Janos Kollar - Quotients of Calabi-Yau Varieties

\[ E_1 : E_1 = (y^2 = x^3 - 1) \]

with Milnor

\[ \mathcal{E} (x,y) \rightarrow (x, y) \]

\[ E_1 / \sigma = \mathbb{P}^1, \quad E_1^* E_1 / (\sigma \sigma) \quad K3 \text{ surface} \]

2. \[ E_2 = (y^3 = x^2 - 1) \]

\[ \sigma : (x, y) \rightarrow (x, \varepsilon y), \quad \varepsilon^3 = 1 \]

\[ E_2 / \sigma = \mathbb{P}^1, \quad E_2^* E_2 / (\sigma \sigma) = E_2^* E_2 / (\sigma_0 \sigma_0) \]

Let's work these down:

\[ \gamma^3 = x^3 - 1 \quad \Rightarrow \quad \text{varieties} \quad u_2, v_2 \]

\[ \gamma_2^3 = x_2^3 - 1 \quad \quad \quad \Rightarrow \quad \quad \quad u_2 = \gamma_2 x_2 \]

\[ u_2^3 = (x_2^3 - 1)(x_2^3 - 1) = K^3 \]

Take triple cover of \( \mathbb{P}^3 \), ramified

\[ \frac{u_2 = \gamma_2 y_2, \quad u_2^3 = (x_2^3 - 1)(x_2^3 - 1) = K^3}{\text{along curve}} \]

Singularity curve of type (3,3) on \( \mathbb{P}^3 \) a rational curve
Claim: both curves $C, C'$ have only two actual ramification points (possible ramification at $z_0$ with $z_0^2 = 1$ but multiplicities cancel, while in $C + C'$ case get lots of ramification points)

find a $P^2$ of rational curves existing after counting number of such $C$.

=> this is a rational surface.

So s.s. quad. $K^3_3$ + some rational.

3. $F_3 = (y^6 = x(x+1)^2(x+1)^3)$

$\omega: (x, y) \mapsto (x, \epsilon y)$ $\text{if } \epsilon^6 = 1$

Claim: $F_3/\langle \omega, \omega \rangle$ is rationally connected $\iff n \leq 5$. 
Problem X CY variety, smooth
(just need $K_X \equiv 0$ numerically)

$G \subset A \times X$ finite subgroup

What kind of variety is $X/6$?

3 cases:
1. $X/6$ is CY (with resolved singularities)
2. otherwise $X/6$ is uniruled
3. stranger: $X/6$ is rationally connected

Aim is to decide among these cases using:

$x \in X \rightarrow G_r = S_{r,x}$

$p_{r,x} : G_r \to X$

- read off above cases from stabilizer representation
- will completely distinguish 1 & 2 &
  already enough for 3.
Def. $g$ satisfies the Reid-Tai criterion if $\text{deg}(g) \geq 1$.

**Theorem 1:** If smooth projective $(Y, \xi, \mathbb{C}^n, K_Y)$, $\mathbb{C}^n$ admits $\exists x \in \mathbb{C}^n$ with $g(x) = 0$.

Let $K_n = 0$.

$\longrightarrow$ $x \in \mathbb{C}^n$ is a smooth point of $g$.

$\longrightarrow$ $g(x) = 0$.

$\longrightarrow$ $\xi$ regular at $x$.

$\longrightarrow$ $\xi$ is a regular divisor.

**Proof of Theorem 1:** Miyaoka-Mori criterion.

$\exists$ variety $Z$ by $\{ Z \}$ family of curves $\mathbb{C}_t \cdot K_Z < 0$.

$Z$ is smooth.
If it rains in color 1 - breakthrough

divide appears in E with negative multiplicity
we take G to be hyperplane scheme,
no problem.

Intensively free is when no rainfalls
in color 1.

\[ E = \sum \alpha_i E_i \]
\[ b \exists \alpha_i \text{ s.t } \alpha_i \leq 0 \]

Try 1: fixed family \( G \)
\[ s.t \quad G \cdot E_i = 0, \quad G \cdot \cap E_i = \emptyset \text{ for } i \neq 0. \]

Don't know how to produce these... but this is a hill.

Try 2: Enough to fix \( G \), s.t.
\[ G \cdot E_i > 0, \quad G \cdot E_i < M \text{ i.e. } 0 \]

So fix \( M \).

In dimension 2 every is easy: \( \forall E_i = 0 \Rightarrow \)
Theorem 2 \[ X \rightarrow Y, \quad g : X \rightarrow Z \]

rational map with \( X(Z) \neq 0 \)

\( \Rightarrow \) can find a finite cover \( X' \rightarrow X \)

which decomposes as \( \mathbb{P}^1 \times F \) with \( Z \rightarrow CY \)

\[ Z \rightarrow F = X', \quad g \rightarrow X', \quad \text{(finite for } X \text{ smooth)} \]

\[ \begin{array}{ccc}
\text{proj} & \downarrow & \\
\mathbb{P}^1 & \longrightarrow & X' \\
\downarrow & & \\
Z & \longrightarrow & F
\end{array} \]

so up to finite cover any such \( g \)

is a projection on a fiber.

If \( X \) is simply connected & not a project

(e.g. a CY3 fold with \( m = 0 \) is always irreducible... so this may happen)

\[ C \subseteq \text{G} \text{ a subgroup which fails RT} \]

\[ \text{ie } \exists g \in C \quad \text{opt}(g) \leq 1, \]

- easy to verify algorithmically for given \( C \).
Truly as we $Z/n \subset G/Z$ open for
Save line if it on all RT.
for $Z/n \subset G/L$, still open to describe
all such satisfying RT.

Reducing: reduce to case where

$L(G \otimes \hat{G}) = \mathbb{C}$ (conjugacy class of $g$)

generates $G$.

2. $\hat{G}$ is additive $\Rightarrow$ can reduce to
case where representation is irreducible.

3. Look of image $G \rightarrow PGL_n$:
   if RT fails for $G \rightarrow PGL_n$, enlarge
   $G$ through in $\mathbb{R}^n$ roots at unity
   - if $\alpha e^x \subset \alpha = 1$ & root
     of unity close to 1
   ...$\Rightarrow$ construct rep into $PGL_n$ where RT fails.

Easy examples where RT fails:
reflection groups!
Theorem 3: If RT fails, \( \langle G G^* \rangle = 6 \)

- have no finite representation

\[ \Rightarrow \text{either } G \text{ is projectively equivalent to a reflection group.} \]

2. Case 1: \( |G_{29}| = 7680, G_{29} \subset G_{28} \)
   - reflection group
   - but have another projective one in class which is not equivalent to a reflection group.

3. Possibly finitely many other exceptions

So for a typical group, the quotient will be CY. Reflection groups are the basic cases where we have branching in codimension 1 to get unimodular quotient.

Theorem 4: X = A simple abelian variety

- \( G \subset Aut A \text{ finite } \Rightarrow A/6 \text{ is a singular CY.} \)
Idea of Theorem 3 \[ G = \text{U}(n) \]

One measure distance \( d \) to

eigenvalue on a circle

\( \lambda \) is not the natural orbit on \( S \).

Write \( \delta = \min \{ \alpha, 1 - \alpha \} \) for \( \alpha \in \mathbb{R} \).

Measure on \( \text{U}(n) \): \( d(\gamma e) = \sqrt{\sum \delta_i^2} \)

Our \( G \) is generated by \( G \cap B(1, e) \) self

neighborhood of identity (gen by \( G_{\delta} \))

Claim: If \( G \) is irreducible,

\[ \exists h \in G \text{ s.t. } d(h(\alpha)) = \sqrt{n} \]

Orthogonality of characters \( \Rightarrow \sum \chi(g) = 0 \)

\[ \Rightarrow \exists G \text{ with } \operatorname{Re} \chi(g) \leq 0 \]

\( \Rightarrow \) need eigenvalues with

negative real part \( \Rightarrow \) simple estimate.

Covering \# of \( G \) : how many elements
of our conjugacy class do we need to multiply to get all of \( G \) —

\[
\min c \geq \infty \left( 1 + \ldots + 1 \right) = \infty \cdot \infty.
\]

So if \( RT \) fails \( \Rightarrow c(g) \geq \frac{1}{2} \infty \).

Known: If \( G \) is a simple group

\[
c(G) \leq C \log |G| \quad \text{(see context)}
\]

Fact: Any \( G \leq GL_n \) is by for the biggest simple group —

\[
|A_n| \sim n^n.
\]

So if understood reps of groups of Lie type,

find that for any other simple group

\[
|G| \leq n^C \log n^D
\]

\[
\Rightarrow c(g) \leq c \left( \log n^D \right)^2
\]

\[\rightarrow \] reduce to reflection groups, which are described using symmetries.
So far questions of Cebeci-Yen this proves to conclude that negative Kalam division ⇒ unlimited!