Gabi Farkas - Koszul divisors on nodal

General method to produce "the best"
divisors on nodal: 1980

Then (80s: Harris-Mumford-Eisenbud)

$\bar{M}_g$ is of good type for $g > 24$

...disproving an ancient conjecture of
Serre that $K(\bar{M}_g) = -\infty$

Conjecture $K(\bar{M}_g) = -\infty$ for $g \leq 22$

Theorem $\bar{M}_{22}$ is of general type

Let $\mathcal{P}_g$ = moduli of Pryms of

$\dim \mathcal{P}_g = \{ (G, \psi) : G \in \mathcal{M}_3, \psi \in \text{Sym}(\mathcal{C}) \}$

\[ \bar{M}_g \to A_{g-1} \to \mathcal{P}_g \to \text{Prym-map} \to \mathcal{C} \]
Theorem 2 \( \text{R}_g \) is of good type \( g \geq 12 \)

... instead of push the sur
& much more sere.

Slogan: look at syzygies of object pair

Recap on Harris-Mumford-Eisenb: 

\( \text{M}_g \cdot \text{M}_g = \Delta_0 \cup \ldots \cup \Delta_g/2 \) boundary cliques

\( \Delta_0 = \{ \sigma_{g+1} \} \quad \Delta_i = \{ x_{g+1} \} \)

\( \delta_i = [\Delta_i] \) in \( \text{Pic}(\text{M}_g) \) Picard of stack

Hodge class \( \lambda \in G(\text{IE}) \) Hodge bundle \( E(c) = H^0(\omega_c) \)

\( \text{Pic} \text{M}_g \) freely generated by \( \lambda, \delta_1 \) \((g \geq 3)\)

For any effective divisor \( D \) on \( \text{M}_g \)

which doesn't come from boundary \( \Rightarrow \)

\( D = a \lambda - \sum b_i \delta_i \) with \( a, b_i > 0 \).
\[ \bar{K}_{m_3} = 13 \lambda - 2\delta_0 - 3\delta; \ldots - 2\delta_{19/21} \]

Is \( \bar{K}_{m_3} \) effective?

(General method)

If \( \exists D \in \bar{m}_3 \) effective divisor with

\[ D = a\lambda - z\delta; \quad \text{s.t.} \quad \text{the slope} \]

\[ s(D) = \frac{a}{\min (z)} < s(K) = \frac{13}{2} \]

\[ \Rightarrow \text{can write} \quad \lambda = x\lambda + y\delta D + \sum_{i} \lambda_{i}, \delta_{i} \]

with \( x, y > 0 \),

\[ \lambda \text{ is big & nef} \quad \Rightarrow \quad \text{entire class is big} \]

\[ \Rightarrow \quad \bar{m}_3 \text{ is good type} \]

Examples of effective divisors on \( \overline{m}_3 \):

1) Horrocks divisors : \( s = 26 - 1 \)

\[ \overline{m}_{3, t} = \{ \{c \} + \overline{m}_3 : \exists \quad \xi \quad \varphi \quad \mu \} \]

Irreducible divisor

\[ [\overline{m}_{3, t}] = \tiny{\text{Proj}} (g^{3,t} - \lambda\delta_0 - \sum_{i} (g_i - \lambda_i)) \]
... real locus of admissible cases is empty.

\[ s \left( \frac{g-1}{2g+1} \right) = 6 + \frac{12}{5g+1} < \frac{13}{7} \quad \text{for} \quad g \geq 25 \text{ odd} \]

2. Gieseker-Peskine-Evian

\[ g = 2k - 2 \quad \exists \text{ six rays \ } \gamma_k \]

Choose \( c \) of \( s(c) \)

\[ \mathcal{G}_k^c = \left\{ \left[ C \right] \in \mathcal{E} \mid \exists \gamma_k \text{ for } \mathcal{A}_c \mathcal{P}_k \mathcal{E}_k^c \quad \text{where} \quad 2 \leq h^0(A^c \otimes E) \geq 4 \right\} \]

\[ \to \text{ good for } \quad \text{ some } s \geq 2 \odot \]

3. K3 divisor (Farkas-Peskine)

\[ g = 10 \quad \kappa_0 = \left\{ \left[ C \right] \in \mathcal{M}_0 \mid \exists \text{ six \ K3s} \right\} = \left\{ \left[ C \right] \in \mathcal{M}_0 \mid \exists \text{ six \ K3s} \right\} \]

\[ = \left\{ \left[ C \right] \in \mathcal{M}_0 \mid \exists \text{ six \ K3s} \right\} \]

\[ \subset \text{ a rank 2 Brill-Noether divisor on } \mathcal{M}_0 \]

expected (and) divisor is -1, find such
not for generic curve but for a divisor of $\omega$.

\[ \chi^{\omega} = 7d - 7 - 5d_1 - 9d_2 - 12d_3 - 14d_4 - 15d_5 \]

\( 5 \left( E_0 \right) = 7 \) < slope of Bulldogs curve.

---

all of these are $\text{formal}$ curves!

(will cover all cases during on $M_g$)

Cornell conjecture

\[ \text{Fix } g \text{ & } \tau > 0 \text{ s.t. } \]

\[ p(g, \tau; c) = g - c + (g - c) \tau = 0 \]

\[ \text{Bull.-Newby } \quad g + rs \tau \quad f = rs + r \]

\[ G_d = \text{stack } \{ C, L \} \cdot [ \mathcal{L} \circ \mathcal{V} r - \text{Leq } \mathcal{L}_{\text{Pic}}^{d}(C), \]

\[ h^0 (L) > r + 1 \]

\[ \text{fibers are } \mathcal{V} \mathcal{V}_{GF}(C) \]

Virtual class of fiber is $p(g, rs; c) = 0$.
Brill-Noether theory: Let $\phi$ be a mapping into $\mathbb{P}^1$ and no components map to a single irreducible.

So get Koszul complex for $(C, L) \in G^r_d$:

\[ \Lambda^r H^0(L)^* \xrightarrow{d_i} \Lambda^{r-1} H^0(L)^* \xrightarrow{d_i} \cdots \xrightarrow{d_i} H^0(L)^* \xrightarrow{d_i} H^0(L)^* \]

\[ k_{ij}(C, L) = \ker d_i / \text{im } d_{i+1} \]

\[ U_i = \{ (C, L) \in G^r_d : k_{i+2}(C, L) \neq 0 \} \]

\[ \cdots k_{i+2}(C, L) = 0 \quad i \geq 3 \]

$\dim k_{i+2} - \dim k_{i+1} = \text{length}$

so above is only stratified, we get $k$-caps.

Set $Z_i = \delta_+ (U_i) \leq M_5$
Can copoly

\[ \tilde{g}_d \rightarrow \tilde{g}_d \]

\[ \text{not a full} \]

compactification over \( \tilde{m}_3 \), there are some
subvarieties at high codimension.

\[ \text{Eq: } K_{0,2} \rightarrow \text{Sym}^2 H^0(L^{02}) \]

\[ K_{0,2}(L,L) \text{ to } \leftrightarrow \text{this map is not surjective} \]

\[ K_{1,2}(L,L) \text{ to } \leftrightarrow I_2(L,L) \otimes H^0(L^{02}) \rightarrow I_2(L,L) \text{ not surjective} \]

Not cut off by quantum

equivalent to Green-Lawson No punch.

Then \( n = 2s + 5t + 3i, \quad g = 5s + 8 \) due to

\( \exists \) 2 vector bundles over stack \( \tilde{g}_d \), i.e.,

of size rank \( b \) map \( \rho: \tilde{A} \rightarrow B \)

s.t. \( \tilde{A} \) degenerates. low of \( \rho \)
where \( \overline{z}_i = \sigma(u_i) \),

with \( \sigma \) defined as:

\[
\sigma = \left[ \sigma_1 \ell_1 (g-1) \right]
\]

\[
= a_2 b f_0 \ldots - b_2 a f_{g/2}
\]

\( \alpha \) explicitly given coefficients

\[ \text{slope} = \frac{6/(s_{ij})}{s(i+1)g(s_{ij})} \quad \text{explicit } f_{ij} \]

\[ \leq 6 + \frac{12}{g} \quad \text{with equality } \iff s = 1. \]

\[ \text{For } s = 1 \quad g = 2i + 3, \quad d = 2g - 2, \quad r = g - 1 \]

\[ g_d = g_{2g-1} \quad \text{i.e. } \quad \text{we consider} \]

\[ = \gamma_2 \]

\[ \Rightarrow \quad \sigma(2i+3, 2g-1) = 6 + \frac{12}{g} \quad \text{in fact Harris-Mumford divisor using Green's conjecture.} \]
\[ \mathbb{Z}_{2;1+3,1} = \left\{ \left[ C \right] : \mathbb{P}^{2;1+3} \cap k_{1,2}(C, \xi) \right\} \]

\( i = 0 \): not projectively normal canonical curves

\( i = 1 \): integral curves...

\[ \mathbb{Z}_{2;3,1} = \left\{ C \in \mathbb{P}^{2;3,1} \ ; \ C \cap \mathbb{A}^1 \right\} \]

Green's conjecture on \( \mathbb{P}^{2;3,1} \).

- so recall Hans' Mumford formula and Green's conjecture.

\[ \begin{cases} s \geq 2 \quad g = 6i - 10 \quad d = 9i + 12 \quad s = 3i + 1 \end{cases} \]

\[ \left\{ \left[ C \right] \in \mathbb{P}^{3;1+4} \ ; \ C \cap \mathbb{A}^1 \right\} \]

\( i = 0 \): case \( s(\mathbb{Z}_{2;0}) \neq 7 \): this is the K3 divisor.

\( i = 2 \quad g = 22 \quad \mathbb{Z}_{2;3,2} = \mathbb{C} \) of genus 22 \( \mathbb{C} \) with \( \mathbb{C}_{2,2} \)
... it's cut out by quadrics, but look for a linear relation among quadrics

\[ s(Z_{22,2}) = 6.500 \pm 3.2 \ldots \]

Can apply this technique to \( M_{3,n}, \) Poincaré models of \( K3, \) etc.

Thus \( M_{3,2} \) is of second type

If constant effective divisor \( D_2 \) on \( M_{3,2} \)

with \( s(D_2) < 6.5. \)

look at \( J_{25} \) over \( M_{3,2} \)

(regard to those where \( E_0 = W_{25}^5 \))

... restrict topological basis \( E, F \)

\( E(c,1) = H^2(C,1), \) \( r_k = 7 \)

\( F(c,1) = H^2(C,E^0) \) \( r_k = 29 \)

Sym² \( E \rightarrow F \)

\( r_k 29 \) \( r_k 29 \)
\( \sigma^{-1}(c) = W_{28}(c) \) curve of large genus >106... 

\( U \) = best dependence: where \( r \neq 27 \), 

virtual cabin = 2.

\[ D = \sigma_{x}(U) \quad \text{virtual cabin} \]

\[ L_{10} \quad \text{is a roof line} \]

\[ \text{cost.} \left( \frac{1712}{2636} \lambda - b_{0} - \frac{14811}{2636} \lambda_{1} - \sum b_{i} \cdot f_{i} \right) \]

\[ s(D) = \frac{1712}{2636} = 6.49 \% < 6.5 \quad b_{1} \geq 1 j22 \]