A. Okounkov - Monodromy of the Quillen

Diffusion Equations to the Hilbert Scheme of Points on \( \mathbb{C}^2 \)

(w/ R. Bezrukavnikov)

\( X \) variety, \( D \) divisor

(for us \( X_n = \text{Hilb} (\mathbb{C}^2, n) \) \( , \mathcal{O}(1) = \mathcal{O}(0/1) \))

Quantum differential equation

\[ \Psi(q) \text{ function valued in } H^0(X) \]

\[ q \frac{d}{dq} \Psi(q) = D \Psi(q) \]

\( \ast = \text{quantum multiplication:} \)

\( q \)-deformation of product on \( H^0(X) \)

derived by \( (\text{exp }, \lambda) = \sum_{\beta \in H_2(X)} \mathcal{P}_\beta \psi(x \lambda) \)

\( \text{exp} \text{ curves: really } \int_{M_{0,1}(\mathbb{C}, \beta)} (\text{---}) \)
We'll have \[ \mathfrak{g} \subset \bigoplus H^*_T(H^1/\mathbb{C}^2, \mathbb{C}) \] \text{located}

\[
\therefore \quad \text{Nakajima } C(1, t_2) \cdot \Lambda
\]

\[ \Lambda = \text{symmetric functions } = \mathbb{C}[x_1, x_2, \ldots] \]

\[ P_k = \sum x_i^k \text{ power sum.} \]

\[ n \rightarrow \text{degree of symmetric polynomial.} \]

\[ \text{Lie } \mathfrak{g}^2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{t_2} \end{pmatrix} \right\} \]

(we localize for convenience — in the ad the answer is integral)

\[
\text{Operators } \xi_k = \text{mult. by } P_k \\
\xi_k = k \frac{\partial}{\partial P_k} \quad k \geq 0
\]

Let \( M \) denote \( \mathcal{O}^* \), quantum mult.
\[ M_D = \left( \sqrt{1 + h^2} \right) \sum_{k=0}^{\infty} \left[ \frac{k}{2} \frac{(\sqrt{2})^k}{(-2)^k} - \frac{1}{2} \frac{(2i)^k}{(-2)^k-1} \right] \frac{x^k}{k!} \]

The operator is diagonal in the basis
\[ \mathcal{P}_m = \prod_{\mu < m} \mathcal{P}_\mu \]

Part studied by Lekz

So the diff of less regular singularities,
can study its monodromy

The intervals between "all singular findings"
known to humans

The monodromy controls all generating
functions in equivalent SU/DT theory
Residues at \( q = 0, \infty \)

\[ \to \text{(second quantized) quantum Calogero-Moser Hamiltonian} \]

Schrödinger equation for quantum CM

With infinite coordinates,

+ time dependent perturbation

non-autonomous integrable system

Residues at minus roots of unity:

\[ \text{Res}_q = (q + x_k) \sum_{k=1}^{\infty} x_k^q \]

\[ (q)^k = 1 \]

So exponents all belong to \((t_1, t_2, \mathbb{Z}^2) \geq 0\)

Theorem \([20]\) The monodromies for

for \((b_1, b_2) \in \mathbb{R}^2\) are isomorphic

(for generic \(b_1, b_2\) with known structure)
M0 is self adjoint (property of Graham product!) so monad is unitary

with hermitian form

\[
\langle P_i, P_j \rangle = \sum Y^i Y^j \prod_i (T_{1, i}^i - T_{1, i}^{i+1}) (1 \leq i \leq k) 
\]

\[ T_i = T_i^{-1} \]

this is the Mac beck inner product

= natural inner product on T-equivalents

K-theory of the Hilbert spaces

(some pointing)

Berezians: monad defined by equivalence on \( D^+_T (\text{Hilb}) \)

& for every point of \( C^* \)

should have a Bridgeland stability condition on \( D^+_T (\text{Hilb}) \) & equivalences are deck transformations
Some cases (in progress) from geometries in characteristic p & rep theory of Chowring abelian group.

[should be Poincaré on equivariant reps]

- currently prove by calculating both sides.

We already know monodromy at 0,0, so concentrate on roots of unity.

look at universal cover $\tilde{\Sigma}$ minus preimages of singularity.

To any interval here we associate a basis of our vector space $K_0$ so as to get basis of Schur functions at -1.

$\dim_i K \longrightarrow C[[\log z]]$

"central charge" $f \mapsto (f, (-z)^i)$
(1) is a nilpotent operator, \((-y^6)\) is a polynomial operator in \(\log y\), can pair with \(0\).

At \(y=1\) this is \(S_+ \rightarrow \sin(x)\)

\text{dim of corresponding image}

\begin{align*}
\text{basis } & \{v_i\} \\
\text{singularity } & \text{to construct the monodromy } \mathcal{R}
\end{align*}

\text{assoicated to the } \text{gen } \text{singularity }

\text{+ the } \text{singularity } \text{in the } \text{next interval.}

Assume the basis is ordered wrt order

of vanishing of \(\ln y(v_i)\) as \(y \rightarrow y_0\)

\(\mathcal{R}\) is the unique unitary operator

with infinite series of \(\mathcal{T}\)'s in descending

\(\mathcal{R} = \begin{pmatrix} \infty & (\mathcal{T})^1 & 0 \end{pmatrix} \) which is unitary.

\(\mathcal{T}\) This is monodromy-so need

something like a \(\mathcal{S}\)-connected
New basis: only $S = JT_A$

$R = \begin{pmatrix} 1 \\ T_1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ T_n \end{pmatrix}$

Procedure: $A \rightarrow$ Gaussian-Jordan Elimination

Factorization: $A = T_A E T_E \rightarrow$ good...