Daniel Huybrechts - Autoequivalences of Derived Categories of K3 surfaces
(with Macri & Stellari)

X: K3 surface = compact complex surface
\( K_X = 0 \), \( H^1(X) = 0 \).

Intersection pairing on \( H^2(X, \mathbb{Z}) \) \( \rightarrow \)
even unimodular lattice of rank 22
\( = 2 \cdot (-E_8) \oplus 3 \cdot U \) (U = hyperbolic plane)

Signature = \((3, 19)\)

\[ H^2(X, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2} \]
\( \cong \mathbb{C}^{22} \oplus \mathbb{C} \)

\[ H^1(X, \mathbb{R}) \cong \mathbb{R}^{22}, \ \text{signature} \ (1, 19) \]

\( \{ \alpha \in H^1(X, \mathbb{R}) : \alpha \cdot \alpha = 0 \} \)

two connected components

\( K_X \)
\( f : X \to X \) automorphism

\[ \Rightarrow f^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z}) \] Hodge isometry;
compatible with intersection pairing & Hodge decomposition.

But also \( f^* (K_X) = K_X \Rightarrow f^* (\mathcal{E}_x) = \mathcal{E}_x \):

... use existence of Kähler classes
to show the two correspds are preserved.

\( f^* \mathcal{E}_x = \mathcal{E}_x \iff f^* \) is orientation preserving.

pick \( F \in H^2(X, \mathbb{R}) \) three-dimensional
self \((a, b) F \) is positive definite.

Pick an orientation of \( F \), \( \sigma \)

\( F, f^* F \in H^2(X, \mathbb{R}) \)
\( \sigma \) \( f^* \sigma \)

Orthogonal projection gives
an isomorphism \( F \to f^* F \).

"Orientation preserving" means this projection identifies the orientations \( \sigma, f^* \sigma \).
Theorem (Donaldson)

If \( f : X \rightarrow X \) is a diffeomorphism
\[ \Rightarrow f^* \text{ is orientation preserving.} \]

So \( \eta : \text{Diff} X \rightarrow \text{Aut} \; H^2(X, \mathbb{Z}) \)
has image \( \mathbb{Q}_+ \langle \text{Diff}^+(X, \mathbb{Z}) \rangle \)

- group of orientation-preserving diffeomorphisms.
- \( \eta \) uses global Torelli theorem (Borel)
- \( \eta \) uses Donaldson's theorem.

Kernel not understood - e.g. is it nontrivial?
related to Witten's strong form of global Torelli for K3, still unknown.

Derived categories

\( \text{Coh} \; X = \text{abelian category of coherent sheaves} \)
\[ D^b(X) = D^b(\text{Coh} \; X) \quad \text{bounded derived category.} \]

Consider \( \phi : D^b(X) \rightarrow D^b(X) \)
always given by \( \oplus \mathcal{E} \in D^b(X \times X) \)
\( \text{Fr} \rightarrow \text{Fr}_*(p^*F \otimes E) \)
\( X \rightarrow Y \)
Mukai: To any sub $\phi$ can associate a Hodge isometry $\phi^H : H^*(X, \mathbb{Z}) \cong \cdots$ for K3 surfaces can do this integrally higher dim only get rational statement.

(note: $\phi^H$ knows what to do with nonalgebraic classes, not just algebra ones!)

$\phi^H : \phi \mapsto g_\phi (\rho = \cdot c_1(3) \cdot \overline{\omega'(X)})$

Isometry for Mukai pairing, not quite the intersection pairing: charge signs on $H^*, H^0$. But we get the Hodge structure on full $H^*(X)$.

$\exists \quad H^{1,1}(X) = H^{1,1}(X) \cap H^0(X) \oplus H^0(X)$

$H^{0,0}$, $H^{1,1}$ don't change. Signature = $(4, 20)$

$\Rightarrow \quad \rho: \text{Aut} D^b(X) \to \text{Out}(H^*(X, \mathbb{Z}))$

group of all Hodge isometries $\rho \mapsto \phi^H$

Q: What are Im $\rho$ & Ker $\rho$?
[Bredon's conjecture for Ker $\rho$ as
a certain fundamental group $J$]

[Torelli: automorphism of $X$ give
all Hodge isomorphisms which preserve
Kähler cone --- don't include
reflectors in $(-2)$ classes !]

On $\mathbb{Q}$, Hasse-Weil-Quillen, Plogg:

$O_+ (\Omega (x, Z)) = \text{Im} \rho$.
--- isometries preserving the four positive directions

Theorem! (in progress!) $\text{Im} \rho = O_+ (\Omega (Z))$
--- every $\rho^+$ is orientation preserving

(Conjecture of Szendrői: explains this
is the mirror of Donaldson's theorem.)

Examples: i) $f: X \cong X$.

$\Theta : D(x) \cong \mathcal{O}_x$. $E = \mathcal{O}_x^\rho$
$\Gamma_t = \text{graph of } f$. 
ii) Shift $\Omega \mathcal{L}$, $E = \mathcal{O}_X(1)$, $\phi^* = -\text{id}$

iii) Let $P \in X$, $L \cong \_$. $\phi^* \in \mathcal{O}_X(L) \_\_\_\_$

iv) Spectra $X$ is a moduli space of stable sheaves on $X$, [Mukai]

$v)$ Spherical twists (Kontsevich-Soibelman)

$T_E : D^b(X) \cong$ with $E = O_X(k)$ a spherical object

\[ E^* \cong (E, E) = H^*(S^2, \mathbb{C}) \]

$T_E(F) = \text{Cone}(\text{Hom}(E, F) \otimes E \xrightarrow{\text{add}} F)$

Eg. any lie bundle $L \in P \in X$ is spherical (but $T_L$ is not $\mathbb{C}$).

Eg. $\mathbb{P} \subset \mathcal{O}_X(-1)$ cone $\implies \mathcal{O}_X(1)$ are spherical.
E spherical => its Multi-pole vector

\[ u(E) = c_n(E) \frac{x}{\|x\|} e^{i \vec{H}(x, n)} \]

is a -2 vector.

\( T_E \) is just reflection in hyperplane orthogonal to \( E \).

So if \( E = O_3(C) \) => reflection \( S_{C3} \)
in -2 class.

\[
\begin{bmatrix}
1 & \text{Line of } c_n(E) \\
1 + c_n(E)^2 + \frac{c_n(E)}{2} + 1
\end{bmatrix}
\]

\[ E = \text{case } (E \otimes E^v \xrightarrow{\text{op}} O_3) \]

Can prove any of the characteristics of types i-iv are orientation preserving, complicated factors: come from spherical twist.

\[ \text{Q: What happens if there are no spherical objects?} \]
Theorem 2. If \( (X, \mathcal{L}) \) is a generic twisted K3 surface, then \( D^b(X, \mathcal{L}) \) does not contain spherical objects.

\[ \text{Im} \mathcal{p} = \text{Ob} \left( \mathcal{H}(\mathcal{L}, \mathcal{L}) \right) \quad (\text{i.e. generic}) \]

for \( p = [\mathcal{L}] \)

\[ \alpha \in H^2(X, \mathcal{O}_X) \] has Brauer class.

An \( \mathcal{L} \)-twisted sheaf is a collection of sheaves on \( X \) with coverings satisfying \( \alpha \)-twisted cocycle condition.

\[ D^b(X, \mathcal{L}) = D^b (\text{Ob} \left( \mathcal{H}(\mathcal{L}, \mathcal{L}) \right)) \]

Sketch of proof:

- No spherical objects,
- No trivial objects (\( Ext^1(\mathcal{L}) = 0 \))
- Any \( \mathcal{L} \times X \) gives a semi-abelian/elliptic object:

\[ Ext^1(\mathcal{L}) = H^1(\mathcal{L}, \mathcal{O}_X) \]
Under any autoequivalence semi-rigid's go to semi-rigid.

\[(\text{Hori}) \times \sum \dim \text{Ext}^i(H^i(F), H^i(F)) \leq \dim E \text{1}(F)\]

for \(F = \phi(k)\) semi-rigid. \[= 2\]

At most one such exists \(= 0\)

The others are rigid, but there are more such \(\Rightarrow\) so \(F\) is a sheaf in deg 0 up to shift.

\(\Rightarrow\) the object \(E\) on \(X\) is actually a sheaf (up to shift), in fact a versal family stable sheaves. So we've reduced to an understand situation. \(\square\)

**Theorem 3** If \(\text{Pic} X = 0\) the \(O_X\) is the only special object up to shift \(2\)

\[\text{Aut} O_X(x) = 
\text{Aut} X \otimes \mathbb{Z} \cdot i \otimes \mathbb{Z} \cdot T_{O_X}\]**
Sketch of proof \( \phi : D^b(X) \to D^b(X) \).

Enough to show \( T_0 \) s.t. 
\( T_0 \phi \in \mathcal{M} : k(x) \to k(x) \) \( \forall x \)

- take points to points up to shift
- twist \( \Rightarrow \) then get automorphism of \( X \).

Construct t-structure on \( D(X) \) with heart

A s.t. point sheaves are only semi-... mild sheaves (no nontrivial sub-sheaves).

\( A' = \phi(A) \)

enough to show \( T_0 \) s.t. \( V \in \mathcal{M} : T_0 V \in \mathcal{M} \)

is in \( A' \) \( \Leftrightarrow \)

- the inequality is close to show
  - all but one cohomologies of \( H^i \) are ... - but we know that \( Q \)

is the only spherical one. Using \( T_0 \),

make \( H^k \) into something of smaller length one level in \( A' \).
- only need to put stability condition & resulting Harder-Narasimhan filtration to reduce to minimal sheaf.

Apply this to Theorem 1:

$\phi^* : D^b(X) \to \text{Der} \phi$ (by definition)

in some finite. If $X$ proper

$\Rightarrow \phi^* \text{ preserves semi-continuit}$

But $\phi^* \text{ constant in } t$

- problem: must deal with convergence issues of definitions!

- instead work with rigid analytic space of fixed scheme $X \to \text{Spf} \mathbb{C}[[T]]$

$\Rightarrow$ closed category $D^b(X)/D^b(\mathbb{C})$
mod of by sheaves supported on some multiple of the general fiber

get model for $\mathbb{D}^b$ of the rigid analytic space: get $\mathcal{A}(\mathbf{A})$ -linear category

prove Theorem 3 in this context.