Improved Bounds on RICs for Gaussian Matrices

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**What is CS?**

**Linear** measurements

- \( x \in \mathbb{R}^N, \text{ } k\text{-sparse} \) (or has sparse representation \( \Phi x \))
- \( A \in \mathbb{R}^{n \times N}, \text{ random sensing (measurement) matrix}, (n \ll N) \)
- **Linear** measurements: \( y_i = \langle a_i, x \rangle, \ a_i \text{ rows of } A, \ y \in \mathbb{R}^n \)

**Nonlinear** reconstruction

- Given (\( A, y \)), solve \( Ax = y \), undertermined system, (\( n \ll N \))
- **Nonlinear** reconstruction of \( k\text{-sparse } x^* \) satisfying:

\[
x^* = \min_{x \in \mathbb{R}^N} \|x\|_0 \quad \text{subject to} \quad Ax = y.
\]

- Algorithms: **greedy** (OMP, CoSaMP, SP, IHT, ...) and regularizations \( (l_q, \ q \in (0, 1]) \)
RIC of a matrix gives a sufficient condition for exact recovery

RIC of $A$ of order $k$ is the smallest $L$ & $U$ for all $k$-sparse $x$:

$$(1 - L(k, n, N; A))\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + U(k, n, N; A))\|x\|_2^2,$$

RIC of $A$: eigenvalues of $A_K^*A_K$, let $\Omega = \{1, 2, 3, \ldots, N\}$

$$1 + U(k, n, N; A) := \max_{K \subset \Omega, |K| = k} \lambda_{\text{max}}(A_K^*A_K)$$

$$1 - L(k, n, N; A) := \min_{K \subset \Omega, |K| = k} \lambda_{\text{min}}(A_K^*A_K)$$

RIC combinatorial, $\binom{N}{k}$ $k$-sets, intractable for deterministic $A$

Goal

Calculate accurate RIC bounds for Gaussian random matrices with entries drawn i.i.d. from $\mathcal{N}(0, 1/n)$
Linear growth or proportional growth asymptotics

Problem instances \((k, n, N)\) considered is where the following ratios converge to nonzero bounded limits:

\[
\frac{k}{n} = \rho_n \to \rho \quad \text{and} \quad \frac{n}{N} = \delta_n \to \delta \quad \text{for} \quad (\delta, \rho) \in [0, 1]^2 \quad \text{as} \quad (k, n, N) \to \infty.
\]

Theorem

\textit{Let} \(A\) \textit{be a matrix of size} \(n \times N\) \textit{whose entries are drawn i.i.d. from} \(\mathcal{N}(0,1/n)\). \textit{For any} \(\epsilon > 0\), \textit{in the proportional-growth asymptotics}

\[
P(L(k, n, N) < L^{BT}(\delta, \rho) + \epsilon) \to 1 \quad \text{and} \quad P(U(k, n, N) < U^{BT}(\delta, \rho) + \epsilon) \to 1
\]

\textit{exponentially in} \(n\).
• Probabilistic methods find smallest $\lambda_{\text{max}}^* > 0$ for which
  \[ P \left( 1 + U(k, n, N; A) > \lambda_{\text{max}}^* \right) = P \left( \max_{K \subset \Omega, |K| = k} \lambda_{\text{max}}^* (A_K^* A_K) > \lambda_{\text{max}}^* \right) \to 0 \]

• Prior derivations used a simple union bound over all $k$-sets
  \[ P \left( \max_{K \subset \Omega, |K| = k} \lambda_{\text{max}}^* (A_K^* A_K) > \lambda_{\text{max}}^* \right) \leq \binom{N}{k} P \left( \lambda_{\text{max}}^* (A_K^* A_K) > \lambda_{\text{max}}^* \right) \]

• We used union bound over groups of $m \geq k$ distinct elements
  \[ P \left( \max_{K \subset \Omega, |K| = k} \lambda_{\text{max}}^* (A_K^* A_K) > \lambda_{\text{max}}^* \right) = P \left( \max_{i=1, \ldots, u} \max_{K \subset G_i, |K| = k} \lambda_{\text{max}}^* (A_K^* A_K) > \lambda_{\text{max}}^* \right) \]

\[ \Rightarrow P \left( \max_{i=1, \ldots, u} \max_{K \subset G_i, |K| = k} \lambda_{\text{max}}^* (A_K^* A_K) > \lambda_{\text{max}}^* \right) \leq u \cdot P \left( \lambda_{\text{max}}^* (A_M^* A_M) > \lambda_{\text{max}}^* \right) \]

where $u = rN$ and $r = \left( \frac{N}{k} \right) \left( \frac{m}{k} \right)^{-1}$, with $|M| = m \geq k$

• Number vs size of groups, $\Rightarrow$ optimizing over $\gamma := \frac{m}{n} \in [\rho, \delta^{-1}]$
Definition

Define $G_i := \{K\}$ for $K \subset \Omega$ and $M_i := \bigcup_{K \subset G_i, |K| = k} K$ with $|M_i| = m \geq k$. Let $u \in \mathbb{Z}^+$ and define $G := \bigcup_{i=1}^u G_i$.

Lemma

Set $r = \left(\begin{array}{c} N \\ k \end{array}\right)\left(\begin{array}{c} m \\ k \end{array}\right)^{-1}$ and draw $u := rN M_i$ sets uniformly at random from the $\left(\begin{array}{c} N \\ m \end{array}\right)$ possible $M_i$ sets. With $G$ defined as above,

$$P\left[\left(\begin{array}{c} |G| \\ k \end{array}\right) < \left(\begin{array}{c} N \\ k \end{array}\right)\right] < C(k/N)N^{-1/2}e^{-N(1-\ln 2)}$$

where $C(p) \leq \frac{5}{4}(2\pi p(1-p))^{-1/2}$.
Groups of \( m \leq k \) distinct elements, contains \( \binom{m}{k} K \subset \Omega \)
\[ \Rightarrow \text{at least} \quad \binom{N}{k} \binom{m}{k}^{-1} =: r \text{ groups to cover each } K \]

For any random group \( M_i & K \subset \Omega ) \> P(M_i \supset K) = 1/r \text{ and } \]
\[ P(G \nsubseteq K) = (1 - 1/r)^u \leq \exp(-u/r) \]

A union bound over all \( \binom{N}{k} \) sets \( K \), yields
\[ P\left[ \left( \binom{|G|}{k} \right) < \binom{N}{k} \right] < \binom{N}{k} e^{-u/r} \]

But RHS of Stirling’s Inequality gives
\[ \binom{N}{pN} \leq \frac{5}{4} (2\pi p(1-p)N)^{(1/2)} e^{NH(p)}, \quad H(p) \leq \ln 2 \text{ for } p \in [0, 1] \]

Substituting \( u = rN \) completes the proof
Compressed Sensing (CS)  
Restricted Isometry Constants (RIC)  
RIC Bounds for Gaussian random matrices

**RIC Bounds & their construction**

- **Covering**
- **Numerical results**

\[ U(\delta, \rho) \]

\[ L(\delta, \rho) \]

\[ U(k, n, N; A) \]

\[ L(k, n, N; A) \]

- Algorithms for \( L(k, n, N; A) \) & \( U(k, n, N; A) \) by C. Dossal et. al. and M. Journée et. al. respectively

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**Improved Bounds on RICs for Gaussian Matrices**
Our bounds within 1.57 of observed values

1st bounds by Candès & Tao, 2nd by Blanchard, Cartis & Tanner, recovered by our bounds if $\gamma = \rho$

CT & BCT bounds within 2.74 & 1.83 of empirical data resp.

Improvement on prior bounds and consistent with data
THANK YOU