.3 to determine some of these properties.

Lemma 19.5. *Let X be an irreducible, projective $\overline{k}$-scheme, then $h^0(X, \mathscr{O}_X) = 1$.*

Proof sketch. The constant functions ensure that $h^0(X, \mathscr{O}_X) \geq 1$, and that a global section s defines a map $\phi : X \to \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$. The map ϕ is projective by the cancellation theorem, hence closed, and so its image is a point, so it corresponds to a constant function. ⊠

This was an exercise in the previous chapter.

Proposition 19.6. *Let X be a k-scheme, $\mathscr{L}$ be a line bundle on X, and K/k be a field extension. Let $L \otimes K$ denote the pullback of $\mathscr{L}$ to $X \times_k K$. Then, $H^0(X, \mathscr{L}) \otimes k \cong H^0(X \times_k K, \mathscr{L} \otimes K)$.*

Corollary 19.7. *For any geometrically irreducible, projective k-scheme X, $h^0(X, \mathscr{O}_X) = 1$.*

One proves this by base changing to $\overline{k}$, then invoking Lemma 19.5 and Proposition 19.6.

Plugging this into (19.4) (with $\mathscr{L} = \mathscr{O}_C$), we recover that $h^0(C, \omega_C) = g$, so it has a basis of g global sections, and plugging in $\mathscr{L} = \omega_C$ implies that $\deg(\omega_C) = 2g - 2$. The divisor associated to this is very useful (the canonical divisor), as we'll see in the differentials chapter.

A lot of things in the past five chapters have been a web of facts about divisors and line bundles. They all begin to play together in a weird way, especially once Serre duality and the Riemann-Roch theorem are added to the mix: you end up with a lot of powerful facts about line bundles, and therefore about the schemes they live on.

Proposition 19.8. *If $\mathscr{L}$ is a line bundle on C with $\deg \mathscr{L} < 0$, then $\mathscr{L}$ has no global sections.*

[21]Over $k = \mathbb{C}$, this is the usual topological genus (the number of holes).

Proof. Suppose s is a global section of $\mathscr{L}$. Then, $\deg \mathscr{L} = \deg(\operatorname{div} s)$; since s has no poles, then $\operatorname{div} s$ is effective, so $\deg(\operatorname{div} s) > 0$. ☒

This is an interesting thing to know about line bundles, and is easy using what we've developed. Next question: when is a degree-0 line bundle trivial?

Proposition 19.9. *Let $\mathscr{L}$ be a degree-0 line bundle. Then, $h^0(C, \mathscr{L})$ is either 0 or 1, and if it's 1, then $\mathscr{L}$ is trivial.*

Proof. We know $h^0(C, \mathscr{O}_C) = 1$. Suppose s is a global section; then, $\deg(\mathscr{L}) = \deg(\operatorname{div} s) = 0$, so s has no zeros and no poles (since it's regular). Hence, it trivializes $\mathscr{L}$, defining an isomorphism $\mathscr{L} \cong \mathscr{O}_C$ sending $s' \mapsto s'/s$. ☒

Okay, what about positive-degree line bundles? It's harder to say anything concrete here, except that Riemann-Roch and Serre duality tell us facts about line bundles of high degree. If we don't really understand $\mathscr{L}$, it'll be hard to understand $\omega_C \otimes \mathscr{L}^\vee$, but if $\mathscr{L}$ has large degree, $\mathscr{L}^\vee$ has large negative degree.

Proposition 19.10. *Suppose $\mathscr{L}$ is a line bundle on C and $\deg \mathscr{L} > \deg(\omega_C) = 2g - 2$. Then, $h^0(C, \mathscr{L}) = \deg(\mathscr{L}) - g + 1$.*

This follows from Proposition 19.3. This also tells us about the degree of the embedding $C \hookrightarrow \mathbb{P}^n$ induced from this line bundle, so degree 2 line bundles correspond to conics, which is pretty cool.

We can use this to turn facts about trivial line bundles into facts about line bundles of degree $2g - 2$.

Corollary 19.11. *If $\deg \mathscr{L} = 2g - 2$, then $h^0(C, \mathscr{L})$ is either g or $g - 1$, and if it's g, then $\mathscr{L} \cong \omega_C$.*

This is a combination of Propositions 19.10 and 19.9.

So now the only line bundles we don't understand have degree bounded by $0 < \deg \mathscr{L} < 2g - 2$. That's pretty good.

This is the big theorem that allows you to classify curves.

Theorem 19.12. *Let k be algebraically closed and $\pi : X \to Y$ be a projective morphism of finite type k-schemes. Then, π is a closed embedding iff*

(1) π is injective on points, and
(2) π is injective on tangent vectors at closed points.

Finiteness implies that the morphism is closed, which is good: we want closed points to map to closed points. That k is algebraically closed ensures all closed points have the same residue field, so we can dualize the cotangent space map $\mathfrak{m}_{\pi(p)}/\mathfrak{m}_{\pi(p)}^2 \to \mathfrak{m}_p/\mathfrak{m}_p^2$ over k. One often refers to the first condition as π **separates (closed) points**, and the second as π **separates tangent vectors**, particularly when discussing curves.

One key technique we've used already is to understand line bundles by inductively removing a point from their associated divisors. If p is a closed point of C, we have a closed subscheme short exact sequence

$$0 \longrightarrow \mathscr{O}_C(-p) \longrightarrow \mathscr{O}_C \longrightarrow \mathscr{O}_C|_p \longrightarrow 0, \tag{19.13}$$

though this hinges on an identification $\mathscr{O}_C(-p)$ with the sheaf of ideals; indeed, both of these capture the functions that vanish at p.

Remark. Recall that if D is a divisor on C and $U \subset C$ is open, then $\mathscr{O}_C(D)(U)$ is the space of rational sections f on U such that $\operatorname{div}|_U(f) + D|_U \geq 0$, and similarly with $\mathscr{L}(D)(U)$ (replaced with rational sections of $\mathscr{L}$). This is a little more fiddly when C isn't irreducible.

Now, apply $- \otimes_{\mathscr{O}_C} \mathscr{L}$ to (19.13). Since $\mathscr{L}$ is locally free, it's still exact:

$$0 \longrightarrow \mathscr{L}(-p) \longrightarrow \mathscr{L} \longrightarrow \mathscr{L}|_p \longrightarrow 0.$$

Now we apply the cohomology long exact sequence:

$$0 \longrightarrow H^0(C, \mathscr{L}(-p)) \longrightarrow H^0(C, \mathscr{L}) \xrightarrow{\alpha} H^0(C, \mathscr{L}|_p) \longrightarrow \cdots$$

We know that $H^0(C, \mathscr{L}|_p) \cong k^{\deg p}$, so if $\deg p = 1$, then $h^0(C, \mathscr{L}(-p))$ is either equal to $h^0(C, \mathscr{L})$ or $h^0(C, \mathscr{L}) - 1$, and we know which if we can tell whether α is the zero map. Doing this iteratively, one can learn a lot about $\mathscr{L}$, and if k is algebraically closed, all closed points are degree 1, which simplifies one's life somewhat. That is:

Corollary 19.14. *If k is algebraically closed and $p \in C$ is a closed point, then*

$$h^0(C, \mathscr{L}) - h^0(C, \mathscr{L}(-p)) = 1.$$

Since sections of $\mathscr{L}(-p)$ are the sections that vanish at p, this further implies

Corollary 19.15. *For any closed point $p \in C$, there exists a section $s \in \Gamma(C, \mathscr{L})$ that does not vanish at p. In particular, $\mathscr{L}$ is basepoint-free.*

If we try this twice, say for two closed points $p, q \in C$, then $h^0(C, \mathscr{L}) - h^0(C, \mathscr{L}(-p-q))$ is either 0, 1, or 2. If it's 2, then it must have decreased by 1 at each step; in particular,

$$h^0(C, \mathscr{L}) = h^0(C, \mathscr{L}(-p)) + 1 = h^0(C, \mathscr{L}(-q)) + 1 = h^0(C, \mathscr{L}(-p-q)) + 2,$$

so there's a section that vanishes at q but not p, and vice versa. If $p = q$, we get that there's a section of $\mathscr{L}$ that vanishes on p, but only to order 1.

All this leads to a great criterion for $\mathscr{L}$ to be very ample, which we'll use over and over.

Theorem 19.16. *Let $\mathscr{L}$ be a line bundle on C over an algebraically closed field k. Then $\mathscr{L}$ is very ample iff for all closed points $p, q \in C$, $h^0(C, \mathscr{L}) - h^0(C, \mathscr{L}(-p-q)) = 2$.*

20. Elliptic and Hyperelliptic Curves: 7/28/16

These are Arun's notes on §19.5 and §19.9, on hyperelliptic and elliptic curves, including the hyperelliptic Riemann-Hurwitz formula, the j-invariant, and the abelian group structure on an elliptic curve.

Throughout this lecture, all curves are projective, geometrically integral, and regular over a field k.

Hyperelliptic curves. In this section, we assume k is algebraically closed, and that $\mathrm{char}(k) \neq 2$.

Definition 20.1. A curve of genus g is **hyperelliptic** if it admits a **double cover**, meaning a degree 2, hence finite, morphism π to $\mathbb{P}^1_k$. π is called the **hyperelliptic map**.

This is not a double cover in the topological sense: closed points of $\mathbb{P}^1_k$ have either 1 or 2 preimages. They are called **branch points** and **ramification points** of π, respectively.

Theorem 20.2 (Hyperelliptic Riemann-Hurwitz formula, Vakil thm. 19.5.1)**.** *Let $\pi : C \to \mathbb{P}^1_k$ be a hyperelliptic map, where C has genus g. Then, π has $2g+2$ branch points.*

There is a more general Riemann-Hurwitz formula; there's also one for Riemann surfaces.

To prove this, we'll need the following.

Proposition 20.3 (Vakil prop. 19.5.2)**.** *Given r distinct points $p_1, \dots, p_r \in \mathbb{P}^1_k$, there is exactly one double cover of $\mathbb{P}^1_k$ branched at those points if r is even, and none if r is odd.*

Proof. Choose points 0 and ∞ for $\mathbb{P}^1_k$ different from the branch points (we can do this because k is algebraically closed, hence infinite). Thus, all of the branch points are in $\mathbb{A}^1_k = \mathbb{P}^1_k \setminus \infty$. If $C' \to \mathbb{A}^1_k = \operatorname{Spec} k[x]$ is a double cover, it induces a quadratic field extension $K/k(x)$.[22] Since $\mathrm{char}\, k \neq 2$, then $K/k(x)$ is Galois with Galois group generated by an involution $\sigma : K \to K$. As a $k(x)$-linear map, this has -1 for an eigenvalue, hence an eigenvector $y \in K$, so $\{1, y\}$ is an eigenbasis for K as a $k(x)$-vector space. Since $\sigma(y^2) = y^2$, then $y^2 \in k(x)$, and without loss of generality (i.e. after multiplying by a suitable rational function), y^2 is a monic polynomial $f(x)$ with no repeated factors.[23]

Let $C'_0 = V(y^2 - f(x)) \subset \operatorname{Spec} k[x, y] = \mathbb{A}^2$. Since $f(x)$ has no repeated roots, then the Jacobian criterion says C'_0 is regular, hence normal, and has the same function field as C'. Thus, C'_0 and C' are both normalizations of $\mathbb{A}^1$ in the extension generated by y, so must be isomorphic. This means we've identified the cover with an explicit equation; the branch points are exactly those where $y^2 = f(x)$ has only one value, i.e. the roots of $f(x)$, so $\deg(f) = r$ and in fact $f(x) = (x - p_1) \cdots (x - p_r)$ (identifying p_i with the maximal ideal $(x - p_i) \subset k[x]$).

[22] Something really cool is going on here. Most people who know Galois theory and covering space theory know that they behave in very similar ways, but here, they are *actually the same*: the covering space theory of the schemes is the Galois theory of their function fields.

[23] Here, we use the hypothesis that k is algebraically closed to ensure that y^2 is monic.

Now, let's return to $\mathbb{P}^1$, by examining C over $\mathbb{P}^1 \setminus 0 = \operatorname{Spec} k[u]$ (so $u = 1/x$). The previous argument shows that this double cover must be of the form

$$\begin{aligned} C'' &= \operatorname{Spec} k[z,u]/(z^2 - (u - 1/p_1)\cdots(u - 1/p_r)) \\ &= \operatorname{Spec} k[z,u]/(z^2 - p_1 p_2 \cdots p_r \cdot u^r f(1/u)). \end{aligned}$$

To have a double cover over $\mathbb{P}^1$, we need to glue this to C', over the gluing $u = 1/x$. That is, in $K(C)$, we would need

$$z^2 = u^r f(1/u) = f(x)/x^r = y^2/x^r.$$

If r is even, we have two possible choices for z, $\pm y/x^{r/2}$, but these describe the same gluing data, so there's exactly one way to produce a branched cover.

Suppose r is odd.

Lemma 20.4 (Vakil ex. 19.5.A)**.** *If $\operatorname{char}(k) \neq 2$ and $f \in k[x]$ is a polynomial with nonzero roots $p_1, \ldots, p_r$, then x doesn't have a square root in the field $L = k(x)[y]/(y^2 - f(x))$.*

Proof. We're trying to adjoin both $\sqrt{x}$ and $\sqrt{f(x)}$ over $k(x)$. However, $xf(x) \notin (k(x))^2$, since its root at 0 has multiplicity 1, so $\sqrt{xf(x)} \notin k(x)$. Since we're away from characteristic 2, this implies that $[k(x, \sqrt{x}, \sqrt{f(x)}) : k(x)] = 4$, so $[L(\sqrt{x}) : L] = 2$, meaning $\sqrt{x} \notin L$. ⊠

Using this, there is no z such that $z^2 = y^2/x^r$ for r odd: if there were, then $x^{\lfloor r/2 \rfloor} y/z$ would be a square root of x; thus, there are no branched covers when r is odd. ⊠

When r is even, this explicit description of the hyperelliptic cover can be summarized in the diagram

$$\begin{array}{ccc} \operatorname{Spec} k[x,y]/(y^2 - f(x)) & \xleftrightarrow[y = z/u^{r/2}]{z = y/x^{r/2}} & \operatorname{Spec} k[u,z]/(z^2 - u^r f(1/u)) \\ \downarrow & & \downarrow \\ \operatorname{Spec} k[x] & \xleftrightarrow[x=1/u]{u=1/x} & \operatorname{Spec} k[u]. \end{array} \tag{20.5}$$

Proof of Theorem 20.2. We'll use (20.5) to compute that the genus of the unique cover branched over r points is $r/2 - 1$. In particular, we continue to use the notation from the proof of Proposition 20.3 and diagram (20.5).

The two schemes in the top row of (20.5) are an affine cover for C whose Čech complex for $\mathscr{O}_C$ is

$$0 \longrightarrow k[x,y](y^2 - f(x)) \times k[u,z]/(z^2 - u^r f(1/u)) \xrightarrow{d} (k[x,y]/(y^2 - f(x)))_x \longrightarrow 0.$$

The monomials $x^n y^\varepsilon$, with $n \in \mathbb{Z}$ and $\varepsilon \in \{0,1\}$, form a basis for the second nonzero term. We'd like to compute the genus $g = h^1(C, \mathscr{O}_C)$, and therefore must understand $\operatorname{coker} d$; the first factor hits all monomials of the form $x^n y^\varepsilon$ for $n \geq 0$, and the image of the second factor is generated by those of the form $u^m z^\varepsilon$ with $m \geq 0$. Here, $u^m z^\varepsilon = x^{-m}(y/x^{r/2})^\varepsilon$, so the only monomials not in $\operatorname{Im}(d)$ are $x^{-1}y, x^{-2}y, \ldots, x^{-r/2+1}y$, so $h^1(C, \mathscr{O}_C) = r/2 - 1$. ⊠

Corollary 20.6. *If $g \geq 0$, there is a hyperelliptic curve of genus g over k, since we can pick the cover branched over $2g + 2$ distinct points.*

There is a sense in which most curves of genus greater than 2 aren't hyperelliptic, which is beyond the scope of this chapter.

Proposition 20.7 (Vakil prop.19.5.6)**.** *If $C \to \mathbb{P}^1_k$ is a hyperelliptic curve of genus $g \geq 2$, then $\mathscr{L}^{\otimes(g-1)} \cong \omega_C$.*

Proof. We can compose the hyperelliptic map with the $(g-1)^{\text{th}}$ Veronese embedding $|\mathscr{O}_{\mathbb{P}^1}(g-1)| : \mathbb{P}^1 \to \mathbb{P}^{g-1}$ to obtain the map $C \to \mathbb{P}^{g-1}_k$ corresponding to $\mathscr{L}^{\otimes(g-1)}$. Since $\deg(\mathscr{L}) = 2$, then $\deg(\mathscr{L}^{\otimes(g-1)}) = 2g-2$, so by exercise 19.2.A (Corollary), it has either $g-1$ or g (linearly independent) sections, and if it has g sections, it's isomorphic to the canonical bundle ω_C.

The pullback map on global sections $H^0(\mathbb{P}^{g-1}_k, \mathscr{O}(1)) \to H^0(C, \mathscr{L}^{\otimes(g-1)})$ is injective: sections of $\mathscr{O}(1)$ describe hyperplanes, so if a section is pulled back to 0, that would mean the image of C is contained in that hyperplane, but this is a rational normal curve in $\mathbb{P}^{g-1}_k$, so cannot be contained in any hyperplane. Since $h^0(\mathbb{P}^{g-1}_k, \mathscr{O}(1)) = g$, then $\mathscr{L}^{\otimes(g-1)}$ has at least g linearly independent sections, so must be ω_C. ⊠

Proposition 20.8 (Vakil prop. 19.5.7)**.** *Any curve C of genus at least 2 admits at most one double cover of $\mathbb{P}^1_k$. That is, if $\mathscr{L}$ and $\mathscr{M}$ are degree one line bundles yielding maps $C \to \mathbb{P}^1_k$, then $\mathscr{L} \cong \mathscr{M}$.*

Proof. The canonical bundle ω_C induces a map $|\omega_C| : C \to \mathbb{P}^{g-1}$, called the **canonical map**.[24] By Proposition 20.7, this is a double cover of the image of the $(g-1)^{\text{th}}$ Veronese map $\mathbb{P}^1 \to \mathbb{P}^{g-1}$, which is isomorphic to $\mathbb{P}^1$, and we saw that this factors as the hyperelliptic map for $\mathscr{L}$ composed with the Veronese map, and also as the hyperelliptic map for $\mathscr{M}$ composed with the Veronese map. Since the Veronese map is a closed embedding, this means the maps induced by $\mathscr{L}$ and $\mathscr{M}$ are the same, so $\mathscr{L} \cong \mathscr{M}$. ⊠

Proposition 20.9 (Vakil ex. 19.5.B)**.** *A curve C of genus at least 1 is hyperelliptic iff it has a degree 2 invertible sheaf $\mathscr{L}$ with $h^0(C, \mathscr{L}) = 2$.*

Elliptic curves.

Line bundles of degree 0. Suppose C is a genus 1 curve. Then, $\deg \omega_C = 2 - 2 = 0$ and $h^0(C, \omega_C) = g = 1$. Thus, ω_C is a degree 0 line bundle with a nonzero section s. This section has no poles, and must have the same number of zeros, so it's a nonvanishing section of ω_C, which therefore trivializes it: $\omega_C \cong \mathscr{O}_C$.

Line bundles of degree 1. Suppose $q \in C$ is a degree 1, k-valued point, so it determines a degree 1 line bundle $\mathscr{O}(q)$. This assignment is bijective: distinct points determine distinct line bundles (since the space of sections of $\mathscr{O}(q)$ is one-dimensional, and different points would define linearly independent sections), and every degree 1 line bundle has a section whose divisor of zeros corresponds to a single point.

Definition 20.10. An **elliptic curve** (E, p) is a genus 1 curve E with a choice of a k-valued point $p \in E$.

The choice of this point is part of the definition — elliptic curves are not the same as genus 1 curves. The choice of basepoint gives us a canonical bijection between the set of isomorphism classes of degree 0 invertible sheaves on E and the set of degree 1 points of E, defined by sending $\mathscr{L} \mapsto \operatorname{div}(\mathscr{L}(p))$, and in the other direction, $q \mapsto \mathscr{O}_E(q - p)$. Now, the degree 0 invertible sheaves are the abelian group $\operatorname{Pic}^0(E)$, meaning:

Proposition 20.11 (Vakil prop. 19.9.3)**.** *This bijection defines an abelian group structure on the degree 1 points of an elliptic curve, with p as the identity.*

In fact, something better is true: elliptic curves are abelian varieties.

Theorem 20.12 (Vakil thm. 19.10.4)**.** *If $(E.p)$ is an elliptic curve, then the multiplication and inversion maps induced from $\operatorname{Pic}^0(E)$ extend to regular maps giving E the structure of an abelian variety.*

The proof is long (though shorter if you only care about regular curves over algebraically closed fields), so I'll omit it. If you want to read it, it's the bulk of §19.10.

Remark.

- This suggests that there's a way to turn $\operatorname{Pic}^0(E)$ into a scheme making the bijection we defined into an isomorphism of schemes. This is true, and in fact the Picard scheme can be defined for any projective variety, but this is beyond the scope of our discussion.
- The Mordell-Weil theorem states that if E is an elliptic curve over $\mathbb{Q}$, the $\mathbb{Q}$-points of E are a finitely generated abelian group, sometimes called the **Mordell-Weil group**.

Line bundles of degree 2. In this (sub)section, we assume that k is algebraically closed and $\operatorname{char}(k) \neq 2$ in order to use Theorem 20.2.

Since $h^0(E, \mathscr{O}(p)) = 2$, then by Proposition 20.9, E is hyperelliptic. p is a ramification point, because one of the sections vanishes to order 2 at p, and by Theorem 20.2, there are 4 total ramification points. Conversely, for any four points of $\mathbb{P}^1$, there's a unique double cover branched at those points, by Proposition 20.3. Thus, elliptic curves correspond to sets of four points in $\mathbb{P}^1$ up to automorphisms of $\mathbb{P}^1$ (or fixing p at ∞, they correspond to sets of three points in $\mathbb{A}^1$ up to maps $x \mapsto ax + b$).

Let p be the image of p under this double cover, and call the other branch points q_1, q_2, and q_3.

Definition 20.13. Since $\mathrm{PGL}_2(k) = \operatorname{Aut} \mathbb{P}^1_k$ is 3-transitive, there's a unique automorphism $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ sending $(p, q_1, q_2) \mapsto (\infty, 0, 1)$. The **cross ratio** of (p, q_1, q_2, q_3) is $\lambda = \varphi(q_3)$.

[24] This is true for all curves we consider in this chapter, and is a useful tool to have.

Proposition 20.14 (Vakil ex. 19.9.B)**.** *The cross ratio classifies sets of four ordered distinct points in $\mathbb{P}^1$, i.e. two sets S and S' of four (ordered, distinct) points have the same cross ratio iff there is an automorphism of $\mathbb{P}^1$ carrying S to S'.*

This can be jazzed up into the fact that $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ (i.e. the image of the cross ratio) is the moduli space of sets of four ordered, distinct points in $\mathbb{P}^1$ up to automorphisms of $\mathbb{P}^1$.

What happens if we permute q_1, q_2, and q_3? The elliptic curve E is determined by p and the *unordered* set of branch points $\{q_1, q_2, q_3\}$, so it doesn't change. But the cross ratio does:

- The automorphism sending $(p, q_2, q_1) \mapsto (\infty, 0, 1)$ sends $q_3 \mapsto 1 - \lambda$.
- The automorphism sending $(p, q_1, q_3) \mapsto (\infty, 0, 1)$ sends $q_2 \mapsto 1/\lambda$.
- The automorphism sending $(p, q_3, q_1) \mapsto (\infty, 0, 1)$ sends $q_2 \mapsto (\lambda - 1)/\lambda$.
- The automorphism sending $(p, q_2, q_3) \mapsto (\infty, 0, 1)$ sends $q_1 \mapsto 1/(1 - \lambda)$.
- The automorphism sending $(p, q_3, q_2) \mapsto (\infty, 0, 1)$ sends $q_1 \mapsto \lambda/(\lambda - 1)$.

So the cross ratio is not the invariant you are looking for.[25] However, it's not useless: given two elliptic curves (E, p) and (E', p') over k, they're isomorphic iff the cross ratios given by writing them as branched covers of $\mathbb{P}^1$ are related by sending λ to one of the five above values.

We'd like to do better, finding an invariant j such that $j(\lambda) = j(\lambda')$ iff λ and λ' are related by one of these six expressions. If it's algebraic (which would be good), it would define a map from the **λ-line** $\mathbb{A}^1 \setminus \{0, 1\}$ to $\mathbb{A}^1$, and hence by the Curve-to-Projective extension theorem, would extend to a map $j : \mathbb{P}^1 \to \mathbb{P}^1$. This is generically $6 : 1$, so we would expect $\deg(j) = 6$.

Alternatively, to every permutation $\sigma \in S_3$, we've associated a map sending λ to the cross ratio of $(p, q_{\sigma(1)}, q_{\sigma(2)}, q_{\sigma(3)})$, which includes S_3 as a subgroup of $\mathrm{Gal}(k(\lambda)/k)$. Let K be its fixed field, so that $k(\lambda)/K$ is Galois of degree 6. We'd like for $K = k(j)$ for some rational function $j : \mathbb{P}^1 \to \mathbb{P}^1$.

To produce something S_3-invariant, you might try multiplying or adding the six images of λ, but these produce constants. Summing their squares or taking the second symmetric function both produce valid j-invariants. The formula that everyone uses is

$$j(\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2 (\lambda - 1)^2}.$$

Line bundles of degree 3*.* In this (sub)section, we remove the assumptions on k unless stated explicitly.

Consider the degree 3 line bundle $\mathscr{O}_E(3p)$. The Riemann-Roch formula for curves shows that $h^0(E, \mathscr{O}_E(3p)) = \deg(3p) - g + 1 = 3$. Thus, $\deg E > 2g$, so by one of our earlier remarks about curves, $\mathscr{O}_E(3p)$ defines a closed embedding $E \hookrightarrow \mathbb{P}^2_k$ as a cubic. $\mathscr{O}_E(3p)$ has a section vanishing to order 3 at p, and this section corresponds to a line in $\mathbb{P}^2$ meeting E at p with multiplicity 3.[26]

We can choose projective coordinates for $\mathbb{P}^2$ such that $p \mapsto [0, 1, 0]$ and the flex line is the line at infinity, $z = 0$. These force some terms to be 0: the cubic is of the form

$$?x^3 + 0x^2y + 0xy^2 + 0y^3 + ?x^2z + ?xyz + ?y^2z + ?xz^2 + ?yz^2 + ?z^3 = 0.$$

The coefficient of x^3 isn't 0, or this would be divisible by z and hence not irreducible. Thus, we can rescale it to 1. The coefficient of y^2z isn't 0, since the cubic is regular at $[0, 1, 0]$; we can scale z such that this coefficient is -1. If $\mathrm{char}(k) \neq 2$, a transformation $y \mapsto y + ?x + ?z$ will set the xyz and yz^2 terms to 0, and if $\mathrm{char}(k) \neq 3$, a transformation $x \mapsto x + ?z$ will kill x^2z. Thus, if $\mathrm{char}(k) \neq 2, 3$, the elliptic curve can be written in the form

$$y^2z = x^3 + axz^2 + bz^3.$$

This is called the **Weierstrass normal form** of (E, p).

This equation is symmetric under the involution $y \mapsto -y$, which allows us to describe the hyperelliptic structure geometrically as projection onto the x-axis. More precisely, over the distinguished open where $z \neq 0$, we can dehomogenize by setting $z = 1$, so $E \setminus p$ is contained in this copy of $\mathbb{A}^2$ (geometrically, the identity is the "point at infinity"). Projecting onto the x-axis defined a morphism $E \setminus p \to \mathbb{A}^1$, which, using the Curve-to-Projective extension theorem, extends to a degree 2 morphism $E \to \mathbb{P}^1$. This also shows where

[25] This sentence best accompanied by a Jedi hand wave (not the same as a mathematical hand wave).

[26] Such a line is called a **flex line**, and the point of tangency is called a **flex point**.

the branch points are if k is algebraically closed: p is the point at infinity for $\mathbb{P}^1$, and the other three are the three places where the curve intersects the x-axis, the solutions to $0 = x^3 + ax + b$.[27]

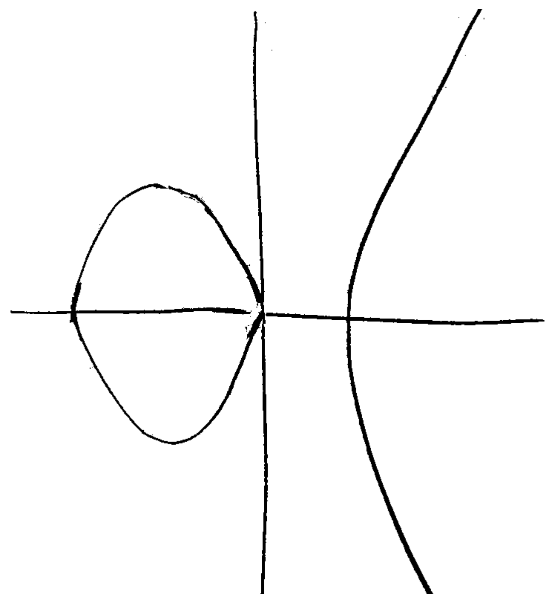

FIGURE 1. An elliptic curve over $\mathbb{R}$ in Weierstrass normal form. The hyperelliptic map extends from projection onto the x-axis, and the branch points are p (the point at infinity) and the three points on the x-axis.

This picture also helps us describe the group law.[28] Suppose a line intersects E at three points p_1, p_2, and p_3; this determines an isomorphism $\mathscr{O}_E(p_1 + p_2 + p_3) \cong \mathscr{O}_E(3p)$, and the tautological section of the latter sheaf corresponds to line at infinity $z = 0$. However, in Weierstrass normal form, the origin p is the only point intersecting this line at infinity, so the group law is $p_1 + p_2 + p_3 = 0$.

Thus, if $q \in E$, $-q$ is the reflection of q across the y-axis: the line through q and $-q$ also hits p at infinity, so $q + (-q) + 0 = 0$ as desired. Addition is by the **chord and tangent method**: any two points q_1 and q_2 determine a line which intersects E at a third point q_3, so $q_1 + q_2 + q_3 = 0$, and therefore $q_1 + q_2 = -q_3$: pictorially, draw the line connecting q_1 and q_2, find its third point of intersection, and reflect. If you interpret intersections with multiplicity, this still works when $q_1 = q_2$, showing that the 2-torsion points are exactly those where the tangent line is vertical, which are exactly the four branch points.[29]

Though we used the Weierstrass normal form to guide our intuition, the chord-and-tangent method works just as well over characteristics 2 and 3: all we need is for $\mathscr{O}_E(3p)$ to induce a closed embedding into $\mathbb{P}^2$.

[27] If you're drawing a picture, keep in mind that $\mathbb{R}$ is not algebraically closed, so not all elliptic curves over $\mathbb{R}$ intersect the x-axis thrice.

[28] Curiously, the explicit geometric description is used in the abstract proof of Theorem 20.12.

[29] Classically, this is often how the group law on an elliptic curve is defined, but then it's a chore to show that addition is associative.