Goal: describe how gauge theory captures a variety of topics in representation theory.

Plan:  
- 2d gauge theory & reps of finite groups
- 3d gauge theory & reps of real & complex Lie groups
- 4d gauge theory & the geometric Langlands program (reps of loop groups, quantum groups etc)

- Demonstrate applications of the powerful results & techniques developed in the lectures by Jacob Lurie & Bertrand Toën.

What is gauge theory?  
Physical theory in which fundamental objects (fields) are sections on principal $G$-bundles on spacetime $M$. 

Classical gauge theory — describe spaces of connections satisfying classical equations of motion (flatness, or Yang-Mills equations) [won't concern us!]

Quantum gauge theory — a quantum field theory in which we study the collection of all connections (with a weighting given by a Yang-Mills type action) by attaching C-linear invariants.

Topological gauge theory — study “coarse” features depending only on the topology of spacetime $M$ (i.e., not requiring a metric, conformal structure, etc.)

Toy model 2d gauge theory with finite gauge group $G$ [variant of Dijkgraaf-Witten theory]
Geometry: to any 0,1,2 manifold $M$ we consider the space of gauge fields $M_\mathcal{G}(M) = \{ G\text{-bundles on } M \} / \sim$

[automatically carry flat connections!]

$= \{ G\text{-Galois covers of } M \} / \sim$

(M corrected) $(m\text{ points}) = \{ \pi_i(M) \rightarrow G \} / \text{conj.}$

...we'll have to keep track of symmetries/stabilizers:

$M_\mathcal{G}(m)$ is a finite orbifold — finite set of points $\pii \circ \circ \circ$

with finite groups $\circ \circ \circ$ attached.

$\cdot M_\mathcal{G}(pl) = \mathcal{G}_c = BG$

$\cdot M_\mathcal{G}(S^1) = \frac{G^G}{G} = \bigsqcup \text{conj. classes}$

$\cdot M_\mathcal{G}(\Sigma_g) = \{ \frac{\mathcal{A}_1, \ldots, \mathcal{A}_g \in G : \sum_1^g [A_i, B_j] = 1}{G} \}$
2d gauge theory \( Z(M) = \int_{\text{fields on } M} e^{-S(\varphi)} \) d\varphi

- calculate volume of \( \text{Fields}(M) \) with measure \( e^{-S(\varphi)} \).

Our case: \( Z_6(\Sigma) = \# \text{M}_6(\Sigma) \):
weighted number of \( G \)-bundles on \( \Sigma \)

\[
= \sum_{\{P\} \text{ stable}} \frac{1}{|\text{Stab} P|}
\]

Locality: calculate \( Z_6(\Sigma) \) via cut & paste in \( \Sigma \)

\( \square \) on manifold with boundary
need to specify boundary conditions to get well defined path integral

\( \varphi_0 \in \text{Fields}(\Sigma_M) \)

\[
Z(M)(\varphi_0) = \int_{\varphi_1 |_{\Sigma_M} = \varphi_0} e^{-S(\varphi)} \, d\varphi
\]

\( \Rightarrow Z(M) \in \text{Functions} (\text{Fields}(\Sigma_M)) =: \text{Z}(\Omega_M) \)
... ie assign a vector space $Z(\mathcal{M})$ to codimension one manifolds.

\[ Z(\mathcal{M}) : Z(\mathcal{M}^0) \rightarrow C \]

\[ f \mapsto \int f(\psi_{\mathcal{M}}) e^{-S(\psi)} d\psi \]

\[ Z(\mathcal{M}_{\text{in}}) \xrightarrow{Z(\mathcal{M})} Z(\mathcal{M}_{\text{out}}) \]

\[ f \mapsto \{ \psi_{\text{out}} \mapsto \int f(\psi_{\text{in}}) e^{-S(\psi)} d\psi \} \]

More generally have a correspondence

\[ T_{\text{in}} : \text{Fields} (\mathcal{M}^{\text{in}}) \xrightarrow{T_{\text{out}}} \text{Fields} (\mathcal{M}_{\text{out}}) \]

\[ T_{\text{out}} \times (T_{\text{in}} f \cdot e^{-S}) \]
Our finite setting: integrals are finite sums, $e^{-S}$ just counts count bundles weighted by automorphisms, Key: functions $= \text{measures}$ can pullback, multiply $\&$ pushforward

\[ Z(S') = \text{Functions}(\text{Fields}(S')) \]
\[ = \mathbb{C}[\mathbb{G}] = \mathbb{C}[\mathbb{G}]^G \]

class functions on the group $G$.

Operators on $Z_G(S')$:

- $Z_G(\text{cylinder}) = \text{id}$ (physically: Hamiltonian = 0 ..., i.e. we're studying quantum mechanics of the vacuum!)

- $Z(0) = \sum \in \mathbb{C}[\mathbb{G}]^G$

- $Z(\partial) = \text{eval}_1 : \mathbb{C}[\mathbb{G}] \to \mathbb{C}$

- $Z(\otimes) = \text{convolution} \colon \mathbb{C}[\mathbb{G}] \otimes \mathbb{C}[\mathbb{G}] \to \mathbb{C}[\mathbb{G}]$

\[ \text{graph of multiplicities} \]

Operator picture: $\mathbb{C}[\mathbb{G}]$
\[ CG^6 \subset CG \text{ group algebra,} \]
with multiplication induced by pullback
\[ \text{by maps } \ \phi \colon G \times G \rightarrow G, \quad \phi_{g_1} \times \phi_{g_2} = \phi_{g_1 g_2} \]

- In fact \( CG^6 \) is a commutative Frobenius algebra:
  \[ \text{tr} (\phi_{g_1}) \text{ is a nondegenerate inner product} \]
  \[ = \mathcal{Z}(G) \text{ inner product} \]
- \( \mathcal{Z} \) this structure deforms \( \mathcal{Z}_6(3) \)
  for all 1,2-manifolds.

To solve the system: [complete integrability]
simultaneously diagonalize the local operators \( \leftrightarrow \) spectral decomposition
\[ \Rightarrow \text{calculate } \text{Spec } CG^6 \]
\[ = \{ \text{homomorphisms } G \rightarrow \mathbb{C} \} \]
\[ = \{ \text{joint eigenvalues } \} = \mathcal{G} \]
\[ = \{ \text{irreducible characters of } G \} \]
\[ \mathcal{G} \]

\[ \mathcal{Z}(\Sigma) = \mathcal{Z} \left( \frac{161}{\dim \chi} \right)^{2g-2} \text{ "Moss formula" \[ \text{[Frobenius]} \] } \]
Codimension 2: \( C \)

Before we assigned a vector space (functions on \( \text{Fields}(N) \)) to an \( n-1 \) manifold. If \( N \) has a boundary, we need to fix boundary values on fields to get a vector space. \( \rightarrow \) \( \mathbb{Z}(N) : \{ \text{fields on } \} \rightarrow \text{vector spaces} \)

ie \( \mathbb{Z}(N) \) can be considered as a vector bundle (or more singular \( C \)-linear sheaf) on \( \text{Fields}(\mathbb{C}N) \)

\( \mathbb{Z}(N) : \text{Vec}(\text{Fields}_N) \rightarrow \text{Vec}(\text{Fields}_{0,N-1}) \)

\[ V \mapsto T_{0,N} \times (T_{n} \times V \oplus \mathbb{R}^{-S}) \]

\[ Z(N) \]

\[ Z(N_1) \]

\[ Z(N_2) \]

Axiomatisms in Jacob's talks...
Our case: $Z_6(\cdot) = \text{Vect}(\cdot_6) = \text{Vect}(B_6) = \text{Rep}_C(G)$ representations = $\text{Mod}(C_6)$ modules for the group algebra.

$Z_6(\cdot) = \text{Vect} : \text{cousin case of closed}(m,1)...$

Examples:

- $A \xrightarrow{B} \text{Hom}_C(A, B)$

- $\text{Hom}(A, B) \otimes \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)$

- $1 \rightarrow \text{Hom}_C(A, A) = \text{End}_C(A)$

- $C \rightarrow \text{End}_C A$ unit

- $1 \rightarrow \text{Id}_A$

- $\text{End}_C A \rightarrow C$:
  - have a canonical trace on endomorphisms of any representative!
  - $\text{Id}_A \rightarrow \dim A$

i.e. $\text{Rep}_C$ is not just any category, it is a "Frobenius" or Calabi-Yau category!
reflects fact that $\mathbb{C} G$ is a (noncommutative) Frobenius algebra, with $tr = eval : \mathbb{C} G \to \mathbb{C}$ nondegenerate trace.

More refined version: \[
\begin{array}{c}
\mathbb{C} G^G \to \text{End}_\mathbb{C} A : \\
\end{array}
\]

$\mathbb{C} G^G$ acts as symmetries of any $G$-representation.

In fact $\mathbb{C} G^G = Z(\mathbb{C} G)$

$= \text{End} \text{Id}_{\text{Mod}(\mathbb{C} G)}$

center of the group algebra

= endomorphisms of the identity of $\text{Rep} G$. More algebraically:

$\mathbb{C} G^G = \text{Hom}_{\mathbb{C} G-\mathbb{C} G}(\mathbb{C} G, \mathbb{C} G)$

domorphisms of $\mathbb{C} G$ as a bimodule over itself.

- toy model for Hochschild cohomology
Dually: \[ A \overset{\cdot}{\otimes} }^\ast \text{End}_G A \overset{2\cdot}{\to} C^G \]
(universal) trace with values in \( C^G \)
\[ \text{Id}_A \mapsto \text{char}_G(A) \]
character of the representation!

In fact \( C^G = C^G \otimes_{C^G} C^G \)
\[ = \text{target of universal map } C^G \overset{\cdot}{\to} V \]
with trace property \( \langle c b \rangle = \chi(\delta a) \)
- toy model for Hochschild homology.

Hecke algebras
Natural source of representations:
\[ K \triangleleft G \text{ subgroup } \Rightarrow V_{6,K} = C[G/K] \]
\[ = \text{Ind}_K^G \in \text{Rep}_G \]

Hecke algebra \( \mathcal{H}_{6,K} := \text{End}_6 V_{6,K} \)
\( V_{6,K} \) represents the functor of \( K \)-invariants:
\[ \text{Hom}_6(V_{6,K}, W) = W^K \overset{\Psi}{\to} \mathcal{H}_{6,K} \]
\[ \varphi \mapsto \varphi (\xi_K) \]
so \( \mathcal{H}_{6,K} = \text{End}(\cdot)^K \)
In particular, $H^*_G,k = \text{Hom}(V_G,k, V_G,k) = V_G,k = C[G/k]^k = C[K\backslash G/k]$

- Subalgebra of group algebra $\mathbb{C}G$
- Consisting of $K$-biinvariant functions.

\[
\begin{array}{cccc}
G \times G & \longrightarrow & G / K \\
K \times K & \longmapsto & K \backslash G / K
\end{array}
\]

TFT picture:
- $G$-bundles on interval with $K$ reductions at the ends

\[H^*_G,k \text{ - mod } \longrightarrow \text{ Reps of } G \text{ generated by } K\text{-invariant reps}
\]
\[= \langle \text{reps appearing in } V_G,k \rangle
\]

0 ➜ fundamental correspondence

0 ➜ 0 ➜ 0 ➜ 0 ➜ 0

\[G / G \longrightarrow G / K \longrightarrow K \backslash G / K
\]

\[\mathbb{C}G \longrightarrow \mathbb{C}G \longrightarrow \mathbb{C}G
\]

\[\text{char}(V_G,k) \longleftarrow 1_k \longleftarrow \pi^* 1_{\text{diag}}
\]
Gauge theory for complex groups

We'd like to replace finite groups by complex (reductive) groups, like $\mathbb{C}^*$, $GL_n \mathbb{C}$, $SO_n \mathbb{C}$ etc.

Then we may variants of G-gauge theory... we'll attempt to linearize the moduli spaces

$$M_6(M) = \{ \text{G local systems on } M \}$$

$$= \{ \Pi_1(M) \rightarrow G \}/\simeq$$

$$M_6(S^\prime) = \frac{G}{G / W}$$

collection of conjugacy classes - generically parameterized by eigenvalues

$$M_6(\Sigma_g) = \left\{ A_1, \ldots, A_g \in G : \Pi_1[\Sigma_g] \mathfrak{g} \right\}/G$$

eg $M_3(\Sigma_1) = $ connected pairs in $G/\simeq$

... interesting singular affine varieties with action of $G$.

How do we "count points" on $M_6(\Sigma_g)$ or "integrate functions" on $M_6(S^\prime)$?
One solution: look over varying finite fields $\mathbb{F}_q \rightarrow 2d$ TFT depending on $q$ …
(cf. work of Hausel, Rodriguez-Villegas)

Algebraic geometry (Weil conjectures) teaches us that numbers of points over finite fields are avatars of the cohomology of the underlying variety, & more generally functions on $\mathbb{F}_q$ points are avatars of sheaves…

\[\implies\]

Another solution: categorify!

Numbers $\rightarrow$ vector spaces
(such as cohomology)

Vector spaces $\rightarrow$ categories
(such as categories of sheaves)

Physics motivation: compactification or dimensional reduction

$\overline{\mathbb{Z}}$ $n$-dim TFT $\rightarrow \overline{\mathbb{Z}}$, $(n-1)$-dim TFT

$\overline{\mathbb{Z}}$ $(M) = \overline{\mathbb{Z}}(M \times S') = \dim M \times S'$
(in Jacob's terminology)

$\dim$ of a vector space is a number
$\dim$ of a category is a vector space — its Hochschild homology $\text{HH}_*(E) \cong \text{char}(E)$ for character = charge of $F$
To linearize we'll need a good theory of functors/ 
measures on \( G, \frac{G}{G} \) etc. with similar 
formal properties (pullback, product, pushforward).

Our setting: replace vector spaces of functions 
by \( \mathcal{D} \) categories of sheaves. 
(cf Bertrand's lectures)

How many variants -- analogs of classical functor spaces!

I. \( X \) variety \( \mapsto \mathcal{O}(X) \) dg category of 
quasi-coherent sheaves on \( X \); basic 
examples are (elsewhere) vector bundles
(finite or infinite rank). More general objects 
come from kernels & cokernels (or complexes)
of maps between bundles.

More formally \( \mathcal{O}(X) \) is defined by

1. assign \( \text{Spec } R \mapsto \text{Mod } (R) \)
chain complexes of \( R \)-modules, with 
quasi-isomorphisms invertible.

2. descent: calculate \( \mathcal{O}(X) \) from 
\( \mathcal{O}(U) \) sheaves on a cover \( U \to X \) 
with gluing data on double overlaps
\( U \times U \), triple overlaps \( U \times U \times U \), & so on...

This definition extends to \( \text{stacks} \) for us 
\( G \otimes X \) variety \( \mapsto \mathcal{O}(X/G) = 
G\text{-equivariant sheaves on } X \)
II. \( X \rightarrow D(X) \) dg category of D-modules on \( X \).

Basic objects: vector bundles with flat connection. More general objects:

- Quasi-coherent sheaves with flat connection
- \( \mathcal{V}: F \rightarrow \mathcal{F} \otimes \Omega \), \( \nabla^2 = 0 \) \( \iff \)
- \( T \otimes F \rightarrow F \) extends to \( D \otimes F \rightarrow F \)

General definition of D-modules parallel:

1. Spec \( R \rightarrow \) module for \( \mathcal{D} \)
   - diffops with \( R \)-coefficients
2. Extend by gluing.

One key motivation for D-modules:

- Tight analogy with classical functions/distributions:

\[ f \in C^\infty(X) \quad \text{or} \quad \sqrt{f} \in C^\infty(X) \]

\[ \rightarrow D \cdot f \in C^\infty(X) \quad \text{or} \quad D \cdot \sqrt{f} \in C^\infty(X) \]

Span under algebraic diffops is a \( D \)-module.

In good cases ("holonomic"), the \( D \)-module captures \( f/f \) up to finite dimensional ambiguity.
A' example: \( e^x \to \mathcal{D}/\mathcal{D}(e^{-x}) \) 
\[ \sum \to \mathcal{D}/\mathcal{D}(x^{-x}) \]

\( \mathcal{D} \) \( x \to \) \( DCA' \to DCA' \)

Fano transform, not on functions but on systems of diffeqs ...

D-modules are also closely related to the "random points over \( \mathbb{F}_q \)" approach, via the Riemann-Hilbert correspondence: the sheaves of solutions of these diffeqs are the characteristic 0 analogs (constructible sheaves) of the sheaves appearing in the Weil conjectures story...
Algebra: Vect is a symmetric monoidal $(\otimes)$ category.
\(\text{dgCat}\) is a ""(\(\infty,1\))" category.

... means we can talk about (associative & commutative) algebra objects in $\text{dgCat}$,
& modules over these algebras, etc - analog of all of basic algebra
(Lurie - DAG II-III)

\(\Omega(X) \& \mathcal{D}(X)\) are both commutative algebras in $\text{dgCat}$, with multiplication given by $\otimes$ of bundles (or sheaves).

Also have notion of pullback of bundles (or bundles w/ flat connection).
\(\Rightarrow \pi : X \rightarrow Y\) induces $\pi^* : \Omega(Y) \rightarrow \Omega(X)$
\(\pi^+ : \mathcal{D}(Y) \rightarrow \mathcal{D}(X)\)

& pushforward: calculate cohomology
(coherent or de Rham) along fibers.
\(\Rightarrow \pi_! : \Omega(X) \rightarrow \Omega(Y)
\pi_* : \mathcal{D}(X) \rightarrow \mathcal{D}(Y)\)

[all operations are derived]

\(\Rightarrow\) integral transforms: \(K : \Omega(X \times Y)\) (or $\mathcal{D}(X \otimes Y)$)
\[ \mathcal{F} \in \mathcal{Q}(X) \mapsto K \ast \mathcal{F} \in \mathcal{Q}(Y) \]

\[ \prod_{x} X \ast Y = \prod_{y} \left( \prod_{x} \mathcal{F} \circ K \right) \]

\[ \mathcal{F} \quad \mathcal{F}(x) K(x, y) \, dx \]

\[ \mathcal{Q}(X) \text{ case: equivalences given this way are known as Fourier-Mukai transforms.} \]

\[ \text{Note these are the kind of operators we need to define "path integrals".} \]

\[ \text{Fields}(N) \quad \text{Fields}(Z_{odd}) \]

\[ \text{Gluing} / \text{Composition} \]

\[ \text{Fields} \left( \frac{M_{1}}{N} \right) \quad \text{Fields} \left( \frac{M_{2}}{N} \right) \]

\[ \mathcal{F}(M_{1}) \quad \mathcal{F}(M_{2}) \quad \mathcal{F}(N) \quad \mathcal{F}(Z_{odd}) \]
Finite analogs: \( X, Y \) finite sets \( \Rightarrow \)

\[
\text{Fun}(X, Y) = \text{Fun}(X) \circ \text{Fun}(Y) = Y \text{ by } X \text{ matrices} = \text{Hom}(\text{Fun}(X), \text{Fun}(Y))
\]

[Note \( \text{Fun}(X) = \text{Fun}(X)^* \) via canonical inner product \( f \cdot g = \sum f(x)g(x) \)]

Relative version: \( X \to Z \to Y \)

\[
\text{Fun}(X, Y; Z) = \text{pairs with same image} = \text{Fun}(X) \circ \text{Fun}(Y, Z) \text{ (impose algebraically)} = \text{Fun}(Z) \text{ - block diagonal matrices}
\]

\[
= \text{Hom}_Z(\text{Fun}(X), \text{Fun}(Y))
\]

**Theorem:** Functors, tensors & integral transforms

\( X \to Z \to Y \) "perverse sheaves" (es scences, most characteristic)

\[
\phi(x) \otimes \phi(y) = \phi(x \times y) = \text{Fun}_Z(\phi(x), \phi(y))
\]

[Toën for \( X, Y \) schemes, \( Z \) - pt., \( BZ \) - Fracis - Nadler in general ... Note fiber product is derived]

\[
\mathcal{O}(X) \otimes \mathcal{O}(Y) = \mathcal{O}(X \times Y) \cong \text{Fun}_Z(\mathcal{O}(X), \mathcal{O}(Y))
\]

\[
\mathcal{O}(Z) \otimes \mathcal{O}(X) = \mathcal{O}(Z \times X) \cong \text{Fun}_Z(\mathcal{O}(Z), \mathcal{O}(X))
\]

for schemes [BZ - Nadler]
3d gauge theory: starting from the point.

Recall for \( \Gamma \) finite, \( \text{Rep}_\mathbb{C} \Gamma = \text{Mod} \mathbb{C} \Gamma \)

- more generally \( \text{Rep}_k \Gamma = \text{Mod} \mathbb{C} \Gamma \)

For infinite groups one specifies different classes of representations (smooth, continuous, measurable, locally constant, ...) by using as modules for different variants of group algebra \( (C_c(G), L^1(G), ...) \)

Categorified setting: G affine algebraic group
we'll look for different notions of
\( G \text{-}(dg) \text{ categories} \) namely want a
functor \( \text{ag} : \mathcal{C} \to \mathcal{C} \) \( \forall g \in G \) with algebras
relating \( \text{ag}_1 \) with \( \text{ag}_2 \) etc.

\( \Rightarrow \) some notion of algebraicity or flatness in g

\( \Rightarrow \) \( QG, DG : \text{qcohant/flat group algebra of G} \)

- monoidal dg category via convolution

\[ \mu : G \times G \to G \]
\[ \xi \ast \xi = \mu \ast (\pi_1^* \xi \otimes \pi_2^* \xi) \]

\( Z(pd) : \) can consider \( \{ G \text{-categories} \} \)
of the appropriate kind:

\( \text{Vect} \Gamma \text{-mod} \) \( (\Gamma \text{ finite}), \)

\( QG \text{-mod} \) or \( DG \text{-mod} \)

(\text{algebraic G-cats}) \( (\text{flat/smooth G-cats}) \)
Source of examples: $G \times X \simeq \text{variety}$

$\Rightarrow \mathcal{O}(X)$ is an algebraic $G$-category

$D(X)$ is a small $G$-category

via $G \times X \rightarrow X$

$G \circ \mathcal{O}(X) \rightarrow \mathcal{O}(X)$ etc.

Prime example $B \subset G$ Borel subgroup

$G = (\mathbb{C}^	imes)^n \subset G_l$.

$G/B = \text{the flag variety of } G$

($G_l$: full flags in $\mathbb{C}^n$)

$\mathcal{O}(G/B)$ is a quasilocal $G$-category

$D(G/B)$ is a flat $G$-category.

**Theorem (Beilinson-Bernstein)**

$\Gamma: D(G/B) \rightarrow \text{og-mal} \quad \text{(via } \mathfrak{g} = \Gamma(G/B, ?))$

is an equivalence to

og-mal's with trivial infinitesimal character (actin of $Z(\mathfrak{g})$)

... ie up to an extra parameter (which is easy to correct, at least generically)

$D(G/B) \simeq \text{og-mal } \chi \neq \mathfrak{h}^*$/goric

What are the symmetries of this G-category?

$$\mathcal{D}(G/B) \xrightarrow{\mathcal{D}(B\setminus G/B) = \mathcal{H}}$$

finite Hecke category.

Action on reps is classical intertwiners for representations.

B orbits on G/B = Schubert cells

Each is contractible.

$$K(\mathcal{H}) = ZW$$, group algebra of Weyl group

So \(\mathcal{H}\) has "bases" (on level of \(K\)-group)

given by different ways of extending

the trivial flat bundle \(C_w\) on each orbit

or bit

- the study of \(\mathcal{H}\) is Kazhdan-Lusztig theory.

Actions of \(\mathcal{H}\) on a category \(C\):

give not Weyl actions, but braid group actions - \(T_s\) (simple reflections) don't square to 1

but satisfy $$T_s; T_s; T_s = T_s; T_s; T_s$$, i.e., (the role in Khovanov link homology)
Examples of \( \mathcal{E} \)-modules: “subs” of \( \text{D}(\mathcal{O}_B) \)

\[
\text{D}(K\setminus G/B) \xrightarrow{B-B} \text{Harish-Chandra } (\mathcal{O}, K) - \text{moduler: }
\]

central objects in rep theory

[eg K symmetric \( \Rightarrow \) reps of real forms of \( G \) !?]

Cobordism hypothesis: Try different assignments for \( \mathbb{Z}(pt) \), see how far up they extend...

\( \text{Vec}_\mathbb{P} - \text{moduler} \Rightarrow 3d \text{ TFT } \mathbb{Z}_p \)

\( Q6 - \text{moduler} \Rightarrow 2d \text{ TFT } [\text{BZ: Nadler} ] \)

\( D6 - \text{moduler} \Rightarrow 1d \text{ TFT } (?!), \text{ but } \)

\( \mathcal{E} - \text{moduler} \Rightarrow 2d \text{ TFT } [\text{BZ: Nadler} ] \)

\( \mathcal{E}_G \) (character theory)

From physics POV, all are part of

3d gauge theory, some better behaved than others... [ rosy: would be DG theory

the union of \( \lambda \)-twisted versions of \( \mathcal{E} \)-theory

over \( \lambda \in \mathbb{C}^* \))
Structures of categorified rep theory

Study modules $\mathcal{O}(G/H)$, $H \leq G$, 
Endomorphisms = $\mathcal{O}(\mu \backslash G/H)$ Hecke category - eg $H = \mathbb{G}$,
$\text{End}_\mathbb{C}(\text{Vec}) = \mathcal{O}(BG) = \text{Rep } G$.

Morita theory:

X finite set, $\text{Fun}(X \times X)$ = algebra of square matrices is Morita equivalent to $\mathbb{C}$:
$\text{Mod}(\text{Fun}(X \times X)) \cong \text{Mod } \mathbb{C} = \text{Vec}$

$X \to Y \Rightarrow \text{Fun}(X; X)$ algebra of block diagonal matrices Morita equivalent to $\text{Fun}(Y)$:
$\text{Mod}(\text{Fun } X; X) = \text{Fun}(Y) \text{mod } = \text{Vec} Y$.

BZ-Francis-Nadler: version for $\mathcal{O}(\text{perfect stack})$:
$\forall H$, $\mathcal{O}(\mu \backslash G/H)$ is Morita equivalent to $\mathcal{O} G$ (same notion of "duelizable modules")

So any of the modules $\mathcal{O}(G/H)$ generates the whole representation theory!

[Special case: Vec $\Gamma \leftrightarrow \text{Vec KG/K}$, theon of Müger & Ostrik]

- Very false for $D$: $\text{Mod } DG$ & $\text{Mod } \mathbb{R}$ differed...
$Z(S')$ has two dual roles (for $Z$ defined on surface):

1. $Z(S') = \operatorname{cl}(\mathcal{Z}(\cdot) = \operatorname{Mod}(\mathcal{A})$
   = Hochschild homology (or abelianization) of group algebra, $A \otimes A$
   $\otimes_{\mathbb{Z}}$
   $Z(S')$ carries characters/characters of $A \in Z(\cdot)$

2. $Z(S') = \operatorname{Endomorphisms of Id}_{\mathcal{Z}(\cdot)}$
   = Hochschild cohomology or center of $A$
   $\operatorname{Hom}_{\mathcal{Z}(\cdot)}(A, A)$
   $Z(S')$ acts on every $A \in Z(\cdot)$

Classical version: Drinfeld center of a monoidal category $C$:

$Z(C): \operatorname{Hom}_{\mathcal{C}}^{\operatorname{op}}(C, C) =$

$\{ F \in \mathcal{C} + F \xrightarrow{\sim} x F \}$

central structure.

$Z(\text{Vect}_F) = \text{Vect} \bigoplus_{I \in I} \text{class bundles}$

$= \bigoplus \mathcal{R} \operatorname{Rep} \mathcal{Z}_p(5)$
Theorem (BZ-Franke-Nadler): 
\[ Q(\mathcal{G}) = \text{center } \& \dim (\mathcal{H}/\mathcal{H}_x) \]
\[ \text{of } Q(\mathcal{G}), \& \text{of } Q(\mathcal{H}/\mathcal{H}/H) \forall H \subseteq G \] (calculate)

Note $Z(S')$ carries a braided tensor product ($E_2$ multiplicative):
\[ \begin{array}{c}
\text{binary operation labeled}
\end{array} \]
\[ \begin{array}{c}
\text{by pairs of discs in}
\end{array} \]
\[ \begin{array}{c}
\text{a larger disc, \& co-representors}
\end{array} \]
\[ \begin{array}{c}
\text{parameterized by } g \mapsto \text{"picture-in-picture"}
\end{array} \]
\[ \begin{array}{c}
\text{doesn't square to the identity,}
\end{array} \]
\[ \begin{array}{c}
\text{but get braid group action on } F \times F \cdots \to F
\end{array} \]

Also, dim $Z(\mathcal{G})$: $Z(S')$ carries action of Diff $S'$.

making it a ribbon category; monodromy automorphism for every $F \in Z(S')$

\[ \text{and } \begin{array}{c}
\text{let } F \mapsto V, V|_{\mathcal{G}} = \frac{1}{\mathcal{G}} \end{array} \]
\[ \begin{array}{c}
\text{and } \text{Rep } Z_+(\mathcal{G}),
\end{array} \]

but $g \in \text{Center}(Z_+(\mathcal{G})) \mapsto g$ gives a canonical automorphism.
Double case: study diagram

\[
\begin{array}{ccc}
\pi & \frac{G}{B} & \rightarrow \\
\downarrow & \frac{(G \times G/B)}{G} & \rightarrow \\
\frac{G}{G} & B \backslash G/B & \\
\end{array}
\]

\([g] \leftarrow (g, B) \rightarrow (B, g \cdot B) \quad \text{rel posn in } W
\]

Restrict to \(1 \in W \leftrightarrow g \cdot B = B \), i.e \(g \in R \):

\[
S \xrightarrow{\tilde{\gamma}/G = \{(g, B) : g \in B\}/G}
\]

\[
\begin{array}{ccc}
G & \rightarrow \\
\downarrow & \\
U & \\
\end{array}
\]

\[
\begin{array}{ccc}
T^*G/B & \text{(or rather affine torsor)} \\
\downarrow & \\
\text{Grothendieck-Springer} & \simultaneous\ resolution \\
\end{array}
\]

Whole diagram is union of \(W\)-twisted versions of this!

Note \(\pi\) is a \(W\)-Galois cover over the dense open subset \(H/W \subset \frac{G}{G}\) (ordering of eigenvalues of \(g\) !)
Basic object in the theory: the Springer sheaf $S \in D(\mathcal{G})$:

- Harish-Chandra's $G$-invariant system of differential equations satisfied by the $G$-invariant distributions on $G$, arising as characters of [admissible, co-dim] representations of $G$; ... explicitly given by \[ \{ L \cdot \mathfrak{g} = 0, L \in \Gamma(G, D) \mathcal{G} \times \mathcal{G} \} \]

- used to show characters are analytic functions with prescribed singularities.

- $S = S \times C \overset{\sim}{\rightarrow}$ pushforward of constant sheaf on Springer resolution
  
  $= T \times C \otimes T_1$ horocycle transform of unit in Iwasawa category

- $S/\mathfrak{g} \otimes C$ is a twisted version of $CW$

*Def (Lusztig) A character sheaf on $G$ is a $D$-module in the image of the horocycle transform $T \times C^\times$ [simple conditions]

- geometric analogs of characters of finite groups $G(T_\mathbb{Q}) \otimes \mathbb{Q}$ ....
**BZ: Nader:** character sheaves are characters of \( H \)-modules!

**Theorem (BZ: Nader)** \( \text{HH}^n(H) = \text{HH}^*_n(H) \)

= image of \( \pi_*' \) in \( \mathcal{D}(\frac{G}{B}) \).

In fact, for 2d TFT, \( \chi_\mathcal{C} : \mathcal{C} \rightarrow \text{mod}_H \)

& \( \chi_\mathcal{C}(S') = \text{character sheaves} \)

(\& \text{other maps given by diagram above)}

\( \mathcal{G} \) is the character of \( \mathcal{D}(\mathcal{G}/B) \)

as a \( G \)-category, or of \( H \) as an \( H \)-module.

**Theorem (Beilinson-Ginzburg-Segal)** \( \mathcal{D}(\mathcal{G}/B) \cong \mathcal{D}(\mathcal{G}/B) \)

**Corollary** Langlands duality for 2d TFTs

\( \chi_\mathcal{C} = \chi_{\mathcal{C}'} \)

- in particular character sheaves for \( \mathcal{G} \& \mathcal{G}' \)

identical.
4d gauge theory & geometric Langlands

Local operators. Key structure in QFT: make measurements on fields near a point $x$: $U(\gamma) = \int \psi(\gamma) \exp \left( - \frac{i}{\hbar} S(\gamma) \right) \, \text{D}\gamma$

- Expectation value of measurement $O$ of $\gamma$ at point $x$ in $\text{D}x$

How to formalize? $U \in Z(S^{n-1})$

- Functional on values of $\gamma$ in punctured neighborhood of $(x,t)$: $Z(S^{n-1}) \otimes Z(M) \to Z(M)$

In general $Z(S^{n-1})$ has a product structure from pair of parts: $E_n$ multiplies:

\[
\begin{array}{ccc}
1 & 
& 1 \\
\circ & 
& \\
\text{n=1} & 
& \\
\circ & 
& \\
\text{n=2} & 
& \\
\circ & 
& \\
\text{n=3} & 
& \\
\end{array}
\]

- more & more commutative as $n$ increases

$Z(M^{n-1})$ is a module.

Loop operators: \(\square\) Given loop $\Rightarrow Z(S^{n-2} \times S^1)$ acts on $Z(M)$:

\[
\text{dim } Z(S^{n-2}) \text{ (8 likewise for other subfields)}
\]
Order operators: \( O(q) = \text{some measurement of } q \). In gauge theory, Wilson loops: \( L \) a loop

\( W_{R,L} = \text{trace of holonomy along loop } L \)

in representation \( R \)

\[ \langle W_{R,L} \rangle = \int W_{R,L}(\mathbf{q}) e^{-S(\mathbf{q})} \, \text{Dq} \]

Disorder operators: impose a particular type of singularity on fields at a point/loop (i.e., insert characteristic function of fields with given singularity)

- e.g., in \( \mathbb{R}^2 \) can require corona to have prescribed monodromy.
- Changes domain of path integral!

- e.g., 2d gauge theory \( \mathcal{C} \subset G \) change conjugacy class \( \Rightarrow \) "disorder operator"

\[ 1_C \subset \mathbb{Z}_G(S) = C \frac{\delta}{\delta} \]

\( \langle 1_C \rangle = \# G \text{-roots with monodromy in } C \text{ around a given marked point.} \)
4d gauge theory I. The B-model $B_6$.
- 4d analog of our 3d theory $Z_6$

Space of gauge fields still $M_6 = G$ (and symplectic)
$B_6 (N^3) = \mathbb{R}^1 (M_6(N^3), \mathcal{O})$

$B_6 (\Sigma) = \mathcal{O}(M_6(\Sigma))$ (dg category of coherent sheaves)

$[B_6 (S^1) = \mathcal{O}(\mathcal{G}_6) = \text{mod-}$-les $= \text{G-equivariant sheaves of dg categories over } G ]$

Loop operators:

$M_6 (S^2) = \begin{array}{c}
\begin{array}{ccc}
\circ & \cdot & \circ \\
\cdot & \circ & \circ \\
\circ & \circ & \circ
\end{array}
\end{array}$

$B_6 (S^2) = \mathcal{O}(M_6(S^2)) \cong \text{Rep } G$ (dg corrected)

$B_6 (S^2 \times S^1) = \text{dim } \text{Rep } G = \text{Rep } G$

$\Rightarrow$ loop operators are Wilson operators $W_{L,R}$, “measures holonomy in rep $R$"
\[ \Gamma \text{ finite (so theory extends to $g$-manifolds)} \]

\[ \mathbb{L} \subset \mathbb{N}^3 \Rightarrow \text{functor } W_{L,R} \text{ on fields on } \mathbb{N}^3 = \text{fibre} \]

\[ \overset{0}{\mathbb{L}} \Rightarrow \text{operator} \]

\[ W_{L,R} : \mathbb{Z}(\mathbb{L}) \rightarrow \mathbb{Z}(\mathbb{N}) \]

sum over gauge fields weighted by $\text{tr} \text{ hol}_L(R)$.

\[ \Sigma \text{ surface} \]

\[ \Rightarrow \text{action of $E_3$ category } \mathcal{B}_g(S^3) \times \text{Rep G} \]

for any $x \in \Sigma$:

\[ W_{x,R} \text{ vector bundle on } M_g(\Sigma \times \mathbb{I}) = M_g(\Sigma) \]

\[ \mathcal{P}_R \text{ fiber of associated bundle } \mathcal{P}_R \text{ at point } x. \]

\[ W_{L,R} : \mathcal{O}(\mathfrak{g} \Sigma) \hookrightarrow \text{given by } \otimes W_{L,R}. \]

- i.e. have huge "commutative algebra" acting on \( \mathcal{O}(\mathfrak{g} \Sigma) \): $\otimes \text{Rep G}_{x \in \Sigma}$.
Another 4d gauge theory $A_6$:

not quite topological, depends on

some extra holomorphic structure, but

we'll treat broadly the same...

closer to $Z_6$. Motivation:

replace rep theory of $G/\mathbb{F}_2$ by rep theory of LG [analogy of particle

grapes]

$[A_6(5') \sim D(L6)-modules...]$

eg log-rep $\in A_6(5')$

or $\mathfrak{g}$-reps

$\Sigma$ now algebraic curve/Riemann surface

Fields now holomorphic $G$-bundles on $\Sigma$

$Bun_6(\Sigma)$ ... harder to describe explicitly!

eg $G=GL_1$, $Bun_6 \Sigma = \text{Pic } \Sigma$

$A_6(\Sigma) \sim D(Bun_6 \Sigma)$ Directly on $Bun_6$

$A_6(N^3) \sim$ de Rham cohomology of moduli

of monopoles [soh? or Bejancu?; egs]

... $G$ bundles with connection satisfying eqns:

for $\Sigma \in T$ this says bundle is holomorphic on $\Sigma$

& complex structure constant in time...
Local operators

't Hooft operators: introduce singularity in (disorder) bundle \( P \) undergoes some transformation as we cross singular part.

4d: introduce magnetic monopole (gauge in \( N^3 \times \text{time} \)) along \( L \).

\( M_6(\mathbb{S}^2) = \text{possible local geometries of singularity of gauge fields} \)

\( = \text{Bun}_6(\mathbb{S}^2) : \text{set theoretically (Grobndard-Birkhoff)} \)

\[ \leftrightarrow \{ \mathbb{C}^* \to G \}/\!\!\sim \]

\[ \leftrightarrow \text{Hom}(\mathbb{C}^*, T)/\!\!\sim \]

\[ \leftrightarrow \text{Hom}(T^*, \mathbb{C}^*)/\!\!\sim \]

\[ \leftrightarrow \text{Irreps of legtady dual group } G^\vee \]

\( A_6(\mathbb{S}^2 \times \mathbb{S}^1) = \text{Rep } G^\vee \text{ representation} \)

- "possible charges of } G\text{-magnetic monopole}"

Morally: \( R^\vee \text{inner } G^\vee \langle H_{R^\vee , L} \rangle = \int_{\text{fields } (R^\vee, L)} e^{-S} \text{d}\rho \)
What is $A_6(S^2) = D(Bun_G(S^2))$?

$D(\text{LG}/\text{LG}_+)$ $\subseteq$ $D(\text{LG}/\text{LG}_+)$

Special Hecke category $\mathcal{H}_{sp}$

Hecke operators: $\text{LG}_+ \backslash \text{LG}/\text{LG}_+ \hookrightarrow \text{imp } G^v$

labels possible relative positions of two $G$-bundles at a point

[just as $B \backslash G/B \hookrightarrow W$ labels

say for floss... ] ie all ways
to modify a bundle at a point.

Theorem (Mirkovic-Vilonen; Lusztig, Drinfeld, Ginzburg)
Benzakour- Finkelberg, Goertz-Lurie

$\mathcal{H}_{sp} \cong \text{O}(\text{Loc}_{G^v}(S^2))$

as $E_3$ categories

Labeling heart: $\text{Rep } G^v$

+ $su(5)$ enhancement.

Hoch $\text{Hoch}$ operators on $A_6(S)$ labelled
by representations of $G^v$ !
\[ R^v \subset \text{Rep} \ G^u, \ H_{R^v} \cdot H_{G^u} \]

\[ H_{R^v} : D(Bun_G \Sigma) \ni \]

\[ F \mapsto H_{R^v} \times F(p) = \int F(p') H_{R^v}(p', p') \]

\[ q : p' \simeq p \text{ away from } x \]

Geometric Langlands Program:

Spectrally decompose \( D(Bun_G \Sigma) \)

under action of \( \otimes_{x \in \Sigma} \text{Rep}(G^u) \)

Hecke operators, ... ie diagonalize commutative action of local operators \( A_6(5^2) \subset A_6(2^z) \).

Geometric Satake: local operators in \( A_6 \) & \( B_{6^u} \) coincide!

\( \Rightarrow \) Strong indication the theories should coincide...
Geometric Langlands Conjecture:
\[ D(Bun_\mathfrak{g} \Sigma) \cong \mathfrak{g} \mathfrak{u} (\mathfrak{loc}_\mathfrak{g} \Sigma) \]

\[ x \in \Sigma \qquad \ll_{\text{gen}} \rightleftharpoons \text{Rep } \mathfrak{g} \]

needs various modifications to not be false for trivial reasons

- Special case of
  Montonen-Olive Electric-Magnetic duality:
  \[ A_\mathfrak{g} = B_\mathfrak{g} \] isomorphism of field theories

Spectral decomposition:
Wilson operators are "diagonal matrices" on \( \mathfrak{loc}_\mathfrak{g} \Sigma \): for any \( \nu \in \mathfrak{loc}_\mathfrak{g} \Sigma \),
skyscraper \( O_\nu \) is an eigensubset for \( W_{x,\mathfrak{g}} \):
\[ O_\nu \otimes W_{x,\mathfrak{g}} = W_{x,\mathfrak{g}} \left|_{\nu_\mathfrak{g}} \right. = (V_x)_\mathfrak{g} \otimes O_\nu \]
fiber at \( x \) of associated bundle \( V_\mathfrak{g} \).