A. Belavin - Quantization of Hitchin Hamiltonians & Hecke Eigensheaves

(c. Drinfeld) \( \text{Objective (D.)} \) - construct Demons on moduli from operas using CFT, which are Hecke eigensheaves...

Local picture - Feigin-Frenkel strange from geometric P.C.V., remainder of special duality of quantum \( W \)-algebras, specified at critical level.

I have given geometric picture only for closed rank...

Operator, \( K = \mathcal{O}([H]), \mathcal{O} = \mathbb{C}[[t]] \), central exts of \( \mathcal{O} \) labeled by symmetric forms on \( \mathfrak{g} \) by \( \mathfrak{c} = \mathfrak{g} / (\mathfrak{g} \circ \mathfrak{r}) \implies X(\mathfrak{c}) \). (Fix \( B \in \mathfrak{c} \))

\( \mathfrak{c} = \mathfrak{h} \) - Killing critical level (not simple) - huge center

So take \( c = -\frac{1}{2} \text{Tr}(\text{adj} \circ \mathfrak{c}) \), \( \mathfrak{c} = \text{center} \).

Final estimate \( \mathfrak{g} \mathfrak{r}_0^\mathfrak{c} \subset (\mathfrak{g} \circ \mathfrak{r})^\mathfrak{c} = \mathfrak{g} \mathfrak{r} + \mathfrak{g} \mathfrak{r}_0^\mathfrak{c} \)

- using regular orbits \( G \) corresponding group \( \mathcal{M}(B, \mathcal{E}) \)

\( \mathcal{E} : \) the above embedding is iso...

\( B \) is deformation of \( \mathfrak{g} \mathfrak{r}_0^\mathfrak{c} 

Consider \( \mathcal{V} \mathcal{E}_c = \text{Ind}_{\mathfrak{g} \mathfrak{r}_0}^{\mathfrak{g} \mathfrak{r}} \mathcal{E}_c \), \( \mathfrak{g} \mathfrak{r} \) acts on \( \mathcal{V} \mathcal{E}_c \), so does \( B \) - huge ideal kills this vacuum \( \implies \) ideal \( I \subset B \) annihilates \( \mathcal{V} \mathcal{E}_c \).

\( \mathcal{E} : \) image of \( B \) in \( \mathcal{E} \mathcal{D}(\mathcal{V} \mathcal{E}_c) \) is all \( X / I \cong \text{End}_{\mathfrak{g} \mathfrak{r}_0} \mathcal{E}_c \)

\( \mathcal{E} = \mathcal{E}/I \), get \( \mathfrak{g} \mathfrak{r}_0^\mathfrak{c} = \text{Funct} on \mathfrak{g} \mathfrak{r}/(\mathfrak{g} \circ \mathfrak{r}) \mathfrak{r}_0^\mathfrak{c} \)

\( \text{Operas} \) \( X \) - curve, \( \mathcal{O} \) oper over \( X \) is a triple \( (\mathcal{F}_1, \mathcal{F}_2, \mathcal{V}) \)

\( \mathcal{F}_1 \) oper over \( X \), \( \mathcal{V} \) connection on \( \mathcal{F}_0 \), \( \mathcal{B} \) \( \mathcal{B} \)-structure on \( \mathcal{F}_0 \)

Standard way of going from system of order-\( c \) opers to order-\( n \) eq.

Very rigid object - no non-trivial auto (besides on \( \mathcal{V} \) - \( \mathcal{V} \)) - i.e. \( \mathcal{O} \) opers are rigid.

\( \text{Description} \) \begin{equation} \text{locally principle} \end{equation} \( B \) \( \rightarrow B \)

\( \mathcal{O} \) opers on rank \( \mathfrak{g} \) - have obstruction to \( \mathcal{P} \) monad \( \mathcal{O} = (\mathfrak{g}/\mathfrak{h})_B \circ \mathfrak{r}_0 \)

Carries obstruction, \( \text{we want our obstruction to } \mathcal{F}_0 \mathcal{B} = \mathfrak{g}/\mathfrak{h} \circ \mathfrak{r}_0 \)

Want to express every \( \mathcal{O} \) oper as related \( \mathcal{O}_2 \) oper + standard correction...

- take \( L_{\mathfrak{g}/\mathfrak{h}} \), generator of \( \mathfrak{g} \mathfrak{r} / \mathfrak{g} \mathfrak{r}_0^\mathfrak{c} \)

\( \mathfrak{g} \mathfrak{r} \mathfrak{r}_0^\mathfrak{c} \) spanned by \( \mathfrak{g} \mathfrak{r}_0^\mathfrak{c} \), \( \mathfrak{g} \mathfrak{r} \mathfrak{r}_0^\mathfrak{c} \)

\( \text{dim} = \text{rank of the algebra} \)

So add to each induced a correction in \( \mathcal{V} \mathcal{O} \)...

\( \text{Fact} \) \( \Gamma(X, (\mathfrak{g}/\mathfrak{h})_B \mathfrak{r}_0^\mathfrak{c} ) \rightarrow \mathcal{O} \)-opers (\( \mathcal{V} \mathcal{O} \) twisted by \( \mathfrak{B}_1 ) \), \( \mathcal{V} \mathcal{O} \) is \( \mathfrak{B}_1 \)-module.

So start with a \( \mathfrak{B}_2 \)-oper, get this map \( B \) on \( \mathfrak{B}_2 \), etc.

- induce, then change the connection by adding this \( B \) part...
Local situation - functions on moduli space of opers have canonical filtration, whose generic graded is \( l \cdot 3 \).
Standard projection (Kostant \( \theta \)) \( \text{Vol} \rightarrow \text{character} \).
\( V \rightarrow \text{class of } L \cdot V \) (\( L \) opposite nilpotent to \( V \) in \( \mathfrak{g} \)).
So up to translation, any opers are h, \( \omega \), or \( \mathcal{E} \) elements of \( \text{Vol} \). \( 
\) \( C \) is \( \mathbb{P}^1 \) for formal punctured disc.)
Now since \( H \) is a \( \mathbb{P}^1 \) space, \( \mathcal{O} \) is a deformation of \( C \).
The \( C \) is a canonical isomorphism between these deformations.
Since \( \mathcal{O} \) is moduli of log opers on \( \text{Spec } \mathcal{O} \), \( \mathcal{O} \) is a deformation of \( \mathcal{O} \).
Canonical - symmetry of \( \text{Spec } \mathcal{O} \) and \( \mathcal{O} \) are on whole picture, and the isomorphism is \( \text{Aut } \mathcal{O} \)-equivariant.
We will define the arrow \( \text{Spec } \mathcal{O} \rightarrow \mathcal{O} \) on \( \text{Spec } \mathcal{O} \).

Prop. 2: Satake equivalence (geometric version of \( \mathfrak{g} \)) - [H. H. Shintani, B. Gross, M. V. Barlow, realizing Satake, etc.]

Visual Satake - \( G \) split semisimple \( \mathfrak{g} \), take Hecke \( (C) = \text{mesures with cusp form} \).
\( \text{Cusp form, } \mathcal{G}(C) \) bi-invariant, perfect ring (commutative) and it's convolution.
Satake identifying this with \( \mathbb{H} \) via \( \text{Rep } \mathbb{R} \cdot \text{rig } \mathfrak{g} \) \( \rightarrow \text{character } \).
Satake - Harish-Chandra (consider unbalanced principal series of \( \mathcal{G}(C) \)), character is \( \text{canon } \mathcal{G}(C) \), canonical vacuum vector, elt of Hecke \( \rightarrow \).

Replace algebras with categories, \( \text{Rep } \mathbb{R} \cdot \mathfrak{g} \) \( \rightarrow \mathcal{G}(C) \) category, \( \text{Rep } \mathbb{R} \cdot \mathfrak{g} \) perverse on dg-sheaves \( \rightarrow \mathcal{G}(C) \).

Replace \( \mathcal{G}(C) \) by \( \mathcal{G}(\mathbb{C}) \) - \( \mathfrak{g} \) points of a group of finite type \( \mathcal{G}(C) \).
\( \mathcal{G}(C) \) principal group, take affine Grass = \( \mathcal{G}(C)/\mathcal{G}(C) \).
\( \text{Perv } \mathcal{G}(C) \) \( \rightarrow \text{cat } \mathcal{G}(C) \) equiv perverse sheaves on \( \mathcal{G}(C)/\mathcal{G}(C) \).

Irreps are numbered by orbits, here are no nontrivial exs between torus, so \( \mathcal{G}(C)/\mathcal{G}(C) \) orbits.
This is a tensor category (moduli on categories of \( \text{Perv } \mathcal{G}(C) \).
\( \mathcal{G}(C) \) monoidal tensor category...

From Lustzig, \( \mathcal{P} \) \( \rightarrow \) \( \text{semi-simple } \), \( \mathcal{P} \) \( \rightarrow \) \( \text{orbits } \), numbered by high weight of irreps.
Fiber functor \( \Phi^* : \text{H}^{-1}(G(k)/G(0), \ast): \mathcal{P} \to \text{Vec} \) (just Hope on D-maps.) \( \Rightarrow \) group of automorphisms of \( \Phi^* \) is canonically \( \mathbb{G}_m \) \( \Rightarrow \) Langlands dual group, via a subtle duality.

What is the distinguished base? Of \( \mathcal{P} \) in this picture?
\( \Rightarrow \) finding a line of principal vectors for each \( \mathcal{P} \to \text{Rep} \).
But corresp are IC sheaves, so not unique — lowest \( \text{IC} \) is \( \mathcal{O} \), so gives a line. \( \Rightarrow \) Base \( \mathcal{O} \) is canonical.

(Twisted) D-modules on \( G(k)/G(0) \): or perverse sheaves at \( \infty \) (and in part, carry through Riemann-Hilbert...). What is a D-module on a scheme, especially, how to limit of singular schemes?

Embed singular \( \to \) smoothy define via Kashiwara equivalence. What are sections of D-module (right)? \( \gamma \to \gamma \) singular \( \text{V smooth} \)

D-module \( \mathcal{M} \) \( \Rightarrow \mathcal{M}(V) \) is same canonical data on singular variety (order of \( \mathcal{M} \), canonically these are crystals (without embedding) ...)

For \( \mathcal{O} \) on glob sections defined inductively via support on limits etc etc etc.

Half-forms on \( G(k)/G(0) \) \( \lambda \)-defined canonically as follows: fix \( \mathcal{L}_0 = \mathcal{O}^*_k \) on \( \text{Spec} \mathcal{O} \), unique up to sign.

Fix \( \gamma \) non-deg scalar product on \( \gamma \) - space \( \text{Vec}(k) \). \( \mathcal{L}_0 \) carries canonical scalar product: twist \( \gamma \) on \( \mathcal{L}_0 \).

If \( \gamma \) \( \mathcal{L}_0 \to \gamma \) gives on \( \text{Vec}(k) \) take \( \mathcal{O}(\gamma) \).

Now define fiber \( \gamma \) of \( \lambda \): \( \gamma = \text{det}(\text{Ad}(\gamma)/\text{Ad}(\mathcal{O}(\gamma))) \).

- relative Pfaffian of two subspaces; depends continuously on \( \gamma \) even though inside spaces jump...
- makes perfect sense on each orbit \( \Rightarrow \) (at finite-dimensional)

**Fact** The restriction of \( \gamma \) to any orbit \( C \) coincide with \( \mathcal{O}(\gamma) \) on that orbit.

\[ \mathcal{O}(\gamma) \to \mathcal{O}(\gamma) \text{ is an isomorphism} \]

\[ \text{restriction of } \lambda \text{ to any orbit } C \]

\[ \text{now twist by } \lambda : \]

\[ \text{spec } \mathcal{M} = H^0 \text{ of } \mathcal{O}(\gamma) \text{ on } (\text{Spec } \mathcal{O}(\gamma)). \]
\[ M_c \times \mathcal{L}^+ = \mathcal{L}(\mathcal{C}) \mathcal{O} = C \] canonical line of section, \( f \)-function corresponding to orbit \( \mathcal{L}(\mathcal{C}) \in \mathcal{G}(M_c \times \mathcal{L}^+) \)

**Theorem.** For any \( M \in \mathcal{P} \), the higher (pervasive) cohomology \( H^i(\mathcal{C}(\mathcal{C}) \setminus \mathcal{C}(\mathcal{C}), \mathcal{M}(\mathcal{L}^+)) \) vanish for \( i > 0 \), and \( \Gamma(\mathcal{C}(\mathcal{C}) \setminus \mathcal{C}(\mathcal{C}), \mathcal{M}(\mathcal{L}^+)) \) is isomorphic to a direct sum of \( \mathcal{C}(\mathcal{C}) \) many copies of \( \mathcal{V}_{\mathcal{C}} \) as a \( \mathcal{L} \)-module. Delta-funcions just give \( \mathcal{V}_{\mathcal{C}} \), and any other \( M \) will just be a direct sum of such \( \mathcal{C}(\mathcal{C}) \) copies of \( \mathcal{V}_{\mathcal{C}} \). [Some remarks about higher cohomology]

**Birth of groups** \( \phi(M) = \text{Hom}_\mathcal{C}(\mathcal{V}_{\mathcal{C}}, \Gamma(\mathcal{C}(\mathcal{C}) \setminus \mathcal{C}(\mathcal{C}), \mathcal{M}(\mathcal{L}^+))) \) for \( \mathcal{L} \)-module of finite type, \( \mathcal{C} \) being again \( \text{End}_\mathcal{C}(\mathcal{V}_{\mathcal{C}}) \).

**Proposition.** \( \phi : \mathcal{P} \to \text{finite free } \mathcal{L} \)-module is a fiber functor (everything finite, unique up to twisting by torsor which is \( \text{Hom}(\text{fiber, standard fiber}) = \text{de Rham cohomology} \).

Thus we defined a canonical \( \mathcal{L}(\mathcal{C}) \)-bundle \( \mathcal{F}_C \) on \( \text{Spec } \mathcal{C} \)

\[ \phi(M) = \text{Hom}_\mathcal{C}(M, \mathcal{F}_C) \]

**Extra structure 1.** There is a canonical reduction \( \mathcal{F}_{\mathcal{C}} \) of \( \mathcal{F}_C \) to \( \mathcal{C} \)

- to do this we need to describe line in any \( \mathcal{F}_C \)-twisted case - this will come from \( \mathcal{C} \in \Gamma(\mathcal{C}(\mathcal{C}) \setminus \mathcal{C}(\mathcal{C}), \mathcal{M}(\mathcal{L}^+)) \).

ii) \( \text{Aut}(\mathcal{C}) \) acts on the picture (i.e., group where containing integrable part + Auto preserving a point + formal part shifting the pt.). \( \mathcal{F}_{\mathcal{C}} \) is \( \text{Aut}(\mathcal{C}) \)-equivariant. Now \( \text{Aut}(\mathcal{C}) \) still acts by \( \mathcal{C} \) if they don't fix point - formal part doesn't preserve \( \mathcal{C} \) - so the reduction \( \mathcal{F}_{\mathcal{C}} \) is only \( \text{Aut}^0(\mathcal{C}) \)-equivariant.

Our aim is to send \( \text{Spec } \mathcal{C} \to \text{functor of } \mathcal{L}(\mathcal{C}) \)-torsors on \( \text{Spec } \mathcal{O} \):

\[ \mathcal{Z} \to \mathcal{C}, \text{ Aut}(\mathcal{C}) \to \text{Spec } \mathcal{C} \to \text{Spec } \mathcal{O} \to \mathcal{Z} \]

Pull back \( \mathcal{F}_{\mathcal{C}}, \mathcal{F}_{\mathcal{C}} \to \mathcal{F}_C, \mathcal{F}_{\mathcal{C}} \to \text{Aut}(\mathcal{C}) \)

\( \mathcal{F}_{\mathcal{C}} \) is equiv w/r.t. \( \text{Aut}(\mathcal{C}) \) - so gives constant (suitable to torsor)

\( \mathcal{F}_{\mathcal{C}} \) constant w/r.t. translations \( \text{Aut}^0(\mathcal{C}) / \text{Aut}(\mathcal{C}) \)-equivariant

- IN OTHER WORDS, \( \mathcal{F}_{\mathcal{C}}, \mathcal{F}_{\mathcal{C}} \) come from the quotient \( \text{Aut}^0(\mathcal{C}) / \text{Aut}(\mathcal{C}) = \text{Spec } \mathcal{C} \) -

constant \( \mathcal{C} \)-bundle on \( \text{Spec } \mathcal{C} \leftrightarrow \text{bundle with connection} \)

**Proof.** This is a \( \mathcal{C} \)-torsor on \( \text{Spec } \mathcal{O} \).
Explanation of Commutativity:

Different def. of $\mathcal{C}$ on $\mathcal{P}_0$ (which will agree with convolution):

reverse shows $G(\mathcal{C}) \setminus \mathcal{C}(\mathcal{D})$ as $\mathcal{D}$ (co) action of even of disc

Now take $X = \text{curve}$ motion (or field)

for commutativity: take operators at different points commute

obviously... try to take pair of form $(x, x' = 0)$

get perverse sheaf on total space $\mathcal{F}_x$ as family

Now take a finite set of points $S_i$... First rate $\mathcal{F}_{S} = \mathcal{C}(\mathcal{C})/\mathcal{C}(\mathcal{C})$

for $X$ affine this doesn't change on formal neighborhood $\Rightarrow$

connection $\Rightarrow$ horizontal section 1 - only horizontal

connection doesn't preserve stratification

Set $\mathcal{F}_x = \prod_{x \in S} \mathcal{F}_x = \mathcal{C}(\mathcal{C})/\mathcal{C}(\mathcal{C})$

bdl $\mathcal{F}_x$ on $X^*$ with fiber $\mathcal{F}_{x,\ldots, x}$

gives true, familiar perverse bundle... over diagonal this bundle is not product but only one copy of $\mathcal{F}_x$ by convolution on the diagonal.

Canonical flat connection $\Rightarrow$ Now before product

$\mathcal{F}_x \rightarrow \mathcal{F}_x \times \mathcal{F}_x = \mathcal{F}_x \times \mathcal{F}_x / U$, $U$ complement of diagonal

Now say $M, M_2 \in \mathcal{F}_x$ (order $M, \mathcal{M}_2 \Rightarrow$ rest shft over $U$ on $\mathcal{F}_x \times \mathcal{F}_x / U$)

Now extend, set $\mathcal{J}_x (M \otimes \mathcal{M}_2 / U)$, perverse sheaf on $X$, whose pullback $\mathcal{J}_x (M \otimes \mathcal{M}_2 / U)$ is

perverse sheaf on $X \Rightarrow M \otimes \mathcal{M}_2$, obvious commutativity

I assoc (like $\mathcal{L}^x$ is intro of Springer convs.)

This agrees w/ convolution...

Small projection $\mathcal{F}_x \rightarrow \mathcal{F}_x$

(consider $\mathcal{F}_x \rightarrow \mathcal{F}_x$, classes on $X$, $\mathcal{F}_x$ horiz.

and there are

isom on $X \times \mathcal{F}_x^2$ - classes of such data are same as $\mathcal{C}(\mathcal{C}) / \mathcal{C}(\mathcal{C})$. Now take a pair at pts $x, y$, consider triple ($\mathcal{F}_x \times \mathcal{F}_y \rightarrow \mathcal{F}_x \times \mathcal{F}_y$)

These give idsh on $X \times X$, which is $\mathcal{F}_x \times \mathcal{F}_y$ with projection to $\mathcal{F}_x$ which is composition $\mathcal{F}_x \rightarrow \mathcal{F}_x$

This way b correct def of $\mathcal{C}(\mathcal{C}) / \mathcal{C}(\mathcal{C})$ as composition -

i.e. in get convolution... twisted (by $\mathcal{F}$) version of $\mathcal{F}_x \times \mathcal{F}_x ...$

$\mathcal{J}_x (M \otimes \mathcal{M}_2) \rightarrow \mathcal{F}_x$ pushes forward b/c on $X^2$ by smallness...

This is geometric generalization of ODE of $V_\alpha$. $V_\alpha$ is $\mathbb{Z}$-analyzing at part $x^2$ -

sections of sheaf of $\alpha$ function - $\Rightarrow$ given two sections of $V_\alpha$ assign

a single section over $x^2$ - i.e. ODE...
Hitchin system - globally uniform on a curve of the center
Gives Hitchin Hamiltonian, passing from quantum objects
to global orbit quantization of Hitchin system as bundles
- which are automatically Hecke eigenstates from anyeway