A. Beilinson - On Langlands Correspondence in the de Rham setting I

Local Picture today. de Rham vers. differs from analytic setting, use methods unavailable locally...

Visual Langlands relates two seemingly unrelated objects.

$G$ split reductive $G_\mathbb{A}$ $\Rightarrow$ local field = $\mathbb{C}(F)$ for us
(usual Langlands: $k$ finite)

Rep theory: $G(F)$ (locally compact group) & Reps of $G$

Galois theory: $G^\vee$ Langlands dual: dual root data to $G$

(- consider $G_\mathbb{Q}$ or $G_\mathbb{C}$ or $G_\mathbb{R}$)

Reps $G(F) \to G_\mathbb{C}$ $\leftrightarrow$

$G^\vee$ local systems on Spec $F$ (Radic)

Expect decomposition of reps of $G(F)$ into series labelled by Galois data.
More precise: Bernstein center = Endo of identity functor of rep category, which is closed under Spec of the Bernstein center.

$\Rightarrow$ expect Spec(Bernstein center) = Set of $G^\vee$-local systems

Principal methods are global - no direct dual local relation.

de Rham vers. $k$ fixed field of char. 0 (e.g. $\mathbb{C}$)

$G^\vee$-local systems: now in de Rham sense = $G^\vee$ bundles with connection on Spec $F$.
- depend on continuous parameters
- formal differential egns (no Stokes parameters), arbitrary irregular singularities allowed!

Rep theory site: $\mathfrak{g}$ split reductive Lie algebra

$\mathfrak{g}(F)$ = dim (topological) Lie algebra...

better consider reps of Kac-Moody central extension, or level $\Delta$ at $\mathfrak{h}$-invariant quadratic form on $\mathfrak{g}$: center $\mathfrak{g}$, $\Gamma = \mathbb{R}^\Delta$ giving central extension...
Note: everything here will be purely local - diagram of definition
(I will act as identity for our reps)
Rep theory depends on $k$. should consider
special $K^v$ - integral & negative in strong
sense (less than critical). & nondegenerate
(e.g., $\mathfrak{g}$ torus: nondegenerate integral scalar product
on corresponding lattice)

Format of conjecture: (rough)

$L^F = \text{module of } G^v$-local systems on $X\in F$
not algebraic (just know what families near)

(a)
Want to define an associative topological algebra (or $O_x$)
$A$ on $L^F$ together with map of Lie algebras
$\mathfrak{o}(F^x) \to A$

Given module over $A \to \text{"quasi-coherent" sheaf}$
on $L^F$, its global sections carry action of $\mathfrak{o}(F^x)$

$\to$ functor $A$-modules $\text{\rightarrow} \mathfrak{o}(F^x)$-modules
Want this to be an equivalence of categories

... so modules on $\mathfrak{o}(F^x)$ decomposed wrt
"Spectral parameters" $L^F$

Very natural construction (e.g., wrt $\mathfrak{o}(F)$-action)

(b)
Given local system $L^F$ can ask which $\mathfrak{o}(F^x)$-modules
are supported here? want explicit geometric
description - at least for some local systems (regular sing.)

Comment: a. will come from natural vertex algebra
associate to $G$ - e.g., torus $\to$ lattice
terminology
- algebra will be equipped $G^v$-action...

so can twist vertex algebra by any $G^v$-local
system & $\to$ new one fibers of $X$
- can't do on level of usual associative algebras
(mixing by $L^F$)
If modulus of $L$ happens to be an affine variety, this would be seen as map $[L]_k \rightarrow \text{center of } \text{orbit}_k$ -- but both sides $[L]_k$, center are trivial! Hope version doesn't work.

Def of vertex algebra analogous to getting lattice from planar enveloping algebra -- add extra generators.

Center is somehow internal usually -- but can construct "external center" -- i.e. with access outside.

Part (5) (for some local systems with unipotent monodromy)

$G(F)$ is an Ind-scheme from $PV$ of $k$ (inductive limit of affine $G$-schemes)

$\varphi: G(F) / \text{Ind-proper Ind scheme: Affine Flag Space}$

$\text{category } \mathcal{M}(\varphi)$ of $G$-modules on $\varphi$: \text{right}$

union of f.d. varieties with closed embeddings, so look at union of $G$-submodules supported on f.d. space.

Right $G$-nodes make sense as sheaves here; these embed into pushforward each other to give unlike left (need to twist).

$\Gamma: \mathcal{M}(\varphi) \rightarrow G(F)$-modules

really should twist by appropriate line bundle!

$K$ defines central extension of $G(F)$ by $G_m$

$G(F)_k = \text{look at equivariant line bundles for } G(F) \otimes_k \varphi$: their form a tensor

over weight lattice of $G$ (after considering, looks by 1) -- comes from choice of affine way
Pick ample line bundle \( L \) from any \( \mathbb{P}^1 \)-orbit

In every way possible

\[ \Gamma \colon M \to \Gamma(\phi, \mathcal{M}_L) \text{ \( \mathcal{O}(L) \)-module} \]

\[ \text{with: } \Gamma \text{ produces equivalence of categories} \]

\[ \phi \text{ \( \mathcal{O}(L) \)-module supported on } \Gamma \text{ \( \mathcal{O}(L) \)-module supported on } \]

\[ \text{all orbits } \]

\[ \text{by } \phi \text{ all \( \mathcal{O}(L) \)-category } \]

\[ \text{Venus et acra two way } (\text{from the local system}) \]

\[ \text{Venus } \Gamma \text{ depend on choice of } \mathcal{L} \text{ but have some constraints} \]

**Case \( G = T \) torus**

\[ A = \text{ lattice Heisenberg } \& \text{ its lifts by local system } \]

\[ \text{Rep heavy side! reps of Lie Heisenberg algebra } \]

\[ \text{decompose its reps wrt reps of all twisted lattice Heisenberg algebras!} \]

**Very brief introduction to vertex algebras:**

**Work over a curve \( X \) (eventually look at a disk)**

\[ A = \text{ quasi-coahed } \mathcal{Q} \text{-module} \]

**Def. “Factorization structure”** on \( A = \text{ collection of } \mathcal{Q} \text{-modules} \] \( \text{all in } \{ A, \ldots \} \) with compatibility data.

**Identity: \( A_x = A \), key property!** \( \forall (x_1, \ldots, x_n) \in X^n \)

Consider fiber \( A_{(x_1, \ldots, x_n)} \), demand that it equals \( \otimes A \)

Where we consider \( (x_1, \ldots, x_n) \) as \( \text{open subset of } \)

\[ X \text{ -- no multiplicity! (one copy for each distinct part).} \]

Precisely on \( X^2 \)

\[ \Delta A^2 \xrightarrow{\Delta} X \]

\[ \text{fused } \Delta A^2 = A \]

\[ \text{j*-action of switching fuses compatible with } \Delta A^2, \]

\[ \text{plus action of switching fuses compatible with } \Delta A^2, \]

\[ U = X \times X \]

\[ j^* A^2 = j^*(A \otimes A) \]
Structure is completely local: glue of $\Delta x A$ off $\Delta$ to $A$ or $\Delta$.

Def. A chiral algebra structure on $A$ is a factorization
structure s.t. 1. all $Ax$ flat transversal
direction to diagonal 2. $A$ has a unit $\phi_0$ such that $\forall a \in A$, $a \phi_0$ off $\Delta$ extends to diagonal $\Rightarrow \exists \phi_1 \in A^2$ s.t. $\phi_1 \phi_0 A \phi_0 = a$.

Note: such structure yields canonically a $\Delta$-mod structure on $A$:

$p_1^* A x \cong P_2^* A x$ pullback back to diagonal.

These maps are isomorphisms so $A x$ off $\Delta$ are flat transversally to diagonal get isomorphisms on formal neighborhoods of diagonal $\cong \Delta$-mod structure.

Operator Product Expansion:

$A x \otimes A x \mapsto j x j x A x \otimes A x = j x j x A x^2$.

$[1, f] \langle a, \phi_0 x \rangle \underset{\text{OPE}}{\mapsto} \langle a_0, \phi_0 x \rangle \rightarrow A x (p_{1,2}) \underset{\text{in local parameter}}{\mapsto} \langle j x j x A x^2 \rangle$.

Algebraic part: take only polar part.

OPE completely determines $A x^2$ hence everything.

Just glue data.