Lecture Notes on Sheaves, Stacks, and Higher Stacks

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Sheaves

A category is a world of objects, all looking at each other. Who you are in a category is completely determined by how you are viewed by all objects, including yourself. This is made precise by Yoneda’s lemma. Let $\mathcal{C}$ be a (small) category, and denote by $\hat{\mathcal{C}} := \text{Cat}(\mathcal{C}^{\text{op}}, \text{Set})$ the category of presheaves on $\mathcal{C}$.

**Theorem** (Yoneda’s Lemma). For any $F \in \hat{\mathcal{C}}$ and $X \in \mathcal{C}$ the map which assigns to each $x \in F(X)$ the natural transformation

$$\left\{ \begin{array}{c} \mathcal{C}(Y, X) \to F(Y) \\ f \mapsto F(f)(x) \end{array} \right\}_{Y \in \mathcal{C}}$$

is a bijection

$$F(X) \cong \mathcal{C}^\wedge(\mathcal{C}(\cdot, X), F),$$

with the inverse given by sending any natural transformation $\eta : \mathcal{C}(\cdot, X) \to F$ to $\eta_X(id_X)$.

As a corollary we obtain that the functor $\mathcal{C} \to \hat{\mathcal{C}}$ sending any object $X \in \mathcal{C}$ to the presheaf $Y \mapsto \mathcal{C}(Y, X)$ is fully faithful; we therefore identify objects in $\mathcal{C}$ by the presheaves they represent. Now, Yoneda’s lemma tells us even more; it tells us that, in addition to the objects in $\mathcal{C}$, all other presheaves on $\mathcal{C}$ are determined by how they are seen by objects in $\mathcal{C}$. This parallels how, when we glue together some simple geometric objects to more complicated geometric objects, these complicated geometric objects are completely determined by how they are seen by the simple ones:

**Example.** Denote by $\text{Cart}_\infty$ the category of open subsets of Euclidean space and smooth maps, and by $\text{Man}_\infty$ the category of smooth manifold and smooth maps, then the embedding $\text{Cart}_\infty \hookrightarrow \text{Man}_\infty$ induces a functor $\text{Man}_\infty \to \text{Cart}_\infty$ by precomposition; composing the Yoneda embedding $\text{Man}_\infty \hookrightarrow \text{Man}_\infty$ with $\text{Man}_\infty \to \text{Cart}_\infty$, we obtain a functor $\text{Man}_\infty \to \text{Cart}_\infty$, which can again be shown to be fully faithful.

Similarly, let $k$ be a commutative ring; denote by $\text{Aff}_k$ the category of affine $k$-schemes, and by $\text{Sch}_k$ the category $k$-schemes, then by composing the Yoneda embedding $\text{Sch}_k \hookrightarrow \text{Sch}_k$ with the functor $\text{Sch}_k \to \text{Aff}_k$, we obtain a fully faithful functor $\text{Sch}_k \to \text{Aff}_k$.

Further pursuing this line of thought, note that glueing just means taking certain colimits, and that all presheaves on $\mathcal{C}$ are obtained as colimits of objects in $\mathcal{C}$ (see Theorem 1.1.1.1);
even better, \( \widehat{\mathcal{C}} \) is the universal cocomplete category generated under colimits by objects in \( \mathcal{C} \) (see Corollary 2.4.0.2). Thus, if we think of the objects in \( \mathcal{C} \) as simple or primitive geometric objects that we would like to glue together to more complicated ones, then it is reasonable to assume these more complicated objects should live in \( \widehat{\mathcal{C}} \).

Unfortunately, the category \( \widehat{\mathcal{C}} \) suffers from a serious defect: Usually we already know how to obtain simple objects by glueing together other simple objects; e.g. in \( \text{Cart}_\infty \) an open interval may be obtained by glueing together two smaller overlapping intervals. More generally, when we say that we are glueing together an object \( U \in \mathcal{C} \) from objects \( U_i \rightarrow U \) in \( \mathcal{C}_U \), we typically mean that \( U \) is the colimit of the diagram \( \coprod_{i,j} U_i \times_U U_j \Rightarrow \coprod_i U_i \) (think of glueing together an open subset of Euclidean space from an atlas, or glueing together affine schemes from open affine subschemes). The datum specifying which such diagrams should constitute glueing diagrams is essentially what constitutes a Grothendieck topology. In \( \widehat{\mathcal{C}} \) the diagram \( \coprod_{i,j} \mathcal{C}(_-, U_i \times_U U_j) \Rightarrow \coprod_i \mathcal{C}(_-, U_i) \rightarrow \mathcal{C}(_-, U) \) is however no longer a colimit in all but the most trivial cases.

There are at least two ways in which we could try to remedy these defects: We could localise \( \widehat{\mathcal{C}} \) along a suitable class of morphisms, forcing the cocones specified by the Grothendieck topology to become colimits. Dually, we could try to restrict to the full subcategory of \( \widehat{\mathcal{C}} \) spanned by objects to which the cocones in question (co-)look like colimits; in detail, let \( F \) be a presheaf on \( \mathcal{C} \), then the diagram

\[
\widehat{\mathcal{C}} \left( \coprod_{i,j} \mathcal{C}(_-, U_i \times_U U_j), F \right) \Rightarrow \widehat{\mathcal{C}} \left( \coprod_i \mathcal{C}(_-, U_i), F \right) \leftarrow \widehat{\mathcal{C}} \left( \mathcal{C}(_-, U), F \right)
\]

is isomorphic to

\[
\coprod_{i,j} F(U_i \times_U U_j) \Rightarrow \coprod_i F(U_i) \leftarrow F(U)
\]

and we could consider the full subcategory of \( \widehat{\mathcal{C}} \) consisting of presheaves \( F \) such that the above two isomorphic diagrams are limits. This is the subcategory of sheaves on \( \mathcal{C} \) and is denoted by \( \widetilde{\mathcal{C}} \). Amazingly, these two notions coincide!

The inclusion \( \mathcal{C} \hookrightarrow \widetilde{\mathcal{C}} \) has a left adjoint, and this is exactly the localisation functor suggested as the first possible remedy. Mercifully, the category \( \widetilde{\mathcal{C}} \) is still cocomplete, and is moreover universal with respect to being generated under colimits by objects in \( \mathcal{C} \), while keeping the colimits of the form discussed above (see Theorem 2.5.0.1). We have thus constructed the universal “container category” for any objects one might like to glue together from objects in \( \mathcal{C} \) according to the rules specified by the Grothendieck topology. More classically, e.g. in the case of schemes, one considers more ad hoc container categories, such as locally ringed spaces. Also one can show that iterating this process does not produce any new spaces; there is a natural way of extending the Grothendieck topology on \( \mathcal{C} \) to \( \widehat{\mathcal{C}} \), and the colimits obtained from objects \( \widetilde{\mathcal{C}} \) can already be obtained as colimits of objects in \( \mathcal{C} \) (see §1.3).

**Example.** Most of the known generalisations of smooth manifolds are sheaves on \( \text{Cart}_\infty \). In order of inclusion we have: smooth manifolds \( \subset \) Fréchet manifolds \( \subset \) Frölicher spaces \( \subset \) diffeological spaces \( \subset \) sheaves on \( \text{Cart}_\infty \). Moving to the right, the objects we consider become less and less tangible, while the formal properties of the categories they span become increasingly good.
Chapter 1

Grothendieck Topologies, and Sheaves: Definitions and the Closure Property of Sheaves - 2.9.2016 - Neža Zager

Throughout this chapter $\mathfrak{C}$ denotes a category, which can usually be assumed to be (essentially) small.

In §1.1 we motivate and state various equivalent definitions of Grothendieck topologies and (categories of) sheaves. We would like to think of $\mathfrak{C}$ as a category of simple or primitive spaces, which we hope to glue together to more complicated ones. As explained in the introduction, these new spaces should be determined by how they are seen by the objects in $\mathfrak{C}$; by Yoneda’s lemma this is true for objects in $\hat{\mathfrak{C}}$, and the examples considered in the introduction show that manifolds (and several generalisations thereof), and schemes embed naturally into $\text{Cart}_\infty$ and $\hat{\text{Aff}}_k$ respectively. In §1.1.1 we give a more useful reformulation of how objects in $\hat{\mathfrak{C}}$ are determined by how they are seen by objects in $\mathfrak{C}$. The subsections §1.1.2 and §1.1.3 then provide the heart of §1.1: We study how to glue together objects in $\mathfrak{C}$ from other objects which cover them, but realise that the colimits involved in formalising this notion of glueing do not commute with the Yoneda embedding $\mathfrak{C} \hookrightarrow \hat{\mathfrak{C}}$. We can attempt to fix this problem by either formally inverting a certain class of morphisms in $\hat{\mathfrak{C}}$, or we can restrict to the full subcategory $\tilde{\mathfrak{C}}$ spanned by objects in $\hat{\mathfrak{C}}$ to which the class of morphisms under consideration already look invertible. In §1.1.3 we will see that the inclusion $\tilde{\mathfrak{C}} \hookrightarrow \hat{\mathfrak{C}}$ has a left adjoint, which is exactly the localisation functor alluded to above, so that the two proposed fixes are equivalent. In §§1.1.5 - 1.1.7 we are naturally led to consider several equivalent notions of Grothendieck topologies\(^1\). We finish section §1.1 with some examples.

In §1.2 we discuss two important formal properties of categories of sheaves: Suitable subcategories of $\hat{\mathfrak{C}}$ are in bijection with Grothendieck topologies on $\mathfrak{C}$, and these subcategories inherit Cartesian closedness from $\hat{\mathfrak{C}}$.

Finally, in §1.3 we make precise the notion that spaces that can be glued together from sheaves on $\mathfrak{C}$ can already be glued together from objects in $\mathfrak{C}$ and give a proof thereof.

\(^1\)The proof that they are indeed equivalent will hopefully be added at some future point.
1.1 Grothendieck topologies and sheaves

1.1.1 Presheaves

Yoneda’s lemma tells us among many other things that presheaves on $C$ are determined by how they are seen by objects in $C$. We re-express this fact in terms of colimits.

**Theorem 1.1.1.1.** [Mac98, Th. I.7.1] Let $F$ be a presheaf on $C$. Denote by $\mathcal{C}_F$ the full subcategory of $\mathcal{C}$ spanned by objects $X \rightarrow F$ with $X \in \mathcal{C}$. The obvious cocone over $\mathcal{C}_F \rightarrow \mathcal{C}$ with $F$ at its apex is a colimit.}

Note that the theorem is trivially true for any object $X \in \mathcal{C}$ because $\mathcal{C}_X$ has a final object.

1.1.2 Coverages and sheaves

In this subsection we discuss the data needed to specify how to glue together objects in $C$ to other objects in $C$, and the failure of this data to be compatible with the Yoneda embedding.

**Definition 1.1.2.1.** Let $U$ be an object in $\mathcal{C}$, then a covering of $U$ is a (small) family of morphisms in $\mathcal{C}$ with $U$ as their codomain.

In the following we assume that the pullback of any two morphisms in a cover exists (this will be true in the examples covered in the seminar). It is straightforward to modify the theory so that this condition is no longer necessary (see [Joh02b, §C.2.1]), but at a first pass the key ideas are more transparent with this extra assumption. Furthermore we assume that the coproduct of the objects in any covering exists as well as the coproduct of the set of all pullbacks of the morphisms in a covering.

**Definition 1.1.2.2.** Let $U$ be an object in $\mathcal{C}$, and let $\{U_i \rightarrow U\}_{i \in I}$ be a covering of $U$, then the glueing diagram\(^2\) associated to $\{U_i \rightarrow U\}_{i \in I}$ is the diagram

$$\prod_{(i,j) \in I \times I} U_i \times_U U_j \Rightarrow \prod_{i \in I} U_i \rightarrow U,$$

where the upper of the parallel arrows is obtained from the projection onto the first objet in the pullback diagrams for all $(i,j) \in I \times I$, and the lower arrow is obtained from the second projection.

In the examples $\text{Cart}_\infty$ and $\text{Aff}_k$, if we take a covering of an open subset of Euclidean space $U$ or an affine $k$-scheme $S$ to be literally a covering of $U$ by open subsets or by open affine $k$-subschemes respectively, then the associated glueing diagram is colimiting\(^3\). Unfortunately the diagram

$$\prod_{(i,j) \in I \times I} \mathcal{C}(\_\times_U U_i \times_U U_j) \Rightarrow \prod_{i \in I} \mathcal{C}(\_ \rightarrow U_i) \rightarrow \mathcal{C}(\_ \rightarrow U) \quad (1.1)$$

in $\mathcal{C}$ is no longer a colimiting diagram in all but the most trivial cases\(^4\).

There are at least two things we could do to try to fix this: If for every $U \in \mathcal{C}$ and every

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\(^2\)Warning, this terminology is nonstandard.

\(^3\)In more sophisticated categorical language: The pullbacks and colimits under consideration commute with each other, so we are asserting that $\prod_{i \in I} U_i \rightarrow U$ is an effective epimorphism.

\(^4\)Even if the glueing diagram in $\mathcal{C}$ is not a colimit, we might still want to force it to be on in $\mathcal{C}$. 

---
covering $\mathcal{U}$ of $U$ we denote by $C_\mathcal{U}$ the colimit of the glueing diagram of $\mathcal{U}$ in $\mathcal{C}$, then we could try to formally invert the canonical morphisms $C_\mathcal{U} \to \mathcal{C}(\underline{U})$ as we range over the coverings $\mathcal{U}$. Otherwise we can try to restrict to the subcategory of $\mathcal{C}$ spanned by those presheaves, to which the diagrams of the form (1.1) look like colimits.

**Definition 1.1.2.3.** Let $U$ be an object of $\mathcal{C}$, and let $\{U_i \to U\}_{i \in I}$ be a covering of $U$, then a presheaf $F : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ satisfies the sheaf condition for $\{U_i \to U\}_{i \in I}$ if the isomorphic diagrams

$$\mathcal{C}\left( \prod_{(i,j) \in I \times I} \mathcal{C}(\underline{U_i \times_U U_j}), F \right) \cong \mathcal{C}\left( \prod_{i \in I} \mathcal{C}(\underline{U_i}), F \right) \leftarrow \mathcal{C}(\underline{U}, F)$$

and

$$\prod_{(i,j) \in I \times I} F(U_i \times_U U_j) \cong \prod_{i \in I} F(U_i) \leftarrow F(U),$$

are limiting diagrams.

**Definition 1.1.2.4.** Let $U$ be an object of $\mathcal{C}$, $\{U_i \to U\}_{i \in I}$ a covering of $U$, and $F : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ a presheaf, then a matching family for $\{U_i \to U\}_{i \in I}$ of elements in $F$ is a sequence $(s_i \in F(U_i))_{i \in I}$, such that for each pair $(i, j) \in I \times I$ we have $F(U_i \times_U U_j \to U_i)(s_i) = F(U_i \times_U U_j \to U_j)(s_j)$. The set of matching families for $\{U_i \to U\}_{i \in I}$ of elements in $F$ is denoted by

$$\text{Desc}(\{U_i \to U\}_{i \in I}, F).$$

With notation as in Definition 1.1.2.4 we see that $\text{Desc}(\{U_i \to U\}_{i \in I}, F)$ is simply the limit of the diagram $\prod_{i,j} F(U_i \times_U U_j) \cong \prod_i F(U_i)$, and that $F$ satisfies the sheaf condition for $\{U_i \to U\}_{i \in I}$ iff for every matching family $(s_i \in F(U_i))_{i \in I}$ there exists exactly one element $s \in F(U)$ such that for all $i \in I$ we have $F(U_i \to U)(s) = s_i$.

### 1.1.3 Reflexive subcategories

Let $\mathcal{B}$ be a category, $\mathcal{A}$ a subcategory of $\mathcal{B}$, and $\Sigma$ a class of morphisms in $\mathcal{B}$. The situation considered in the last section, in which for any object $X \in \mathcal{A}$ and any morphism $f : A \to B$ in $\Sigma$ the map $\mathcal{B}(B, X) \to \mathcal{B}(A, X)$, $g \mapsto g \circ f$ is a bijection will now be studied systematically:

**Definition 1.1.3.1.** The subcategory $\mathcal{A}$ is **reflexive** if the inclusion functor $\mathcal{A} \hookrightarrow \mathcal{B}$ admits a left adjoint. The left adjoint is called the **reflection** of $\mathcal{A}$.

**Remark 1.1.3.2.** It is readily verified that a left adjoint of $\mathcal{A} \hookrightarrow \mathcal{B}$ may be chosen such that it restricts to the identity on $\mathcal{A}$.

**Proposition 1.1.3.3.** [Bor94a, Prop. 5.4.4] Assume $\mathcal{A}$ is reflexive in $\mathcal{B}$; denote the reflection by $R$ and the unit of the adjunction by $\eta : \text{id}_\mathcal{B} \Rightarrow R$. Assume furthermore that $\Sigma$ consists precisely of the morphisms in $\mathcal{B}$ which are inverted by $R$.

1. For an object $X \in \mathcal{B}$ the following are equivalent:
   
   (I) $X \in \mathcal{A}$;

   (II) for all $f : A \to B$ in $\Sigma$ the map $\mathcal{B}(B, X) \to \mathcal{B}(A, X)$, $g \mapsto g \circ f$ is a bijection;
(III) for all $B \in \mathcal{B}$ the map $\mathcal{B}(R(B), X) \to \mathcal{B}(B, X), \ g \mapsto g \circ \eta_B$ is a bijection.

(2) For a morphism $f$ in $\mathcal{B}$ the following are equivalent:

(I) $f : A \to B$ is in $\Sigma$;

(II) for all $X \in A$ the map $\mathcal{B}(B, X) \to \mathcal{B}(A, X), \ g \mapsto g \circ f$ is a bijection.

\[ \square \]

Theorem 1.1.1.1 tells us that in a precise sense the category $\widehat{\mathcal{C}}$ is generated by a small category, namely $\mathcal{C}$. Such categories are so-called locally presentable categories (see [Bor94b, Def. 5.2.1]). This allows us to apply a partial converse of Theorem 1.1.3.3:

**Theorem 1.1.3.4.** Assume $\mathcal{B}$ is locally presentable and that $\mathcal{A}$ corresponds exactly to the objects $X$ such that for all $f : A \to B$ in $\Sigma$ the map $\mathcal{B}(B, X) \to \mathcal{B}(A, X), \ g \mapsto g \circ f$ is a bijection, then the subcategory $\mathcal{A}$ is reflective, and the reflector of $\mathcal{A}$ is the localisation of $\mathcal{B}$ along $\Sigma$.

Note on the proof. The construction of the reflector is essentially an application of the small object argument (see [Gar09]). For a full proof put together [Bor94b, Ex. 5.5.5.b, Prop. 5.2.10] and [Bor94a, Th. 5.4.7].

Explicitly, Theorem 1.1.3.4 tells us that if we are given a family $P$ of coverings on the objects in $\mathcal{C}$, then restricting to the subcategory of $\widehat{\mathcal{C}}$ spanned by presheaves which satisfy the sheaf condition for every covering, or taking the localisation of $\widehat{\mathcal{C}}$ along the morphisms $C_U \to \mathcal{C}(\_, U)$ for covering $U$ in $P$ produce equivalent categories.

Note that the class of morphisms in $\widehat{\mathcal{C}}$ corresponding to $\Sigma$ in Proposition 1.1.3.3 generally contains more morphisms than the morphisms $C_U \to \mathcal{C}(\_, U)$ for every covering $U$ in $P$.

**Definition 1.1.3.5.** Using the same notation as above the localisation of $\widehat{\mathcal{C}}$ along the morphisms $C_U \to \mathcal{C}(\_, U)$ for every covering $U$ in $P$ is called the $P$-sheafification functor, or simply sheafification functor, when $P$ is understood from context. The image of any presheaf under the $P$-sheafification functor is referred to as its $P$-sheafification, or simply sheafification.

We will see some concrete constructions of this left adjoint Chapter 2.

### 1.1.4 Coverages and Grothendieck pretopologies

Consider a morphism $f : X \to Y$ between manifolds (resp. schemes), then we can find atlases (resp. coverings by open affine subschemes) such that $f$ restricts to morphisms between open subsets of Euclidean space (resp. affine schemes), and furthermore $f$ is completely determined by these restrictions. This is in particular true if $X$ and $Y$ are themselves open subsets of Euclidean space (resp. affine schemes). That maps between spaces are determined locally in this sense is fundamental in geometry, which is why we axiomatise this property in the following definition. The notion of a Grothendieck topology given in §1.1.5 will be obtained by adding closure axioms to the axiom defining coverages.

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5 Using the same notation as in the previous subsection.
6 Proofs in this subsection will hopefully be added later.
7 The axiom will manifest itself in a surprising way in Theorem 1.2.0.4.
Definition 1.1.4.1. [Joh02b, Def. C.2.1.2.] A coverage $T$ on $\mathcal{C}$ is a collection of coverings of objects in $\mathcal{C}$, (where for each object $U \in \mathcal{C}$ the collection of coverings of $U$ in $T$ is denoted by $T_U$) such that

(C) for each object $U \in \mathcal{C}$ and covering $\{f_i : U_i \to U\}_{i \in I}$ in $T_U$ we have that for any morphism $g : V \to U$ in $\mathcal{C}$ there exists a covering $\{h_j : V_j \to V\}_{j \in J}$ in $T_V$ such that for each $j \in J$ there exists an element $i \in I$ such that the map $g \circ h_j$ factors through $f_i$.

Assume $\mathcal{C}$ is equipped with a coverage $T$, and let $g : V \to U$, $\{f_i : U_i \to U\}_{i \in I}$, $\{h_j : V_j \to V\}_{j \in J}$ be as in Definition 1.1.4.1, then if for each $j \in J$ we choose an element $i \in I$ and a factorisation of $f_i$ through $g \circ h_j$, we obtain by the universal property of coproducts a morphism $\tilde{g} : \bigsqcup_{j \in J} V_j \to \bigsqcup_{i \in I} U_i$ making the bottom square in (1.2) commute. Similarly, $\tilde{g}$ determines a morphism $\tilde{g} : \bigsqcup_{(j_1,j_2) \in J \times J} V_{j_1} \times_V V_{j_2} \to \bigsqcup_{(i_1,i_2) \in I \times I} U_{i_1} \times_U U_{i_2}$ making the upper squares in (1.2) commute containing only the left arrows and the right arrows of the vertical pairs respectively. In this sense $g$ is completely determined by $\tilde{g}$.

\[
\begin{array}{ccc}
\bigsqcup_{(j_1,j_2) \in J \times J} V_{j_1} \times_V V_{j_2} & \xrightarrow{\tilde{g}} & \bigsqcup_{(i_1,i_2) \in I \times I} U_{i_1} \times_U U_{i_2} \\
\downarrow & & \downarrow \\
\bigsqcup_{j \in J} V_j & \xrightarrow{\tilde{g}} & \bigsqcup_{i \in I} U_i \\
\downarrow & & \downarrow \\
V & \xrightarrow{g} & U
\end{array}
\]  

(1.2)

Definition 1.1.4.2. Let $T$ be a coverage on $\mathcal{C}$, then a presheaf $\mathcal{C}^{\text{op}}$ is called a $T$-sheaf, or simply a sheaf if $T$ is understood from context, if $F$ satisfies the sheaf condition for for every covering given by $T$.

The subcategory of $\mathcal{C}$ of $T$-sheaves is denoted by $\mathcal{C}_T$, or simply by $\tilde{\mathcal{C}}$ if $T$ is clear from context.

Assume we are given coverage $T$ on $\mathcal{C}$, then any $T$-sheaf may satisfy the sheaf condition for additional coverings not in $T$. We shall proceed to systematically investigate this matter.

Lemma 1.1.4.3. Let $U$ be an object in $\mathcal{C}$, then every presheaf on $\mathcal{C}$ satisfies the sheaf condition for the covering of $U$ consisting only of the identity morphism $U \xrightarrow{id} U$.

Lemma 1.1.4.4. [Joh02b, Lm. C.2.1.6.i] Assume that $\mathcal{C}$ is equipped with a coverage $T$. Let $U$ be an object in $\mathcal{C}$, $\{U_i \to U\}_{i \in I}$ a covering in $T_U$, and $\{V_j \to U\}_{j \in J}$ a covering, not necessarily in $T_U$, such that for each $i \in I$ there exists a $j \in J$ such that $U_i \to U$ factors through $V_j \to U$, then any $T$-sheaf satisfies the sheaf condition for $\{V_j \to U\}_{j \in J}$.

Lemma 1.1.4.5. [Joh02b, Lm. C.2.1.6.i] Assume that $\mathcal{C}$ is equipped with a coverage $T$. Let $U$ be an object in $\mathcal{C}$, $\{f_i : U_i \to U\}_{i \in I}$ a covering in $T_U$, and for each $i \in I$ let $\{h_{ij} : U_{ij} \to U_i\}_{j \in J_i}$ be a covering in $T_{U_i}$, then any $T$-sheaf satisfies the sheaf axiom for the covering $\{f_i \circ h_{ij} : U_{ij} \to U\}_{i \in I, j \in J_i}$.

\[\text{The lemma as sated in [Joh02b] is incorrect. See [Low].}\]
Lemma 1.1.4.6. Assume $\mathcal{C}$ admits pullbacks, and that it is equipped with a coverage $\mathcal{T}$. Let $U$ be an object in $\mathcal{C}$, and $\{U_i \to U\}_{i \in I}$ a covering in $\mathcal{T}_U$. For any morphism $V \to U$ in $\mathcal{C}$ any $\mathcal{T}$-sheaf satisfies the sheaf condition for the covering $\{V \times_U U_i \to V\}_{i \in I}$.

Proof. Exercise. \qed

We are now ready to give the classical definition of a Grothendieck pretopology.

Definition 1.1.4.7. Assume $\mathcal{C}$ admits pullbacks. A Grothendieck pretopology or simply a pretopology $\tau$ on $\mathcal{C}$ is a collection of coverings of objects in $\mathcal{C}$, (where for each object $U \in \mathcal{C}$ the collection of coverings of $U$ in $\tau$ is denoted by $\tau_U$,) such that

(I) For each object $U \in \mathcal{C}$ the identity morphism $U \xrightarrow{id} U$ is in $\tau_U$.

(C_p) For each morphism $V \to U$ in $\mathcal{C}$ and each covering $\{U_i \to U\}_{i \in I}$ in $\tau_U$ the covering $\{V \times_U U_i \to V\}_{i \in I}$ is in $\tau_V$.

(L) For each object $U \in \mathcal{C}$, if $\{f_i : U_i \to U\}_{i \in I}$ is a covering in $\tau_U$ and for each $i \in I$ the covering $\{h_{ij} : U_{ij} \to U_i\}_{j \in J_i}$ is in $\tau_{U_i}$, then $\{f_i \circ h_{ij} : U_{ij} \to U\}_{i \in I, j \in J_i}$ is in $\tau_U$.

By the axiom (C_p) any grothendieck pretopology is a coverage.

Proposition 1.1.4.8. Assume $\mathcal{C}$ admits pullbacks. For any coverage $\mathcal{T}$ on $\mathcal{C}$, the intersection of all Grothendieck pretopologies containing $\mathcal{T}$ is again a Grothendieck pretopology with the same sheaves as $\mathcal{T}$.

Proof. The intersection $\tau$ of all Grothendieck pretopologies containing $\mathcal{T}$ is again a Grothendieck pretopology, and is contained in the intersection $\tilde{\mathcal{T}}$ of all Grothendieck topologies containing $\mathcal{T}$. As $\tilde{\mathcal{C}}_T \subseteq \tilde{\mathcal{C}}_{\tilde{\mathcal{T}}} \subseteq \tilde{\mathcal{C}}_{\tau}$, and $\tilde{\mathcal{C}}_T = \tilde{\mathcal{C}}_{\tau}$ by Proposition 1.1.5.17\footnote{We note that, although Proposition 1.1.5.17 is the next section, it does not logically depend on this proposition.} we have $\tilde{\mathcal{C}}_T = \tilde{\mathcal{C}}_{\tilde{T}}$. \qed

1.1.5 Sieves

In the previous section in Lemmas 1.1.4.3 - 1.1.4.5 we studied, given a coverage $\mathcal{T}$ on $\mathcal{C}$, for which coverings not in $\mathcal{T}$, any $\mathcal{T}$-sheaf nonetheless satisfies the sheaf condition. In this subsection we will study how can add morphisms to a given covering without affecting which presheaves satisfy the sheaf condition.

Definition 1.1.5.1. Let $U$ be an object in $\mathcal{C}$, then a covering $\mathcal{U}$ of $U$ is called a sieve if for any morphism $f : V \to U$ in $\mathcal{U}$ and any morphism $g : W \to V$ the composition $f \circ g : W \to U$ is in $\mathcal{U}$.

Any covering generates a sieve in an obvious way.

Remark 1.1.5.2. With notation as in Definition 1.1.5.1, if $\mathcal{U}$ is a sieve, then we could define a simpler, “sifted” glueing diagram: Viewing $\mathcal{U}$ as a subcategory of $\mathcal{C}_U$, we can simply compose the functors $\mathcal{U} \to \mathcal{C}_U \to \mathcal{C}$. If $U$ is an open subset of Euclidean space or an affine scheme, and $\mathcal{U}$ is the sieve generated by a covering by open subsets or open affine subschemes respectively,
then the colimit of the described diagram is again $U$. We will see in this section that we can replace coverings by sieves, so that the discussion motivating sheaves in §1.1.4 could have been held with sieves and sifted glueing diagrams.

Lemma 1.1.5.3. Let $U$ be an object in $\mathcal{C}$, then sieves on $U$ correspond bijectively to subfunctors of $\mathcal{C}(\_ U)$: any sieve $S$ on $U$ corresponds to the subfunctor $S \subseteq \mathcal{C}(\_ U)$ which sends any object $V \in \mathcal{C}$ to those morphisms $V \to U$ which lie in $S$.

Notation 1.1.5.4. For any object $U \in \mathcal{C}$ and any sieve on $U$, we will use the same symbol to denote the sieve and the corresponding subfunctor of $\mathcal{C}(\_ U)$, as well as full subcategory of $\mathcal{C}$ spanned by morphisms in $S$.

Notation 1.1.5.5. Let $f : V \to U$ be a morphism in $\mathcal{C}$, then for any sieve $S \subseteq \mathcal{C}(\_ U)$ we denote the pullback of $S$ along $f$ by $f^* S$.

Note that for any sieve $S$ on any object $U \in \mathcal{C}$, the pullback of $S$ along any morphism $f : V \to U$ is given by all morphisms $g : W \to V$ such that $fofS = g(tf)$ for every $g : W \to \text{Dom}_f$ yields a sequence in $\lim_{(f : V \to U) \in S} F(V)$. These two maps are inverse to each other, yielding a bijection

$$\text{Desc}(U, F) \cong \lim_{(f : V \to U) \in S} F(V).$$

Lemma 1.1.5.6. Let $U$ be an object in $\mathcal{C}$, $\mathcal{U}$ a covering of $U$, and $S$ the sieve generated by $\mathcal{U}$, then for any presheaf $F : \mathcal{C}^{\text{op}} \to \text{Set}$ the the restriction of any element $(s_f) \in \lim_{(f : V \to U) \in S} F(V)$ to $(s_f)_{(f : V \to U) \in \mathcal{U}}$ is a matching family for $\mathcal{U}$, and for any matching family $(t_f)_{f \in \mathcal{U}}$, setting $t_{g \circ f} = g(t_f)$ for every $g : W \to \text{Dom}_f$ yields a sequence in $\lim_{(f : V \to U) \in S} F(V)$. These two maps are inverse to each other, yielding a bijection

$$\text{Desc}(U, F) \cong \lim_{(f : V \to U) \in S} F(V).$$

Lemma 1.1.5.7. Let $U$ be an object in $\mathcal{C}$, $\mathcal{U}$ a covering of $U$, and $S$ the sieve generated by $\mathcal{U}$, then any presheaf on $\mathcal{C}$ satisfies the sheaf condition for $\mathcal{U}$ iff it satisfies the sheaf condition for $S$.

Proof. Let $F : \mathcal{C}^{\text{op}} \to \text{Set}$ be a presheaf. By the previous lemma we have the following chain of canonical bijections:

$$F(U) \cong \text{Desc}(U, F) \cong \lim_{(f : V \to U) \in S} F(V) \cong \text{Desc}(S, F).$$

We thus see that we could replace coverings by sieves and define sheaves as presheaves which take colimits to sifted glueing diagrams to limits.

Lemma 1.1.5.8. Let $U$ be an object in $\mathcal{C}$, and $S$ a covering of $U$. A presheaf $F : \mathcal{C}^{\text{op}} \to \text{Set}$ satisfies the sheaf condition for $S$ iff the map

$$\hat{\mathcal{C}}(\_ U, F) \to \hat{\mathcal{C}}(S, F)$$

given by precomposition with $S \hookrightarrow \mathcal{C}(\_ U)$ is a bijection.
Thus $F$ satisfies the sheaf condition for $S$ iff every morphism $S \to F$ descends uniquely to a morphism $\mathcal{C}(\ _, U) \to F$.

Proof. Firstly, $\mathcal{C}(\ _, U), F) \cong F(U)$ by Yoneda’s lemma. Next, we obtain the chain of isomorphisms

$$\mathcal{C}(\ _, U), F) \cong \mathcal{C}(\lim_{(V \to U) \in S} \mathcal{C}(\ _, V), F) \cong \lim_{(V \to U) \in S} \mathcal{C}(\ _, V), F) \cong \lim_{(V \to U) \in S} F(V),$$

where the first isomorphism follows from Theorem 1.1.1.1, and the last isomorphism from Yoneda’s lemma. Finally, we note that the square

$$\begin{array}{ccc}
F(U) & \longrightarrow & \lim_{(V \to U) \in S} F(V) \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{C}(\ _, U), F) & \longrightarrow & \mathcal{C}(\ _, U), F)
\end{array}$$

commutes by chasing around any element $s \in F(U)$:

$$\begin{array}{c}
s \longleftarrow (F(f)(s))_{f \in S} \\
\downarrow \quad \downarrow \\
(f \mapsto F(f)(s))_{f \in C_U} \longleftarrow (f \mapsto F(f)(s))_{f \in S}.
\end{array}$$

Remark 1.1.5.9. If we are given a collection $P$ of coverings in $\mathcal{C}$, the $P$-sheafification functor can thus be obtained as the localisation along the morphisms in $\mathcal{C}$ given by first replacing any cover of any object $U \in \mathcal{C}$ in $P$ by the sieve it generates, and then considering the canonical morphism $S \hookrightarrow \mathcal{C}(\ _, U)$.

The following three lemmas are analogues of Lemmas 1.1.4.3 - 1.1.4.5 respectively.

Lemma 1.1.5.10. Let $U$ be an object in $\mathcal{C}$, then every presheaf on $\mathcal{C}$ satisfies the sheaf condition w.r.t. the sieve generated by the identity morphism $U \to U$.

Lemma 1.1.5.11. [Joh02b, Lm. C.2.1.6.ii] Assume that $\mathcal{C}$ is equipped with a coverage $T$. Let $U$ be an object in $\mathcal{C}$, and $S$ a sieve in $T_U$, then for any sieve $S'$ on $U$ containing $S$ any $T$-sheaf satisfies the sheaf condition for $S'$.

Proof. This is a special case of Lemma 1.1.4.4.

Lemma 1.1.5.12. [Joh02b, Lm. C.2.1.7.ii] Assume that $\mathcal{C}$ is equipped with a coverage $T$. Let $U$ be an object in $\mathcal{C}$, let $R, S$ be sieves on $U$ with $R$ in $T_U$, and assume that $f^*S$ is in $T_V$ for every $f : V \to U$ in $R$, then any $T$-sheaf satisfies the sheaf condition for $S$.

Proof. This is an immediate consequence of Lemma 1.1.4.5 and 1.1.4.4.
Definition 1.1.5.13. A coverage on $\mathcal{C}$ is called sifted if all the coverings given by the coverage are sieves.

Convention 1.1.5.14. Let $T$ be a sifted coverage on $\mathcal{C}$, then if $T$ is understood from context, we will often refer to sieves given by $T$ as covering sieves.

Note that in a sifted coverage the axiom (C) translates into saying that the pullback of any covering sieve contains a covering sieve. By Lemma 1.1.4.4 any sheaf satisfies the sheaf condition for any pullback of any covering sieve, so we might as well assume that covering sieves pull back to covering sieves.

Definition 1.1.5.15. A sifted coverage $T$ on $\mathcal{C}$ is called a Grothendieck topology if

(Cₚ) covering sieves pull back to covering sieves;

(M) for every object $U \in \mathcal{C}$ the sieve generated by the identity morphism $U \to U$ is in $T_U$;

(Lₚ) let $U$ be an object in $\mathcal{C}$; if $R$, $S$ are sieves on $U$ with $R$ in $T_U$, and $f^*S$ is in $T_V$ for every $f : V \to U$ in $R$, then $S$ is in $T_U$.

Definition 1.1.5.16. A site is a category equipped with a Grothendieck topology.

Proposition 1.1.5.17. For any coverage $T$ on $\mathcal{C}$, the intersection of all Grothendieck topologies containing $T$ is again a Grothendieck pretopology with the same sheaves as $T$.

Sketch of proof. It is easily verified that the intersection of all Grothendieck topologies containing $T$ is again a Grothendieck topology, and that the maximal collection of sieves, for which any $T$-sheaf satisfies the sheaf axiom, is a Grothendieck topology.

In the following sections, §§1.1.6 - 1.1.7, we give equivalent definitions for Grothendieck topologies. We will refer to a category equipped with a Grothendieck topology or any of the equivalent structures considered in these sections as a site. Also, more abusively, we will refer to a category with a coverage or Grothendieck pretopology as a site.

1.1.6 Local isomorphisms

Assume $\mathcal{C}$ is equipped with a Grothendieck topology $T$, and view $T$ as a subset of $\text{Mor}(\mathcal{C})$. By the discussion in §1.1.3 and Remark 1.1.5.9 we can view the category of $T$-sheaves $\tilde{\mathcal{C}}_T$ as the localisation of $\mathcal{C}$ along $T$, and that the class or morphisms inverted by the left adjoint to $\tilde{\mathcal{C}}_T \to \mathcal{C}$ is completely determined by $\tilde{\mathcal{C}}_T$.

Definition 1.1.6.1. Assume $\mathcal{C}$ is equipped with a Grothendieck topology $T$, then a morphism in $\tilde{\mathcal{C}}$ which is sent to an isomorphism by the $T$-sheafification functor is called a $T$-local isomorphism, or simply a local isomorphism, when $T$ is understood from context.

Local isomorphisms can be described axiomatically.

---

10It is often convenient to assume that this category is (essentially) small.

11We will see in Theorem 1.2.0.3 that this topology is unique.
**Definition 1.1.6.2.** A family of local isomorphisms $I$ on $\mathcal{C}$ is a class of morphisms in $\hat{\mathcal{C}}$ satisfying the following three axioms:

(LI 1) Every isomorphism belongs to $I$.

(LI 2) The class $I$ satisfies the two-out-of-three property.

(LI 3) The class $I$ is stable under base change along morphisms with domain in $\mathcal{C}$.

A morphism in $I$ is referred to as an $I$-local local isomorphism, or simply a local isomorphism when $I$ is understood from context.

**Remark 1.1.6.3.** By axioms (LI 1) and (LI 2) a family of local isomorphisms in fact forms a category. Also, it is true in general, that when we consider a functor $\mathcal{C} \to \mathcal{D}$, the class of morphisms in $\mathcal{C}$ which are sent to isomorphisms form a category, and that moreover these morphisms satisfy (LI 1) and (LI 2).

**Theorem 1.1.6.4.** Let $T$ be a Grothendieck topology on $\mathcal{C}$, then the class of morphisms in $\hat{\mathcal{C}}$ sent to isomorphisms by the $T$-sheafification functor is a family of local isomorphisms. Conversely, the subclass of a family of local isomorphisms on $\mathcal{C}$ consisting of monomorphisms with codomain in $\mathcal{C}$ is a Grothendieck topology. These two operations determine a bijection between Grothendieck topologies and families of local isomorphisms on $\mathcal{C}$.

**Proof.** Remains to be written. \(\square\)

### 1.1.7 Local epimorphisms

**Definition 1.1.7.1.** Assume $\mathcal{C}$ is equipped with a Grothendieck topology $T$, then a morphism in $\hat{\mathcal{C}}$ which is sent to an epimorphism by the $T$-sheafification functor is called a $T$-local epimorphism, or simply a local epimorphism, when $T$ is understood from context.

Local epimorphisms can be described axiomatically.

**Definition 1.1.7.2.** A family of local epimorphisms $E$ on $\mathcal{C}$ is a class of morphisms in $\hat{\mathcal{C}}$ satisfying the following four axioms:

(LE 1) Every identity morphism belongs to $I$.

(LE 2) The class $I$ is closed under composition.

(LE 3) Given any morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ in $\hat{\mathcal{C}}$, if $g \circ f$ is a local epimorphism, then $g$ is a local epimorphism.

(LE 4) The class $I$ is stable under base change along morphisms with domain in $\mathcal{C}$.

A morphism in $I$ is referred to as an $I$-local local isomorphism, or simply a local isomorphism when $I$ is understood from context.
Remark 1.1.7.3. By axioms (LE 1) and (LE 2) a family of local epimorphisms in fact forms a category.

Also, it is true in general, that when we consider a functor \( C \to D \), the class of morphisms in \( C \) which are sent to epimorphisms form a category, and that moreover these morphisms satisfy (LE 1) - (LE 3).

The following is an important class of examples of local epimorphisms.

**Proposition 1.1.7.4.** Assume \( C \) is equipped with a coverage \( T \), then for any object \( U \in C \) and any covering \( \{ U_i \to U \}_{i \in I} \) in \( T_U \), the map \( \coprod_{i \in I} C(-,U_i) \to C(-,U) \) a local epimorphism\(^{12}\).

**Proof.** The image of \( \coprod_{i \in I} C(-,U_i) \to C(-,U) \) is the covering sieve generated by \( \{ U_i \to U \}_{i \in I} \), which is a local epimorphism, as it sent to an isomorphism, which is a fortiori an epimorphism, by the sheafification functor. \( \square \)

We will require the following, auxiliary definition.

**Definition 1.1.7.5.** Assume \( C \) is equipped with a family of local epimorphisms \( E \), then a morphism \( A \to B \) in \( \hat{C} \) is called a local monomorphism if \( A \to A \times_B A \) is a local epimorphism.

**Theorem 1.1.7.6.** If \( C \) is equipped with a family of local isomorphisms, then the class of morphisms \( f : A \to B \) in \( \hat{C} \), such that the image morphism\(^{13}\) \( \text{Im}_f \to B \) is a local isomorphism, determines a family of local epimorphisms.

Conversely, if \( C \) is equipped with a family of local epimorphisms, then the class of morphisms in \( \hat{C} \), which are both local epimorphism and local monomorphisms, forms a family of local isomorphisms.

These two operations determine a bijection between families of local isomorphisms and families of local epimorphisms on \( C \).

**Proof.** Remains to be written. \( \square \)

We can conclude from Theorem 1.1.6.4 and 1.1.7.5 that there is a bijection between Grothendieck topologies and families of local epimorphisms on \( C \). We give an explicit bijection in the following theorem.

**Theorem 1.1.7.7.** If \( C \) is equipped with a family of local epimorphisms, then the collection of subfunctors of representable presheaves on \( C \) which are local epimorphisms, form a Grothendieck topology on \( C \).

If \( C \) is equipped with a Grothendieck topology, then the collection of morphisms \( A \to B \) such that for all morphisms \( U \to B \) with \( U \in C \) the morphism \( A \times_B U \to U \) is a covering sieve, forms a family of local epimorphisms on \( C \).

These two operations determine a bijection between families of local epimorphisms and Grothendieck topologies on \( C \). \( \square \)

\(^{12}\)Here the local epimorphisms are of course the ones given by the Grothendieck topology generated by the coverage.

\(^{13}\)The image of a morphism of presheaves \( A \to B \) is defined as the objectwise image of sets; it is left to the reader to verify, that this indeed forms a subpresheaf of \( B \).
Using a family of local isomorphism we can define sheaves as those presheaves to which the local isomorphisms look like isomorphisms. The reader may thus be lead to suspect that given a family of local epimorphisms, sheaves correspond to those presheaves to which the local epimorphisms look like epimorphisms. This is however not the case.

**Proposition 1.1.7.8.** Assume $\mathcal{C}$ is equipped with a family of local epimorphisms, then a presheaf $F : \mathcal{C}^{\text{op}} \to \text{Set}$ is a separated presheaf$^{14}$ iff for every local epimorphism $A \to B$ the map $\hat{\mathcal{C}}(B, F) \to \hat{\mathcal{C}}(A, F)$ is a monomorphism.

**Sketch of proof.** Assume $F$ is a separated presheaf, and that $B = U \in \mathcal{C}$, then $A \to U$ factors as $A \to S \hookrightarrow U$, where $S \subseteq U$ is a covering sieve, and $A \to S$ is an epimorphism. By assumption $\hat{\mathcal{C}}(S, F) \to \hat{\mathcal{C}}(A, F)$ is a monomorphism, and $\hat{\mathcal{C}}(B, F) \to \hat{\mathcal{C}}(S, F)$ is a monomorphism by the definition of epimorphisms. The general case can then be obtained by writing a general presheaf $B$ as a colimit of representables, and reducing to the case just described.

Conversely, if for every covering $\{U_i \to U\}_{i \in I}$, such that $\coprod_{i \in I} \mathcal{C}(_, U_i) \to \mathcal{C}(_, U)$ is a local epimorphism, the morphism $\hat{\mathcal{C}}(\mathcal{C}(_, U), F) \to \hat{\mathcal{C}}(\coprod_{i \in I} \mathcal{C}(_, U_i), F)$ is a monomorphism then $F$ is a separated presheaf by definition. \qed

If $\mathcal{C}$ is equipped with a Grothendieck topology, then every epimorphism $A \to B$ in $\hat{\mathcal{C}}$ is in fact an effective$^{15}$ epimorphism. It turns out that a presheaf on $\mathcal{C}$ is then a sheaf iff it views local epimorphisms as effective epimorphisms$^{16}$. (So maybe the term "local effective epimorphism" would have been a better choice, but "local epimorphism" already seems to be standard.)

**Theorem 1.1.7.9.** [KS06, Prop. 17.3.4.] Assume $\mathcal{C}$ is equipped with a family of local epimorphisms, then a presheaf $F : \mathcal{C}^{\text{op}} \to \text{Set}$ is a sheaf iff for every local epimorphism $A \to B$ the functor $\hat{\mathcal{C}}(_, F)$ maps the sequence

$$A \times_B A \rightrightarrows A \to B$$

to an equaliser diagram.

**Note on proof.** The “if” part is not particularly hard prove; nevertheless we refer the reader to [KS06, Prop. 17.3.4.], as the proof is somewhat unenlightening when studying this material the first time around.

The “only if” part of the statement is easy to verify: Let $\{U_i \to U\}_{i \in I}$ be a covering in a coverage corresponding to the family of local epimorphisms, then the coequaliser diagram corresponding to $\coprod_{i \in I} \mathcal{C}(_, U_i) \to \mathcal{C}(_, U)$ is isomorphic to

$$\coprod_{(i,j) \in I \times I} \mathcal{C}(_, U_i) \times \mathcal{C}(_, U_j) \rightrightarrows \prod_{i \in I} \mathcal{C}(_, U_i) \to \mathcal{C}(_, U),$$

and a sheaf was defined precisely to be a presheaf taking such diagrams to equaliser diagrams. \qed

---

$^{14}$See Definition 2.1.0.1 for the definition of a separated presheaf.

$^{15}$Recall that $A \to B$ is an effective epimorphism if it is the coequaliser of $A \times_B A \rightrightarrows A$.

$^{16}$Thinking back to our discussion of local isomorphisms, Proposition 1.1.3.3 gives two equivalent ways of how to view them: As the morphisms which look like isomorphisms to sheaves, or as those morphisms which are sent to isomorphisms by the sheafification functor. Viewing local epimorphisms as “local effective epimorphisms” we have only discussed the first viewpoint discussed for local isomorphisms. The second viewpoint is also still valid, which is not hard to check using that the sheafification functor for a Grothendieck topology is right exact (see Theorem 1.2.0.1).
While the discussion of this subsection, and in particular the proof of Theorem 1.1.7.9 might suggest that local epimorphisms could be viewed as generalised coverings, we would like to make the case that simply viewing local epimorphisms as local effective epimorphisms is more elegant. Say we are given a map of sets \( X \to Y \), and that we would like to know, given a real valued function \( f: X \to \mathbb{R} \), when there exists a unique real valued function on \( Y \) which pulls back to \( f \). One necessary condition is clearly that \( X \to Y \) be surjective. A sufficient condition is then that \( f \) maps any two elements in a give fibre of \( X \to Y \) to the same element in \( \mathbb{R} \). But the set of pairs of elements in the same fibres of \( X \to Y \) is exactly \( X \times Y \), so \( f \) descends uniquely to map \( Y \to \mathbb{R} \) iff \( X \to Y \) is an effective morphism\(^{18}\) and \( f \) equalises \( X \times Y \to X \). The point here is that an effective epimorphism is given by a (canonical) equivalence relation; it appears to be a fundamental feature of a nice theory (i.e. category) of spaces that all epimorphism can be described as a being a quotient arising from an equivalence relation.

1.1.7.1 A short note maximal coverages

Assume \( \mathcal{C} \) is equipped with a coverage. We saw in Proposition 1.1.7.4 that \( T \)-coverings \( \{U_i \to U\}_{i \in I} \) furnish local epimorphisms \( \coprod_{i \in I} \mathcal{C}(\_ \to U_i) \to \mathcal{C}(\_ \to U) \). Conversely, by Theorem 1.1.7.9 we see that if \( \{U_i \to U\}_{i \in I} \) is any covering then \( \coprod_{i \in I} \mathcal{C}(\_ \to U_i) \to \mathcal{C}(\_ \to U) \) is a local epimorphism precisely when all \( T \)-sheaves satisfy the sheaf condition when \( \{U_i \to U\}_{i \in I} \).

In Proposition 1.1.5.17 it was shown that any coverage can be completed to a Grothendieck topology\(^{19}\); a more naive completion would be to try to find the maximal coverage with the same sheaves as \( T \), and our discussion of local epimorphisms, shows exactly that this coverage is given by those coverings \( \{U_i \to U\}_{i \in I} \) such that \( \coprod_{i \in I} \mathcal{C}(\_ \to U_i) \to \mathcal{C}(\_ \to U) \) is a local epimorphism. For the the case when \( \mathcal{C} \) admits pullbacks, an axiomatic description of maximal coverages is given in [KS06, Def. 16.1.5].

1.1.8 Examples of Grothendieck topologies

We start off with a formal class of examples before turing to some concrete examples.

**Example 1.1.8.1.** A coverage \( T \) on \( \mathcal{C} \) is called subcanonical iff every representable presheaf is a \( T \)-sheaf. It is readily verified that a coverage is subcanonical iff the associated glueing diagrams are coequalising. Also note that the Yoneda embedding followed by \( T \)-sheafification is fully faithful iff \( T \) is subcanonical.

**Example 1.1.8.2.** The category 1 consisting of only one object and one morphism admits only one Grothendieck topology, the trivial one. This example is not as vacuous as may originally be thought, both conceptually and practically. The category of sheaves on 1 is canonically equivalent to the category of sets. A set is thus precisely what you obtain by freely glueing together its points. Also, \( \textbf{Set} \) is atomic in the sense that it admits no subcategories of sheaves (see Theorem 1.2.0.3), and in topos theory, maps\(^{20}\) from \( \textbf{Set} \) to \( \tilde{\mathcal{C}} \) are thought of as “points” of \( \tilde{\mathcal{C}} \).

**Example 1.1.8.3.** Let \( X \) be a topological space, then assignment to every open subset of \( X \) the set of coverings (in the classical sense) determines a Grothendieck pretopology on the category \( \text{Ouv}_X \) of open subsets of \( X \).

\(^{17}\)Really, any non-empty target set would do.

\(^{18}\)Note that all epimorphisms in \( \textbf{Set} \) are effective.

\(^{19}\)In Theorem 1.2.0.3 we will see that this Grothendieck topology is unique.

\(^{20}\)By maps we mean geometric morphisms, which we may or may not define at some point.
Example 1.1.8.4. The category $\text{Top}$ admits an obvious Grothendieck pretopology where we assign to every $U \in \mathcal{C}$ the coverings $\{U_i \to U\}_{i \in I}$ such that for each $i \in I$ the continuous map $U_i \to X$ is a homeomorphism to an open subset of $X$ and the images of all continuous maps in $\{U_i \to X\}_{i \in I}$ cover $X$.

We have treated $\text{Cart}_\infty$ and $\text{Aff}_k$ as sites throughout the whole chapter, and now we finally make it official.

Example 1.1.8.5. The assignment to every object in $\text{Cart}_\infty$ the class of coverings by open subsets is a Grothendieck pretopology.

Example 1.1.8.6. For a commutative ring $k$ the category of affine $k$-schemes admits an abundance of interesting Grothendieck pretopologies of which we describe but three, in increasing order of fineness:

1. To any affine $k$-scheme $X$ we can assign the class of coverings $\{U_i \to X\}_{i \in I}$ such that for each $i \in I$ the morphism $U_i \to X$ is an open affine subscheme and the (underlying sets of the) images of the morphisms in $\{U_i \to X\}_{i \in I}$ cover (the underlying set of) $X$. This defines the Zariski pretopology on $\text{Aff}_k$ and is denoted by Zar.

2. To any affine $k$-scheme $X$ we can assign the class of coverings $\{U_i \to X\}_{i \in I}$ such that for each $i \in I$ the morphism $U_i \to X$ is étale, and the (underlying sets of the) images of the morphisms in $\{U_i \to X\}_{i \in I}$ cover (the underlying set of) $X$. This defines the étale pretopology on $\text{Aff}_k$ and is denoted by ét.

3. To any affine $k$-scheme $X$ we can assign the class of coverings $\{U_i \to X\}_{i \in I}$ such that for each $i \in I$ the morphism $U_i \to X$ is given by a faithfully flat ring homomorphism of finite presentation, and the (underlying sets of the) images of the morphisms in $\{U_i \to X\}_{i \in I}$ cover (the underlying set of) $X$. This defines the the fppf pretopology on $\text{Aff}_k$, and is denoted by fppf, where fppf stands for “fidèlement plat, présentation finie”.

It is still true that the functor $\text{Sch}_k \to (\text{Aff}_k)_{\text{fppf}}$, and thus also $\text{Sch}_k \to (\text{Aff}_k)_{\text{ét}}$, is fully faithful, but unlike the case $\text{Sch}_k \to (\text{Aff}_k)_{\text{Zar}}$, this is a deep theorem.

1.2 Some formal properties of categories of sheaves

We discuss three important formal properties of categories of sheaves.

Theorem 1.2.0.1. Assume $\mathcal{C}$ is equipped with a collection $P$ of coverings. If the corresponding sheafification functor is right exact, then the collection of morphisms sent to isomorphisms (resp. epimorphisms) by the $P$-sheafification functor is a family of local isomorphisms (resp. epimorphisms); or equivalently, the family of sieves sent to isomorphisms form a Grothendieck topology.

Conversely, if the collection of sieves sent to isomorphisms by the $P$-sheafification functor is closed under pullbacks along morphisms in $\mathcal{C}$, then $P$-sheafification is right exact.

Morally the theorem thus says that the $P$-sheafification functor is left exact iff $P$ satisfies (C), i.e. it is a coverage.
Discussion of proof. Assume the \( P \)-sheafification functor is right exact. If we wish to prove the stated consequence for local isomorphisms (resp. local epimorphisms), then by Remark 1.1.6.3 (resp. 1.1.7.3), it is enough to show LI 3 (resp. LI 4). We will discuss local epimorphisms; the statement for local isomorphisms could then be deduced from the case of local epimorphisms using local monomorphisms (as in [SGA 4_I, Th. 5.5]), or it could be proved directly by writing presheaves as quotients of coproducts in \( \mathcal{C} \) similarly as in the proof we are about to give (see [Rez99, 8,9]). Let \( A \to B \) be a morphisms of presheaves on \( \mathcal{C} \); we wish to show that if the pullback \( A \to B \) along \( U \to B \), for all \( U \to B \) in \( \mathcal{C} \), is a local epimorphism, then \( A \to B \) is a local epimorphism. As pullbacks commute with coproducts in \( \text{Set} \), they do so in \( \widehat{\mathcal{C}} \), so the pullback of \( \coprod e_B U \to B \) is given by

\[
\begin{array}{ccc}
\coprod e_B U 
\times_B A & \longrightarrow & A \\
\downarrow & & \downarrow \\
\coprod e_B U & \longrightarrow & B.
\end{array}
\]

As sheafification commutes with colimits the coproduct of a small family of local epimorphisms is again a local epimorphism. Sheafification maps the composition of the bottom left morphisms in the pullback to an epimorphism, so the rightmost vertical morphism has to be mapped to an epimorphism.

Let us also briefly consider the statement for sieves. That (M) is satisfied is clear. The proof that (L) holds is more difficult, but it is similar to (LI 3) and (LE 4) (see [Rez99, 2]). Proving (Cs) establishes the close link between the right exactness of \( P \)-sheafification and the axiom (C). For any morphism \( f : V \to U \) and any sieve \( S \subseteq U \) we have by assumption that

\[
\begin{array}{ccc}
f^* S & \longrightarrow & S \\
\downarrow & & \downarrow \\
V & \longrightarrow & U
\end{array}
\]

gets sent to a pullback diagram, where the vertical morphism to the right is an isomorphism, and isomorphisms get pulled back to isomorphisms.

Finally, assume that \( P \) is a coverage, then we will see in Corollary 2.2.0.1 that the \( P \)-sheafification functor is constructed using limits and filtered colimits, and is thus right exact.

Remark 1.2.0.2. From the proof of Theorem 1.2.0.1 it is easy to conclude that the reflector of a reflective subcategory of \( \widehat{\mathcal{C}} \) is right exact if it preserves pullbacks.

Theorem 1.2.0.3. [SGA 4_I, Th. 5.5] The map sending any Grothendieck topology on \( \mathcal{C} \) to its category of sheaves determines a bijection between Grothendieck topologies on \( \mathcal{C} \) and isomorphism-closed subcategories of \( \widehat{\mathcal{C}} \) admitting a right exact left adjoint\(^{21}\).

Proof. By Theorem 1.1.3.4 and Theorem 1.2.0.1 isomorphism-closed reflective subcategories of \( \widehat{\mathcal{C}} \) with a right exact reflection are in bijection with families of local isomorphisms, which are in bijection with Grothendieck topologies by Theorem 1.1.6.4.

\(^{21}\)Such adjunctions are “geometric embeddings” of topoi.
We have now seen that the right exactness of sheafification is intimately related to axiom (C). In Theorem 1.1.7.9 and the surrounding discussion we saw that right exactness is crucial for ensuring that local epimorphisms are local effective epimorphisms. We finish this section with another important consequence of the right exactness sheafification.

**Theorem 1.2.0.4.** [Joh02a, Prop. A.4.3.1] Let \( S \subseteq \hat{\mathcal{C}} \) be a reflective subcategory, then \( S \) is Cartesian closed with \( \hat{\mathcal{C}}(F,G) \in S \) for any presheaves \( F,G \in S \) iff the reflection of \( S \subseteq \hat{\mathcal{C}} \) preserves finite products.

Thus the axiom (C), perhaps unexpectedly, also guarantees for any collection \( P \) of coverings that \( \tilde{\mathcal{C}}_P \) has internal hom in the simplest possible way.

### 1.3 The closure property of the category of sheaves on a site

Throughout this section we assume \( \mathcal{C} \) is equipped with a Grothendieck topology \( T \). As we view \( \tilde{\mathcal{C}} \) as the category of spaces that can be glued together from spaces in \( \mathcal{C} \) according to the rules specified by \( T \), one should expect to be able to extend \( T \) to \( \tilde{\mathcal{C}} \) in some canonical way. Furthermore it is natural to ask whether we can glue together spaces in \( \tilde{\mathcal{C}} \) to obtain even more spaces. These two considerations are closely related: If we require of the extended topology to be subcanonical, and more importantly, that sheaves on \( \tilde{\mathcal{C}} \) are still determined by how they are seen by objects in \( \mathcal{C} \), then there is indeed only one extension. Moreover sheaves on \( \tilde{\mathcal{C}} \) with this topology are representable (Theorem 1.3.0.4), which shows that \( \tilde{\mathcal{C}} \) can be viewed as a sort of closure of \( \mathcal{C} \).

Using Theorem 1.1.1.1 we reformulated the fact that every presheaf is determined by how it is seen by objects in \( \mathcal{C} \) in terms of colimits. This natural leads to another reformulation in terms of sieves.

**Lemma 1.3.0.1.** Let \( F : \mathcal{C}^{\text{op}} \to \text{Set} \) be a presheaf, and let \( S \subseteq \hat{\mathcal{C}}(\_, F) \) be the sieve generated by \( \mathcal{C}_F \), then \( \varinjlim_{(G \to F) \in S} G \cong F \).

**Proof.** Let \( (G \to Z)_{(G \to F) \in S} \) be a cocone, then this cocone restricts to a cocone over \( \mathcal{C}_F \), so we obtain a unique morphism \( F \to Z \) of cocones over \( \mathcal{C}_F \) by Theorem 1.1.1.1. It is thus enough to show that this morphism extends to a morphism of cocones over \( S \). Thus let \( G \to F \) be in \( S \), then by assumption there exists an object \( U \in \mathcal{C} \) such that \( G \to F \) factors through \( U \), and we obtain the diagram

\[
\begin{array}{ccc}
F & \to & Z \\
\downarrow U & & \downarrow \\
G & \to & U \\
\end{array}
\]

where the three triangles having a \( U \) as a corner commute by assumption, so that the outer triangle must then also commute.

**Definition 1.3.0.2.** [Joh02b, Def. C.2.2.1] A full subcategory \( \mathcal{D} \subseteq \mathcal{C} \) is called dense if every object \( U \in \mathcal{C} \) has a covering sieve generated by objects in \( \mathcal{D} \).
**Theorem 1.3.0.3.** [Joh02b, Th. C.2.2.3] Let $\mathcal{D} \subseteq \mathcal{C}$ be a small, dense subcategory, then the restriction functor $\tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{D}}$ is an equivalence of categories.

As an immediate corollary we obtain:

**Theorem 1.3.0.4.** Assume $T$ is subcanonical. If there exists a topology on $\tilde{\mathcal{C}}$ which restricts to $T$ on $\mathcal{C}$, and contains all sieves generated by $\mathcal{E}_F$ for all $F \in \tilde{\mathcal{C}}$, then any sheaf on $\tilde{\mathcal{C}}$ is representable.

So if an extension of $T$ with the desired properties exists, then it must be the canonical topology; by assumption it is subcanonical, and if it is coarser than the canonical topology, by Theorem 1.2.0.3 there are guaranteed to be sheaves on $\tilde{\mathcal{C}}$, which are not representable, contradicting Theorem 1.3.0.4.

It turns out the canonical on $\tilde{\mathcal{C}}$ does have the desired properties, in fact we have the following remarkable result:

**Theorem 1.3.0.5.** [Joh02b, Th. 2.2.8] A category $\mathcal{E}$ is equivalent to the category of sheaves on a site iff it has a small dense subcategory for the canonical topology on $\mathcal{E}$, and every sheaf on $\mathcal{E}$ is representable.

To get an idea of just how canonical the canonical topology on a category $\mathcal{E}$ equivalent to a category of sheaves on a site is, one should dwell upon the fact that no matter which subcanonical site $\mathcal{D}$ one chooses in order to realise $\mathcal{E}$ as $\tilde{\mathcal{D}}$, the topology on $\mathcal{D}$ is the restriction of the canonical topology on $\mathcal{E}$.

We finish by showing that canonical topology on $\tilde{\mathcal{C}}$ can be described in an very natural way. If $\mathcal{C}$ is sufficiently nice, and $T$ is subcanonical, then for every $T$-covering $\{U_i \rightarrow U\}_{i \in I}$ the glueing diagram exists, and we see that $\coprod_{i \in I} U_i \rightarrow U$ is an effective epimorphism. The sheaves on $\mathcal{C}$ are then those presheaves to which the morphisms $\prod_{i \in I} \mathcal{E}(\mathcal{C}, U_i) \rightarrow \mathcal{E}(\mathcal{C}, U)$ look like effective epimorphisms, and in $\tilde{\mathcal{C}}$ are genuine effective epimorphisms. In fact the coverings $\coprod_{i \in I} U_i \rightarrow U$ for which $T$-sheaves satisfy the sheaf condition are characterised as those coverings such that $\prod_{i \in I} \mathcal{E}(\mathcal{C}, U_i) \rightarrow \mathcal{E}(\mathcal{C}, U)$ is an effective epimorphism in $\tilde{\mathcal{C}}$. There is now an obvious extension of $T$ to a coverage on $\tilde{\mathcal{C}}$:

**Theorem 1.3.0.6.** [Joh02b, Ex. C.2.1.12.e] The canonical topology on $\tilde{\mathcal{C}}$ is given by the coverage $T'$ such that a (small) family of morphisms $\{G_i \rightarrow F\}_{i \in I}$ is a covering in $T'_F$ iff $\prod_{i \in I} G_i \rightarrow F$ is an effective epimorphism.

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\textsuperscript{22}Which exists by Theorem 1.3.0.5.

\textsuperscript{23}This might explain why there is a great deal of literature on canonical topologies on categories of sheaves.
Bibliography


