(after ideas of Kontsevich, Seidel, Manin-Vafa,)
part joint w/ Katzarkov & Orlov

Mirror symmetry for Calabi-Yau

\((X, J, \Omega, \omega) \rightarrow (X^*, J^*, \Omega^*, \omega^*)\)

\(\bigoplus_{n=0}^{N} \Omega_n\), \(K\)-\(\text{Kähler form}\)

\(\text{Rel. : SYZ (after Kontsevich-Siebert & Gross-Siebert }\)
\(\text{dual torus fibrations with singularities over common base}.

Recip. look for SLAG toric fibration

(with singularities)

\(T^* \rightarrow X\)

\(\text{if}\)

\(B = \text{moduli of SLAG}\)

\(\text{tori in } X\)
Structure: $T_b \mathcal{B} = H^i(f^{-1}(b), \mathbb{R}) = H^{n-i}(f^{-1}(b), \mathbb{R})$

Canonical identification

Z-lattice -> natural integer
affine structure on $\mathcal{B}$
2 of them, B minor sparse

2. tentative mirror

$M = \{ (L, \Omega) \mid L \text{ Slog } T^n = \Omega \}
\downarrow \\ B \ni \Omega \text{ (not UC) commut/symmetric}$

3. Instanton corrections: really close to make sense of the geometry near the singularity.

Homological Mirror Symmetry (kontsevich)

$D^b(C_h \times \mathbb{I}) \cong D^b(F(x^\nu))$
\( \text{Karasi complete derived Fukaya category} \)

\[ D^b \mathcal{F}(X) \cong D^b \text{coh}(X^r) \]

Motivation for SYZ from this:

- Op \( \in D^b \text{coh} X \) point objects
- hope it corresponds to \((L, V)\) on \(X\).
- look at moduli for \(L\)
- so points of mirror moduli should parametrize pairs \((L, D)\) on \(X\).

\( \underline{\text{Fukaya}}(\mathbb{A}_\mathbb{C}) \text{ category } F(X, \omega) \)

- \(X\) convex symplectic
- \( \omega \) canonical \( \omega \)
- \( L \) exact Lagrangian

\( V = \) flat connections on trivial unitary line bundle
( + B field: shall be precisely the
not that: should have controllable budding
shall have condensing objects so...)

\[ \text{Hom}(l_1, l') = \text{CF}^* (l, l') \text{ for some} \]
\[ = \mathcal{C}_{l_1} \text{ when } l \neq l' \]

(very likely from modular theory...)

have differentials, composition, k-fold composition...

\[ w_k : \text{Hom}(l_0, l_1) \otimes \ldots \otimes \text{Hom}(l_{k-1}, l_k) \]
\[ \rightarrow \text{Hom}(l_0, l_k) [2-k] \]

[\[ w_1 = 0, \quad w_2 = \text{product}, \quad w_k = \text{Messy product}\]]

\[ w_k (P_{l_0}, P_{l_1}, \ldots, P_{l_k}) \text{ in the product} \]

= \[ \sum_{P_{l_0} \otimes l_1 \otimes \ldots \otimes l_k} \left( 2 \times M((P_{l_0}, l_1), \ldots, (P_{l_{k-1}}, l_k), (l, l)) \right) \text{Pol} \]
Choose J almost-complex structure

\[ M((p, i\mathbb{R}), B) \text{ moduli of pseudo-Kähler discs of this type with class } B \]

Today: \((X, \omega, J)\) smooth complex Kähler

\[ D = X - \text{anti-canonical divisor } -K_X \]

with normal crossing singularity (later will be smooth)

\[ [X \text{ Fano works great, unknown for } X \text{ of good/}] \]

\[ X - D \text{ open Calabi-Yau} \]

\[ \Omega \text{ hol. volume form on } X \text{ with poles on } D \]

(week 17: \(\Omega\) does not have constant form)
Look for SLAG $T^n$ fibration, all fibers compact.

$$T^n \rightarrow X - D$$

$B$ affine base w/ singularities locally.

$\Rightarrow$ build the mirror $M = \text{opn } C^\ast$ (with instanton correction).

Points of $M \rightarrow (L, D), \ L \in X - D$.

How to build the mirror to $X$ itself from this?

$\Rightarrow$ equip mirror $M$ with a superpotential

(LG model): $w: M \rightarrow \mathbb{C}$ holomorphic function.

Meaning of $w$: what happens to $w$ if we add the divisor?

Their Floer homology gets messed up;

$L$ will have holomorphic discs passing through $O$. 
... in fact Fukaya category has an
\( m_0 \) : curved Aoo category, has
an obstruction term:

For each Lagrangian \( \mathcal{L} \)
\( m_0 : \text{End}((\mathcal{L}, \mathcal{L})) \) (end of identity)

\( \& \quad m_i^2 = m_2 (m_{i-1}, -) - m_2 (\cdot, m_0) \)

on \( \text{Herm} (\mathcal{L}_0, \mathcal{L}_1) \)

instead of \( m_i^2 = 0 \).

In our case \( m_0 \) is a scalar multiple
of the identity

\( m_0 = 1 (\mathcal{L}, \mathcal{L}) \cdot \text{Id} (\mathcal{L}, \mathcal{L}) \)

If we look only at SLAGs in CYs

\[ \text{Neveu index} = 0 \quad \& \quad \text{don't care discs} \]

with non-zero degree ending on \( \mathcal{L} \),

but otherwise do use to force \( m_0 \).
Define $W(l, \theta) = \sum_{y = \epsilon l \theta} \deg \text{ev}_y M(\mu)$

$\mu(\mu) = 2 \quad e^{-\frac{1}{\mu}}$

$(\mu(\mu) = 2 \iff y \circ \theta = 1) \quad \text{when} \quad \theta \circ \theta = 2$

degree of $\mu$-clash on $L$
which is number of discs
through a sing point of $L$

\[\Theta \]

(would space be $\mu$-in $n-3 + \text{matt}\$

This is ill defined holomorphic function
on $M = \{ (l, \theta) \}$ - have intersection crossing
= modifications of minor as analytic space
which makes e.g. GL invariant well defined.

\[W \text{ depends on choice of point } p \in L\]
holomorphic study - need to change
$M$ to get well defined function.
2. Even if $HF(L,L) = H^*(H^0(L,L, E))$ is well-defined, it's usually $= 0$.

$HF(L,L) = 0$ unless $(L,E)$ is a critical point of $W$ (ie category is localized on singularities of $W$; LG model)

$m_i^2 \neq 0 \quad \implies \quad \text{matrix A has 2 distinct eigenvalues.}

\hline
\text{HMS in this context (Kontsevich)}
\hline
\begin{align*}
\text{1. } & D^b G^X = D^b F(M,W) \\
& = D^b F(M,F) \quad \text{relative Fukaya with fiber } F \\
& \text{of } W \\
\text{2. } & D^b F(x) = DB(M,W) \\
& \text{a low category of singularities}
\end{align*}

\hline
\text{From now on assume } D \text{ is small, } D \subset X \text{ is itself a CY hypersurface.
holomorphic volume form of $D = \text{residue}$ of that on $X - D$. $\Omega_0 = \Pi_0 - L$.

So SYZ maps $D$ as a Lagrangian fibration (by $T^{n+1}$).

e.g. $X = \text{torus}$, $D = \text{U-dicke subset [not smooth]}

$X - D = \mathbb{C}^n$, $\Omega = \frac{dz_1 \wedge \cdots \wedge dz_n}{\prod_i (z_i - z_i^0)}$;

$T^n$ orbits are Lagrangian.

As we go to walls (give corners!)

they collapse to $T^{n-1}$, orbits on boundary

(Conj. Near its boundary, the Slag $T^n$ fibres on $X - D$ consist of tori $S^1$ - fixed and a Slag $T^{n-1}$)

(i.e. I Slag torus fibration with such properties...
\[ L/\mathcal{D} \]

\[ \mathcal{L} \text{ of } L \text{ lands} \]

\[ \text{the fiber over } \mathcal{A} \]

\[ \text{L then as the minor one} \]

\[ \mathcal{E} = \text{class of meridian disc} \]

\[ \mathcal{S} = \mathcal{S}^2 \mathcal{F} \mathcal{B}(\mathcal{F}) \text{ (fiber) function on } \mathcal{M} \]

\[ \text{near boundary} \]

\[ \text{near } \Gamma, \quad \mathcal{W} = \mathcal{Z}_{\Gamma} \times \mathcal{Z}_{\mathcal{A}} \]

\[ \text{... bounded right by } \mathcal{L} \text{ near } \mathcal{M}. \]

\[ \mathcal{E} = \{ \mathcal{Z}_{\Gamma} = 1 \} \neq \mathcal{F} = \{ \mathcal{Z}_{\Gamma} = 1 \} \]

\[ \text{“fiber of } \mathcal{W} \text{”} \]

\[ \mathcal{L} \text{ is collapsed on } \mathcal{D} \text{ & } \mathcal{D} \text{ is pulled} \]

\[ \text{back from } \mathcal{A} = \mathcal{D} \]

So \[ \mathcal{F} = \mathcal{S} \mathcal{Z}_{\Gamma} \neq \text{ minor to } \mathcal{D}! \]

\[ \text{End restriction functor for relative Euler} \]

\[ \mathcal{F}(\mathcal{M}, \mathcal{F}) \rightarrow \mathcal{F}(\mathcal{F}) \]
Theorem (Awan-Ober-Rabin, Katzarkov-Ober)\nTrue for $X = \mathbb{CP}^2$ blown up at k points
on cube $D_0$, proper transform of cube $D$, this is Failekhet-Zaslavskii

Def (Kontsevich, Seidel after Hori-Iqbal-Vafa)\n$M$ symplectic manifold with convex boundary,\ncorresponds near $M$ a symplectic Khovanov $F \to M$ (outside a compact subset)\n\[ J \circ f \\
1 \in D^1 \]
\[ L \in M \text{ properly embedded Lag submanifold is} \]
admireful if $\nabla L \subset F$ (eg. empty)
1. For $x \in \mathbb{R}$, $f(x) = x^2$.

2. Hamiltonian perturbations:
   \[ H: M \to \mathbb{R} \] supported near $\Omega$
   \[ H = H_{\text{lf}} \] depends only on $\ell$

   $E^+$ - hamilton flow

   Def: Objects of $\mathcal{F}(\omega, \mu)$ are admissible $(\ell, \rho)$

   \[ \text{Hom}_{\mathcal{F}(\mu, \nu)}(L, L') = \]
   \[ \text{CF}_\mathcal{F}(L, E^+(L')) \]

   too small.
   - no intersecting near-boundary

   For restriction, need to fix the boundary
   pointwise $\Rightarrow$ use $E^+$ to rotate