$\mathcal{X} - \text{smooth}/\mathbb{C}$

$\mathcal{D}^{\mathcal{X}} - \text{ring of diff. operators}$

$\mathcal{M}^{\mathcal{D}}(\mathcal{X}) = \mathcal{M}^{\mathcal{D}}(\mathcal{X}) - \text{category of left } \mathcal{D} \text{-mod}$

Example (i) $\mathcal{X} = \text{Spec} A$, $\mathcal{D}_A = \Gamma(\mathcal{X}, \mathcal{D})$

$\Gamma : \mathcal{M}^{\mathcal{D}}(\mathcal{X}) \rightarrow \mathcal{D}_A - \text{mod}$

- Coherent $\mathcal{D}$-module $: = \text{loc. fin. gen. } \mathcal{D} \text{-module}$
- Locally free of finite rank $\mathcal{D}_X$-mod.

(ii) $\mathcal{M}^{\mathcal{D}}(\mathcal{X}) \leftrightarrow \mathcal{M}^{\mathcal{O}_X}(\mathcal{X})$

Quasicoherent sheaves

(iii) $\Gamma^\nabla : \mathcal{M}^{\mathcal{D}}(\mathcal{X}) \rightarrow \text{Vect}$

$\Gamma^\nabla(M) = \text{Hom}(\mathcal{O}_X, M)$

$\mathcal{M}^{\mathcal{D}}(\mathcal{X})$ is an abelian tensor category

$M_1 \otimes M_2 \quad \mathcal{O}_X$ is a unit

(i) $\mathcal{X} = \text{Spec} A \quad \mathcal{M}^{\mathcal{D}}(\mathcal{X}) = \mathcal{D}_A - \text{mod}$

(ii) $\otimes$ does not preserve coherence

(iii) Lack of duals

\text{Ex}! $M$ has dual iff $M$ is smooth
\[ O_n = \mathbb{C}[t_1, \ldots, t_n] \]

\[ \text{Aut}_C(O_n) \] is the group of \(\mathbb{C}\)-points of an affine pro-algebraic group \[ = \text{Aut}^0(\mathcal{O}_n) \]

\[ \text{ieAut}^0(\mathcal{O}_n) = \text{Der}^0(\mathcal{O}_n) = \text{Der}(\mathcal{O}_n) \]

\[ \text{Aut}(\mathcal{O}_n) = \text{group ind-scheme} \]

\[ \text{Aut}(\mathcal{O}_n) = (\text{Der}(\mathcal{O}_n), \text{Aut}^0(\mathcal{O}_n)) \]

\[ \text{Kaneva-Vasquez pair} \quad (\mathcal{G}, \mathcal{K}) \quad \text{Lie} \mathcal{K} \hookrightarrow \text{G} \quad \text{action of } \mathcal{K} \text{ on } \mathcal{G} \]

\[ \text{Aut}(\mathcal{O}_n)(\mathcal{R}) = \text{Aut}_{\text{top} \text{c-alg}} \mathcal{R}[[t_1, \ldots, t_n]] \]

\[ \text{it is represented by ind-scheme} \]

\[ \text{Aut}^0(\mathcal{O}_n)-\text{mod} \xleftarrow{\text{ind}} \text{Aut}\mathcal{O}_n-\text{mod} \xrightarrow{\text{X-\mathcal{U}-mod}} \]

\[ \text{Ind}(V) = U(\text{Der}(\mathcal{O}_n)) \otimes V \]

\[ U(\text{Der}^0(\mathcal{O}_n)) \]

\[ \dim X = n \]

\[ (x, x') = (\text{Spec} O_n \to X) \]

\[ X \xrightarrow{\text{formal coord. system at } x} \]

\[ \tilde{x} \to X \quad \text{Aut}^0(\mathcal{O}_n) \text{-action on } \tilde{x} \]

\[ \tilde{x} \text{ is a principal } \text{Aut}^0(\mathcal{O}_n) \text{-bundle over } X. \]
\[ \text{Aut}^0(\mathcal{O}_n)\text{-mod} \leftrightarrow \Gamma_\sim \xrightarrow{\Delta_\sim} M^2_\mathcal{D}(X) \]

\(\Delta_\sim\) is left adj. to \(\Gamma_\sim\)
\[ \Gamma_\sim(N) := \Gamma(X_\sim, \pi^*N) \]
\[ \Delta_\sim(V) := V_{X_\sim} = (V \otimes \pi^*_X \mathcal{O}_X)_{\text{Aut}^0(\mathcal{O}_n)} \]

\(\Delta_\sim\) is a tensor functor

Claim: The \(\text{Aut}^0(\mathcal{O}_n)\) action extends canonically to an \(\text{Aut} (\mathcal{O}_n)\)

Important: \(\text{Der} \mathcal{O}_n\) action on \(X_\sim\) is formally simply trans.

\[ \text{Remark} \quad (\alpha, K) \]
\[ X \text{ dim } X = \text{dim} (\alpha / \text{Lie } K) \]

Definition: \(A(\alpha, K)\) - str. on \(X\) is a fibration \(X^\sim \to X\) together with \(\alpha\) - action on \(X^\sim\) s.t. \(X^\sim\) is a prime \(K\) - bundle over \(X\) and \(\alpha\) acts simply trans.

\[ \text{Aut} (\mathcal{O}_n)\text{-mod} \leftrightarrow \Gamma_\sim \xrightarrow{\Delta_\sim} M^2_\mathcal{D}(X) \]

Those \(\mathcal{D}\) - modules which appear this way are called "Natural."
A. Beilinson - Vertex Operator Algebras

Fall '95

Let $X$ be a smooth algebraic variety, $\dim n$.

$(O_{X/k}, K)$ a H-Ch. pair.

$\dim (O_{X/k}) = n$.

$(O_{X/k})$ structure on $X$.

$\mathfrak{g}$ acts transitively on $X$.

$(O_{X/k}, K)$ action on $X$, $X$ $K$-torsor $/X$, $\mathfrak{g}$ acts simply transitively on $X$. $\mathfrak{g}$ acts simply transitively on $\text{Alt} X$.

Example:

1) $\mathfrak{g} = \mathfrak{gl}_6$, $\mathfrak{g}_{2x} :$ take $X = K \setminus \mathcal{G}$, $\mathfrak{g} = 6$. "Integrable example."

2) $(O_{X/k}) = \text{Alt} \mathfrak{g}$.

3) $(O_{X/k}) = \text{Der} \mathfrak{g}$.

4) $(O_{X/k}) = \text{Alg} \mathfrak{g}$.

$(O_{X/k}, K)$ structure on any smooth $X$.

To any $(O_{X/k}, K)$ structure on $X$ assign sheaf of Lie algebras $\mathfrak{O}$ on $X$ acting on $X$ — i.e., morphism $\hat{\Theta} \rightarrow \Theta X$.

$\Theta (U) = \mathfrak{O}_X (U)$ — vector fields in $U$, $(O_{X/k}, K)$ acts — commute with $\mathfrak{g}$ action.

inf. symmetries of the $(O_{X/k}, K)$ structure

1) $\hat{\Theta} = \mathfrak{g}$ via right translations.

2) $\hat{\Theta} = \mathfrak{g} X$.

$\Rightarrow$ D-modules on $X$ from H-Ch $(O_{X/k})$ modules:

$$\hat{\Theta} X \rightarrow \text{Mod}_K X$$

$\hat{\Theta} X$ := pull back D-mod to $X$, take global sections.

$\Delta$: twist by our $K$-torsor $\Delta$.

Comment:

$\Delta (O_{X/k} X)$ = action of $\mathfrak{g}$ on $\Delta X$ is given by:

$\Delta X = \hat{\Theta} (O_{X/k})$ (normal kind of Lie's theorem is $O/k$, $\text{Der}$ on twist as above).

$\Delta X = (\mathfrak{g}_{O_{X/k}} \otimes O_{X/k})^K$.

$\mathfrak{g}_{O_{X/k}}$ acts on $\text{Der} \mathfrak{g}_{O_{X/k}} = \Delta (O_{X/k}) = (\mathfrak{g}_{O_{X/k}}^* \otimes O_{X/k})^K$.

$\Delta X$ := forgetful.

$\Delta$ is a tensor functor.

The $D$-modules we study, $\Delta (O_{X/k})$, are equipped with action of $\hat{\Theta}$.

Two separate $\Theta$ actions, as part of $D$ and as part of symmetries of the structure.

$D_X$-algebra := associative commutative unital algebra in $\text{Tor}_D^*$.

$D_X$ is a $D$-mod $A$, $A \otimes D_X \rightarrow A$, $D_X \rightarrow A$ is $D$-algebra with flat connection, horizontal unit, etc.
Category of $\mathcal{D}$-algebras: $\text{ComD}(X, \text{ Ver}_\mathcal{D}(X))$, versus $\text{ComD}(X)$
- tensor categories (the tensor product is a categorical product; $\otimes$)
- $\otimes$ commutes with $\square$ (tensor in ComD is $\otimes$)

Jets

Lemma
1) $\mathcal{D}$ admits a left adjoint $F: (\text{ComD}(X)) \rightarrow (\text{ComD}(X))$
2) $\mathcal{D}$ commutes with $\square$

Proof:
1) $\mathcal{D}$ an $\mathcal{O}$-algebra: $\text{Hom}_{\text{D-gen}}(JR, \mathcal{A}) = \text{Hom}_{\mathcal{D-gen}}(R, \mathcal{O}A)$
2) so take $JR$ as follows: need minimal version $R \rightarrow \mathcal{O}X$

$JR = \text{Sym} (Dx \otimes R) / [\text{ideal gen. by } 1_x - 1, 1_xf_r - (a_0)(1 x f_2)]$

$JR$ is the "jet algebra" for $R$.

Prop: $B$, $R$ $\mathcal{O}$-algebras, then $\text{Hom}_{\mathcal{OX}}(\square \circ JR, B) = \lim_{\text{Hom}_{\mathcal{OX}}(\square \circ B, \mathcal{O}X)}$

Here $\mathcal{O}X$ is $O(X \otimes \mathcal{O}_X \otimes X) = \mathcal{O}_X / I_{\text{red}}$, $I_{\text{red}}$ diagonal ideal.

Sketch of proof: $\text{RHS} \overset{\text{lim}}{=} \text{Hom}_{\text{ring of } \mathcal{O}X} (\square \circ B, \mathcal{O}X) = \lim_{\text{Hom}_{\mathcal{O}X}(\square \circ B, \mathcal{O}X)}$

$\square \circ B \overset{\text{lim}}{=} \text{Hom}_{\mathcal{O}X}(\square \circ B, \mathcal{O}X)$: $b \circ f \rightarrow (b \circ f) \circ (\square \circ g)$

above = $\text{Hom} (Dx \otimes R, B) \Rightarrow \text{Hom}_{\text{D-gen}} (JR, B)$; map determined by action on generators.

Cor. $C$-points of $\square \circ JR$ are the same as pairs $(C, r)_C \times X$, $r$ a section
of Spec $R/X$ on the formal neighborhood of $X$.

Thus Spec $JR$ is the union of infinite jets of sections of Spec $R$.

- General principle: $\mathcal{O}X$-algebra, infinite jets of sections have canonical
flat connection, as above construction.

Globalize: $D_X$-scheme = $X$-scheme with a flat connection along $X$.
(of which above is affine case.)

Claim the $\square$-construction is compatible with Zariski or Tate
localization, hence passes to $D_X$-schemes: $D_X$-sen $\rightarrow \mathcal{O}X$.
\[ \text{Sch}_D(x) \xrightarrow{\phi} \text{Sch}(x) \]

Ex. (a) K \& pair commuting finite algebras in category of (g, k)-sets:
\[ \text{commute with } y, \text{ show } \Delta \Gamma \text{ commute with } y, y. \]

Make our \( D_x \) algebraic algebras: \( A \) a \( D_x \)-algebra \( \Rightarrow \) sheaf of rings
\[ A[D_x] : \text{sheaf of algebras on } X \text{ with } \text{algebras } A \rightarrow A[D_x] \]
with obvious compatibilities: \( A[D_x] \) generated by \( A, D_x \) with only these relations. (\( A \in \mathcal{A} \).)

Ex. \( A[D_x] = A \otimes D_x \) as \( A \)-bimodule.

A \( D_x \)-algebra allows one to multiply these \( A \)-valued diffs.

Claim \( A \)-modules = sheaves of \( A[D_x] \)-modules \( \Rightarrow \) \( A \)-modules.
\[ \rightarrow \text{ define } \text{ge} \text{ modules on } D \text{-schemes, projectivity} \]
\( \text{not very good to generalize in ind-finite; want proj. projectivity} \)

Lemma For gen projective \( A[D_x] \)-modules are "local" objects.
\( \text{i.e., projective on some covering } \Rightarrow \text{projective on true } \).

Sheaf of (killing) differentials \( A \rightarrow \Omega_A \) is given any \( A \)-module \( M \).
\( \text{a differential } d \rightarrow M \) is a morphism satisfying \( \text{Lie}_d(ab) = a\text{d}b + bd\text{d} \).

1) Universal differential \( A \rightarrow \Omega_A \)
2) For an algebra \( A \)
3) \( \text{d}A \rightarrow \Omega_A \), \( \text{d}a \equiv a \text{e}_1 \oplus a \mod I^2 \).

Smooth \( D_x \text{-schemes} \): \( A \) a \( D_x \)-algebra.
For \( A \) is formally smooth if for \( x \in \overline{C} \Rightarrow \text{I} \in \overline{C} \), \( \text{I}^2 = 0 \).
any morphism \( A \rightarrow C[I] \) lifts to a morphism \( A \rightarrow C \).

Ex. \( A = \text{Sym } M. \) Then \( A \)'s formally smooth iff \( M \) is projective \( D \)-mod.
\( \text{iff } M \) is proj., \( A \rightarrow C[I] \) is mor of \( D \)-mod \( M \rightarrow C[I] \) liftable, conversely (exercise)
\( \Rightarrow \) lets you check projectivity.

Assume \( X \) affine - huge supply of proj. \( D \)-mod (\( \text{e.g. }, \text{R} \).
\( \text{A any } D_x \)-alg. can be written as quotient of formally smooth \( J \sqcup B \rightarrow A \).
We may unit \( \Omega B \rightarrow \Omega A \)
\[ 0 \rightarrow \Omega^2 A \rightarrow \Omega^2 B \rightarrow \Omega B \otimes A \rightarrow \Omega A \rightarrow 0 \quad \text{exact.} \]
\( \Omega^2 A \) depends on \( A \) only for \( B \) formally smooth.

**Show (using \( Y \)) formally smooth \( \text{Gr} B \) is formally smooth \( D_X \)-alg.

- Then \( \Omega^2 A \) is just \( \Delta^{(2)} A \) as \( O \)-module.

   **Categorial:** \( A \) is formally smooth iff
   1. \( \Delta^{(2)} A = 0 \),
   2. \( A \) is projective

   (as \( A[D_X] \)-module)

   (A formally sm. 1) obvious, ii) the \( B \)-poly. alg. ...

\[ \mathfrak{m} \rightarrow B^{1/2} \rightarrow A \rightarrow 0 \quad \text{for } \mathfrak{m} \text{ sm. alg.}, \quad \mathfrak{m} \in \text{direct summand of } \Omega^2 A \rightarrow \text{proj.}

Conversely \( \Delta A \text{ proj} \Rightarrow \text{splitting } \Rightarrow \text{splitting } B^{1/2} \rightarrow A

\[ \mathfrak{m} \rightarrow \mathfrak{m} \rightarrow 0 \quad \text{etc.} \]

\( A \) is fin. sm. as \( D_X \)-alg iff it is fin. sm. as \( O \)-alg and \( \Delta A \) is projective.

**Ex.** \( A \) is fin. sm. as \( \text{Gr} B \) alg iff it is fin. sm. as \( C \)-alg.

**Def.** \( A \) is smooth if it is sm. and finitely generated (as \( D_X \)-alg -

- quotient of symm. alg. of \( E \), free \( D \)-module.)

Claim smoothness is local \( \Rightarrow \) notion of a smooth \( D \)-sch.

**Ex.** \( Y \) smth. \( C \)-sch \( \Rightarrow \) smth. \( D_X \)-alg and smth. scheme \( \Rightarrow \) smth. \( D \)-sch.

**Question.** Is any smooth \( D_X \)-alg is finitely presented? (as \( D_X \)-alg)

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**Ex.** of fin. gen. not fin. gen. \( D_X \)-alg: \( \text{Sym}(D_X^2) / \text{Sym}^2 D_X = \text{sym}^2 \) is not finitely generated...

**Pitfall:** Above we have \( Y \) smooth \( D \)-sch. Are more words on \( Y \)?

- A usual symm alg \( Y \) is a morph of \( D \)-alg \( Y \rightarrow \text{Sym}(D_X^2) \)

  \[ \text{st. } \text{if } f: Y \rightarrow \text{Sym}(D_X^2) \text{ is an isomorphism.} \]

  \[ \text{always have local coordinates on } Y \text{ scheme.} \]
Not true for smooth $\mathcal{L}$-sheaves, e.g. smooth $\mathcal{L}$-jets, smooth scheme - these do have coads - but not true for arbitrary $\mathcal{L}$-sh

on generic point.

Reason assume $\mathcal{L} = \text{Spec } A$, $A$ integral. $\text{Spec } A$ has finitely many elements: $A[c] = A^*$ - invertible differential operator must have

deg 0 (say $\text{dim } X = 1$, top symbol is function, degrees of symbols add...).

Now assume $A$ is smooth and $\mathcal{L} = \text{Spec } A$ for an algebra (not scheme). $A \subset \mathcal{L}$ corresponds to some $x \subset X$, well defined linear up to right mul of invertible operator $\Rightarrow A^*$ which fixes $A$!

so $x \subset X$ is canonical. - Line subbundle in $\mathcal{N}$, field of hyperplanes... is it integrable?

Integrability: $d(\mathcal{L}) \subset \mathcal{N}X$ must sit in $\mathcal{L} \subset \mathcal{N}$...

Example/Exercise $V + uV = 0$, $X = \text{Spec } \mathbb{C}[V]. B = \mathbb{C}[x^i, u, \ldots, V, \ldots, V']$

$F$ gives $\mathcal{L}$-scheme. Take $\mathcal{O}_V$ ideal $I \subset B$ generated by $\mathcal{O}_x = V + uV$,

$A = B/I$, so the $\mathcal{L}$ is satisf $\mathcal{L}$ is integrable.

Claim $A$ is smooth and $\mathcal{L} \subset X$ is not integrable.

$\mathcal{O}_x = dV + d(uV) = du' + dV + d(uV') = dV + dV + VdV + V' dU + VdV + V' dU$

Horizontal Sections (classical analog of space of non-formal blocks)

Com Assoc (aln) Com $\rightarrow$ Com$\mathcal{D}(V)$

$A \rightarrow R \otimes O_X$

Easier right adjoint: $\text{Com } A \rightarrow \Gamma(X, A)$ global horizontal sections

by lemma this is so being

Left adjoint: space of horizontal sections. Take maximal constant quadratic of $\mathcal{L}$! project $\alpha$ to $A$.

call this $\text{H}_p (X, A) = \text{H}_p (A)$

"constant algebra" = image of $R \rightarrow R \otimes O_X$.

Is there a smallest ideal that will do the trick?

Proposition If $X$ is compact then any $A$ has a smallest ideal $I$ st $A/I$ is small.

[Will return to proof]

What is "space of sections"? $\text{R-sections} \rightarrow$ morphisms $X: \text{Spec } R \rightarrow \text{Spec } A \rightarrow \text{Spec } R$.

So we want such morphisms $A \rightarrow R \otimes O_X$ which are $R$-morphisms, i.e.

horizontal.
Via functor of jets, this contains usual notion of sections (by adjunction).

Right D-modules $M^r_\mathcal{D}(X)$ are $M^s_\mathcal{D}(X)$

a. If left $M$ right, then $L_{\mathcal{D}^m}M$ is naturally a right $D$-module: $\mathcal{L} \cdot \mathcal{D} \cdot \mathcal{L} \cdot \mathcal{D} = -L_{\mathcal{D}^m}(-1) \mathcal{D} \mathcal{L} \mathcal{D} \mathcal{L} \cdot \mathcal{D}^m$

b. Canonical right and $\mathcal{L}\mathcal{D} = \mathcal{D}^{m-1}$ implies $\mathcal{D} \mathcal{L} = -1 \mathcal{L}$

c. $M^r_\mathcal{D}(X) \to M^s_\mathcal{D}(X)$, $\mathcal{L} \mapsto \mathcal{L} \mathcal{D} \mathcal{L} \mathcal{D}$ is an equivalence of categories.

$D^b_{\mathcal{D}^m}(X) \to D^b_{\mathcal{D}^s}(X)$

$\mathcal{L} \mapsto \mathcal{L} \mathcal{D} \mathcal{L} \mathcal{D}$ should shift naturally

Thus $D^b_{\mathcal{D}^m}(X)$ has two natural cores (above-subcategories)

$M^r \to M^r$, the left core $M^r$ coincides with $M^s \mathcal{D}^{m-1}$.

Pull-back functors $f : Y \to X$, $M^s_\mathcal{D}(Y) \to M^r_\mathcal{D}(X)$

$M^s_\mathcal{D}(Y)$ as $D$-module: pull-back of bundle with connection is finite with connection.

$\Rightarrow$ Right exact functor $f^* : M^s_\mathcal{D}(Y) \to M^s_\mathcal{D}(X)$, left derived is

$Lf^* : D^b_{\mathcal{D}^s}(Y) \to D^b_{\mathcal{D}^s}(X)$, which we denote by $f^*$

Exer.

$D^b_{\mathcal{D}^s}(X) \xrightarrow{f^*} D^b_{\mathcal{D}^s}(Y) \xrightarrow{Rf^*} D^b_{\mathcal{D}^s}(Y)$

$\downarrow \mathcal{L} \mathcal{D}$

$\downarrow \mathcal{L} \mathcal{D}$

Derivations of regular sheaves: $M^s \Omega^r = \text{sheaf of vector spaces}$ ($Z$-rank $r$)

$\Delta(M) = M^s \Omega^r \to \mathcal{O}_X \otimes \mathcal{D}_X$ (collinearity with respect to $\mathcal{O}_X$).

Projection $\pi : M \to \Delta(M)$

Deligne complex: $L \Delta M^r$, $D(R)(L) = L \Delta \mathcal{D} \mathcal{L} \Delta \mathcal{L}$, integrable

connection gives differentiable.

$M^s \Omega^r$, $D(R)(M) = M \Omega^r \mathcal{L} \Delta \mathcal{L}$ (in negative degree).

$M = \mathcal{L} \mathcal{D} \mathcal{L} \mathcal{D}$ these get identified (contract $\mathcal{L} \mathcal{D}$ with $\mathcal{L} \mathcal{D} \mathcal{L} \mathcal{D}$).

So canonical functor $D^b_{\mathcal{D}^m}(X) \xrightarrow{DR} D^b\mathcal{D}(X)$
Inducion \[ M^D(x) \xrightarrow{\text{Diff}(x)} M^D(x) \]

Lemma: \[ h(F) = F \] (Prove by reducing to local, reduce to \( F \) first by inductive limits and exactness in first argument.)

Example: Introduce category \( \text{Diff}(x) = M^D(x) \), some objects but morphisms are differential operators. Lemma ii. strict:

\[ \text{Diff}(x) \xrightarrow{\text{fully faithful}} M^D(x) \]

Example: \( M \in M^D(x) \). \( \text{DR}(M) \) is a complex in \( \text{Diff}(x) \). So we may consider \( \text{DR}(M) \) as complex of \( D \)-modules:

\[ \text{DR}(M) = (\ldots \to M \otimes \mathcal{O}_x \otimes \mathcal{O}_x \to M \otimes \mathcal{O}_x) \]

Claim: this morphism \( \text{DR}(M) \to M \) is a quasi-isomorphism, i.e., \( \mathcal{O}_x \) gives canonical left resolution of a right \( D \)-module \( M \).

Exercise: Define \( \text{D}^\bullet \text{Diff}(x) \) \( \text{D}^\bullet \text{Mod}(x) \) are mutually inverse equivalences of categories.

Fundamentality: Direct image (somewhat better for right pullback [left]).

Case a) Open embedding \( j : U \to X \). \( j_* \mathcal{N}_U = j_\ast j^* \mathcal{N}_U \) as sheaves.

- Meromorphic continuation;
- Closed embeddings \( i : Y \to X \) closed smooth subvariety, \( \mathcal{O}_Y = \mathcal{O}_X/I \)

\[ i^! : M^D_Y \to M^D_X, \quad i^! M = \{ m \in M | I m = 0 \} = \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_Y, M) \]
To make this a $D^m$-mod, extend vector field arbitrarily to $X$, action (from right) independent of extension. Pushforward is now $i^*: M^D_D(Y) \to M^D_D(X)$ left exact to $i^*$.

$M^D_D(Y) : i^* N = N \otimes_{D^m} i^* D^m$.

Let $D^m$ a left $D^m$-module, pull back to $Y$, $i^* D^m = D^m / i^* D^m$ (can $= \mathcal{O}(Y)$)

Alternatively define $D^m_{pY} \hookrightarrow D^m$ as $\mathcal{O}(D^m Y) \hookrightarrow \mathcal{O}(D^m Y)$

So set $i^* N = N \otimes_{D^m_{pY}} D^m$ and show it's the same.

$i^*$ is left exact and $i^*$ is in fact exact.

What is $\text{Hom}(i^* N, M)$ - natural sends $N$ to subobject of section killing $i^*$? $\text{Hom}(\mathcal{O}(Y), M) = \mathcal{O} M$.

Locally $X = Y \times_Z Z$ in étale topology. (y-scheme) $Y = Y \times_Z \cdot$, $\cdot \to Z$.

$D^m_{pY}$ is reductive. $\text{Hom}(\mathcal{O}(Z), C) = C$, $i^* N = N \otimes C$.

Lemma (Kashiwara) $M^D_D(Y) \to M^D_D(Z)$ $Y$ is again of categories, with inverse $i^*$, $i^* M^D_D$.

Proof: Need to show natural transforms $i^* \to i^*$ are equivalences.

Case $Y$ point - algebraic version of Stone-von-Neumann (???)

Support at 0 means null by $X$ is locally nilpotent, $Z$ rep of Heisenberg algebra locally, $\Rightarrow$ $\text{Fock} = Z$, add parameters

This functor is compatible with $D^m R^m$ (???) $D R^m N \to i^* D^m R^m N$ is a quasi-isomorphism in particular $h(\cdot N) = i^* h(N)$

Also compatible with induction $F(\cdot)$ mod: $i^* F = i^* F$

(Follow from adjunction/universality $\cdots$)

Open vs. closed: $Y \hookrightarrow X \setminus U$ complement

$i^* i_{K i}^* M \to i^* i_{K j}^* N$ is an exact triangle.
Formal definition of formal pullback $f^* : \mathcal{D}^b(M)^A \to \mathcal{D}^b(M)^B$ via $f_* : \mathcal{D}^b(M)^C \to \mathcal{D}^b(M)^D$ pullback of $f$ as left $\mathcal{D}^b$-adjoint.

Alternatively decompose $f$ as $\gamma \xrightarrow{\pi} X \xrightarrow{\pi_0} X$ - compatible with connections.

$f_0$ (chooses $\mathcal{T}_f \to N$ exact), take $\pi_0$ for projection! take $\pi_0$ as delRham complex t.b.w. to $\pi_0$ along $\gamma$ $\mathcal{T}_f \to \mathcal{T}_f$.

When $X$ is point, then $f_0^* N = R\pi_0^* \mathcal{D}^b(X \otimes \mathcal{O}(N))$ deRham sheaf.

Natural morphism $R\pi_0^* \mathcal{D}^b(X \otimes \mathcal{O}(N)) \to \Gamma(X, \mathcal{D}^b(X \otimes \mathcal{O}(N)))$ commutes with $\mathcal{O}(N)$ of our previous $f$.

$f_0^* N$ doesn't have expression as single derived functor neither left nor right exact, complex of $R\pi_0$ and $\mathcal{O}$.

Theorem (key property of $\otimes$) $f : Y \to X$ proper, then $\mathcal{D}^b(M)^C(Y) \xrightarrow{f_*} \mathcal{D}^b(M)^D(X)$ then $f_*$ is left adjoint to $f^*$.

[Boer ...

Particular case $X$ compact $\to \text{pt.} \quad M \in \mathcal{D}^b(X)$.

$H^0(\text{pr}_1^* \mathcal{O}_M) = H^0(X, \mathcal{D}^b(M))$.

Lemmas: There is a canonical morphism of $\mathcal{D}$-modules $\mathcal{M} \to H^0(\text{pr}_1^* \mathcal{O}_M) \otimes \mathcal{O}_X$ (constant) chasing the RHS with the maximal constant quotient of $\mathcal{M}$.

Theorem tells us how to compute $\text{Hom}((\mathcal{M}, \mathcal{N}))$ - it is

$\text{Hom}(\mathcal{M}, \mathcal{N})$ by definition:

$x \in X \mapsto \mathcal{U}_x \in \mathcal{O}_{X}$. $M_x = M^{\alpha \beta}_{x}M$ fiberwise.

Fibrewise $\mathcal{M}_x \mapsto H^0(\mathcal{O}_x \mathcal{K}_M)$, constructed from our canonical triangle

$0 \to \mathcal{K}_x \otimes \mathcal{M} \to \mathcal{M}_x \to \mathcal{K}_x \to 0$

$M_x$ in $\mathcal{K}_x$ look at formal neigh where our connection trivializes $M$ as $M_x$.

So set now $\otimes_{\mathcal{K}}^\mathcal{D} M \to \mathcal{M}$, surjective since $\mathcal{U}_x$ is open, no local.

The kernel can be computed via

$H^{-1}(\mathcal{U}_x, M) \xrightarrow{\text{Res}} M_x \to H^0(\mathcal{O}_x \mathcal{K}_M) \to 0$. 


Assume \( \dim X = 1 \), \( X \) compact, \( L = U^1 \mathcal{O}(X) \)
What is the maximal constant quotient of \( L \) (\( \mathcal{O}(\) of copies of \( \mathcal{O} \) ...)

Proof construction: \( H^0_{\mathcal{O}}(\mathcal{O}(X) \otimes \mathcal{O}) \).
There is a canonical morphism of left \( \mathcal{O} \)-modules \( L \rightarrow H^0_{\mathcal{O}}(\mathcal{O}(X) \otimes \mathcal{O}) \).

Proof: May assume \( L \) has no torsion as \( \mathcal{O} \)-mod - can quotient out by torsion without changing the question. Take \( \text{ker} \mathcal{O} : j : V = X \otimes \mathcal{O} \rightarrow X \)
Consider the fibres \( L_x = \mathcal{O}_x / \mathcal{O}_x \cdot \mathcal{O}_x = \text{ker} \mathcal{O}_x (L \rightarrow j^*L) \otimes \mathcal{O}_x \)
- \( \alpha : \mathcal{O}_x \rightarrow \mathcal{O}_x \rightarrow \text{Vec} \)
- \( \mathcal{O}_x \)

Why is \( \alpha \) an identification? Multiplication by \( \frac{dt}{t} \).
- take element of \( \mathcal{O}_x \), extend locally, tensor with \( \frac{dt}{t} \) - different extensions differ by \( \mathcal{O}_x \) (regular forms)
- i.e., residue map.

\[ 0 \rightarrow L \rightarrow j^*L \rightarrow L_x \otimes \mathcal{O}_x \rightarrow 0 \]

Long exact: top dir of \( j^*L \) remains as \( \text{ker} \mathcal{O} \) \( \otimes \mathcal{O}_x \) ... so for
\[ H^0(C, \mathcal{O}) \rightarrow L_x \rightarrow H^1_{\mathcal{O}}(\mathcal{O}(X) \otimes \mathcal{O}) \rightarrow 0 \]
\( V \) is a fibre so that we may quotient \( \text{ker} \mathcal{O} \rightarrow H^0(C, \mathcal{O}) \rightarrow H^0(C, \mathcal{O}) \)

Restrict \( L \) to formal neigh of \( x \), \( V \) trivializes, i.e.
\[ L^\times = \lim \mathcal{O}(\mathcal{O}/\mathcal{O}^m) \rightarrow L_x \otimes \mathcal{O}_x \]

The map above is just \( \text{Res}_x \).
\[ H^0_{\mathcal{O}}(\mathcal{O}(X) \otimes \mathcal{O}) \) is kernel of residue map.

Do this in fam to cover \( x \) (look at diagonal in \( X \times X \)).

so the above becomes stalk of the map we were to construct.

Maximality of quotient: say \( L \rightarrow V \otimes \mathcal{O}_x \), \( \alpha \)
is the factor through \( H^0_{\mathcal{O}} \): apply \( H^2 \rightarrow 0 \).

\[ H^2(\mathcal{O}(X) \otimes \mathcal{O}) \rightarrow V \otimes H^2(\mathcal{O}(X) \otimes \mathcal{O}) = V \]

which is the map we wanted ...

Con: application we used before: \( A \) a \( \mathcal{O} \)-algebra, \( \dim X = 1 \), \( X \) compact
then \( A \) has maximal constant quotient \( H^0_{\mathcal{O}}(X, A) \):
\[ R < A \rightarrow H^2_{\mathcal{O}}(X, A) \otimes \mathcal{O}_x \rightarrow R \otimes \mathcal{O}_x \) isn't 2nd.

so maximal quotient is \( A/R \)-ideal gen by \( R \).

\[ H^0_{\mathcal{O}}(X, A) = A/R, \] The spec of this is spec of bosonic section ...
\[ A = \text{res} \phi(C(X \times X), A \otimes \mathcal{O}_x), \ A/R \rightarrow \text{spec} \phi(C(X \times X), A \otimes \mathcal{O}_x) = H^0_{\mathcal{O}}(X, A) \)

In quantized situation this will be conformal blocks, standard coimultants definition.
New pseudotensor structure on $M_5^g \Omega X$.

**Categories:** Example $M_5 \Omega$ tensor category (symmetric monoidal -

strictly commassoci.) Given $[M; \Omega; I]$ finite nearly

$\Rightarrow \{ M_i \}$ need not demand $I$ ordered etc.

Polynomial operad $P_\Omega ([M; \Omega]; L) = \text{Hom} (\{ M_i \}, L)$

composition $\times I \Rightarrow I$, family $\{ K_\Omega \}$

$\Phi_i \in P_\Omega (\{ K_\Omega \}; M_i)$, $\chi \in P_\Omega (\{ M_i \}, L)$

$\Phi (\chi_i) = \chi \circ (\phi^i \chi_i) \in P_\Omega (\{ K_\Omega \}, L)$ associative.

**Definition:** A category, or $\Omega$ structure on $\Phi$ is a lax $P_\Omega (\{ M_i \}, L)$.

More generally just assume $M$ is a set

set law of composition $P : \text{Hom} (M, L) = P_\Omega (\{ M_i \}, L)$,

assumed we have an identity in here.

*Difference from usual tensor category:*

Assume further $P_\Omega (\{ M_i \}, L)$ is representable — by $\cdot_{\{ M_i \}}$.

Composition law gives canonical morphism $\cdot_{\{ M_i \}} \Rightarrow \text{Hom} (\{ M_i \}, L)^{\otimes I}$

Now we have

**Lemma:** A tensor category is a representable $\Omega$-cat such that all

$\Phi_i$'s are isomorphism.

**Example:** $\Omega$ category with single object: $I \times \times \times \Rightarrow P_\Omega$ composition operations $\Rightarrow$ operation

$\in \{ \Rightarrow P_\Omega \}$ composition law. + $\times$ action. $\Rightarrow P = P_{\times 2}$, etc.

So acts transitively on the composition.

This will be tensor if all $P_i = M$, commutative monoidal composition

are products.

Obvious notion of $\Omega$ functor $\Rightarrow$ sub $\Omega$-cat etc.

Examples of subcat: Full subcategories, take any collection of

objects and their operations e.g. any object $\Rightarrow$ object.

$\Phi_\Omega$ on $M_5^g \Omega X$ which is subcat (poly)linear: $P_\Omega (\{ M_i \}, L)$ =

$\text{Hom} (\{ M_i \}, \Delta^\text{op} \Omega L)$ — purely local (concentrated on

diagonal, form sheaf on $X$. )
We say a $*$-structure on an abelian category $\mathcal{M}$ is abelian if $\mathcal{P}$ are left exact. (Generalize usual exactness of $\mathcal{T}$.)

Let $\mathcal{M}$ be an abelian category. Define:

- $\mathcal{M}^0(x) := \mathcal{M}^0(x)$ (Categories of Chain Complexes).

On the full subcategory $\text{Diff}(\mathcal{A})$ of $\mathcal{M}^0(x)$ closed under:

- $M_i = \mathbb{F} = \mathbb{F} \oplus \mathbb{F}$
- $L = \mathcal{E} = \mathcal{E} \oplus \mathcal{E}$
- $\mathcal{P}^\mathbb{F}(\mathbb{F}, \mathbb{F}) = \text{Hom}(\mathbb{F}, \mathbb{F}) = \text{Hom}(\mathcal{E}, \mathcal{E})$

so $\text{Diff}(\mathcal{E}, \mathcal{F}) = \text{Poly-differential operator}$

We have $h: \mathcal{M}^0(x) \to \mathbb{S}(\mathbb{H})$, a tensor category with usual tensor product.

Claim $h$ is a $*$-functor $\mathcal{M}^0(x)^\ast \to \mathbb{S}(\mathbb{H})$.

Let $\mathcal{P}^\mathbb{F}(\mathbb{F}, \mathbb{F}) = \text{Hom}(\mathcal{E}, \mathbb{F})$, $h(x)$ is a generator of $\mathcal{M}$, the ext product. $\Delta_\mathcal{E}$ commutes with $h$ i.e. natural maps $\Delta_\mathcal{E} h(M) \to h(\Delta_\mathcal{E} M)$, $h(\Delta_\mathcal{E} M) = \Delta_\mathcal{E} h(M)$, compatible with compositions.

Assume $\mathcal{O}$ is an operad. We have action of an algebra $\mathcal{O}$.

Let $\mathcal{O}$-algebra $\mathcal{V}$ with $\mathcal{O}$-action i.e. maps $\mathcal{O} \to \text{End}(\mathcal{V})$. End algebra of $\mathcal{V}$.

Take $\mathcal{O}$-functor, an $\mathcal{O}$-alg $\mathcal{M} = \mathbb{S}$ (functor from $\mathcal{O} \to \mathcal{M}$, i.e. an object $\mathcal{V} \in \mathcal{M}$ with a morphism $\mathcal{O} \to \mathcal{V}$.

Example 1: For us the most important is the Lie algebra. Lie - space which consist of all natural $\mathcal{O}$-Lie operators, generated by $\mathcal{F}$, $\mathcal{I}$ in degree 1 freely with two relations - Jacobstrew-symmetry.

Example 2: Comm - just $\mathcal{C}$ in each degree, only one way to $\mathcal{C}$-commute.

Example 3: Ass - just $\mathcal{F}$ in degree 2, no relations at all except associativity.

Example 4: Poiss
Given $O$ can define free $O$-algebra on rings (universal property), which is the space of $O$.

A Lie algebra in $k$ is $L \otimes M$ with $LL \subseteq [L, L, L]$. Use symplectic & Jacobi identity in $k \otimes [L, L, L]$. 

An $L$-module is $V \otimes M$ with $v \otimes L \subseteq [L, V, V]$. Use

show it gives an $O$-algebra category $L$-mod for $O$-algebra $k$. 

In particular we have Lie algebras.

If $L$ is a Lie algebra then $h(L)$ is a Lie algebra on $X$, 

Good $L$-module $\Rightarrow h(V)$ is an $h(L)$ module.

is not faithful, but not too far from it.

Ex. For induced modules, a Lie algebra in $\text{Diff}(X) = O$-mod +

the bracket given by a bidifferential operator.

This gives a huge study - vector fields, diffops, algebras of vector bundle $\mathcal{A}$ ...

Ex 2. Simplest example of non-induced Lie $^*$-algebra: by simple Lie algebra $L = SO(3)$.

Assume dim $X = 1$, consider $\mathfrak{g} = \mathfrak{so}(3)$, Lie algebra.

with $\mathfrak{g}$-flow exist, induce to $X$.

This gives a central extension by means of $\mathfrak{w}_X$.

$0 \rightarrow \mathfrak{w}_X \rightarrow \mathfrak{g} \otimes \mathfrak{dx} \rightarrow \mathfrak{g} \otimes \mathfrak{dx} \rightarrow 0$.

Induced action $\otimes$ bilinear action on $\mathfrak{g} \otimes \mathfrak{dx} \otimes \mathfrak{w}_X \otimes \mathfrak{dx}$

get almost $\mathfrak{w}_X$, but not full symmetry.

define $[\mathfrak{g} \otimes \mathfrak{dx} \otimes \mathfrak{w}_X \otimes \mathfrak{dx}]$.

with $\mathfrak{w}_X$-action (a, b) — operation in

$\mathfrak{p}^2(\mathfrak{g} \otimes \mathfrak{dx}, \mathfrak{g} \otimes \mathfrak{dx}, \mathfrak{w}_X \otimes \mathfrak{dx})$, push it forward.

Explicitly, take the product operation $\mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$, what is the corresponding operation in $\mathfrak{p}^2(\mathfrak{g} \otimes \mathfrak{dx}, \mathfrak{dx})$?

map $\mathfrak{g} \otimes \mathfrak{dx} \rightarrow \mathfrak{g} \otimes \mathfrak{dx}$. Use the map by $s$ with relation $s$ is killed by diagonal.
Diff(C) A diffeo between C-modules $F_1 \to F_2$ is the same as a morphism between the reduced modules $\tilde{F}_1 \to \tilde{F}_2$:

$\tilde{F}_1 \to \tilde{F}_2 = \text{Diff}(\mathcal{O}_X, F_1) \ni \phi \mapsto (q \mapsto \phi(q) F_2)$

$\text{Diff}(\mathcal{O}_X, F_1) \to \text{Diff}(\mathcal{O}_X, F_2) : \text{composition}$ from left with $\tilde{\phi}$.

Diff(C)* $\to M(C)^*$ is a Y# functor, given

$L \to \text{Diff}(C)^* - \text{ Lie algebra } \mathfrak{g} \to \text{L}^*$ bidifferential

$\mathfrak{g} \to \text{L} \to \text{L}^* \to \text{L}^*$ differential

$i : \mathfrak{g} \to \text{L}$ differential (1)

$(L \mathfrak{g} \mathfrak{g} \mathfrak{g}) \otimes (L \mathfrak{g} \mathfrak{g}) \to \mathfrak{g} \otimes \mathfrak{g} (\mathfrak{g} \otimes \mathfrak{g})$ I ideal of diagonal.

Examples (i) $\mathfrak{g}$ is a linear:

$[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$ sends $L \otimes L \subset (L \mathfrak{g} \mathfrak{g} \mathfrak{g}) \otimes \mathfrak{g} \otimes \mathfrak{g}$ to $L \otimes L \otimes \mathfrak{g} \otimes \mathfrak{g}$ and coincides on it with $\mathfrak{g}$.

e.g. K-M case $G_2 :$ bracket $\{g \otimes g\} \otimes \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ (Kac \wedge \mathfrak{g})

$k \mathfrak{g} \mathfrak{g} \mathfrak{g} \mathfrak{g} \mathfrak{g}$ - diagonal operators $\otimes$ with $Y$-functions in transverse directions - constant in transverse $X_3$ direction $- \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$.

\begin{enumerate}
\item $L = O_X$ with standard bracket, $\otimes$ bracket on $O_X$
\end{enumerate}

On $O_X \otimes O_X$, the * bracket is $[\xi_1, \xi_2] = [\xi_1, \xi_2] \otimes 1 + \xi_1 \otimes \xi_2^{\otimes 2}$ in $O_X \otimes (\mathfrak{g} \otimes \mathfrak{g})$

$\subset$ (2) means $\subset$ as DIFF

acting along the second variable

e.g. $[\xi_1, \xi_2] = 2 \otimes (\xi_1 \otimes 2)$

\begin{enumerate}
\item How do we $*$ this: $Q_i \otimes Q_2^{\otimes 2} \to [Q_1, Q_i, Q_2]^{\otimes 2}$ is the new diffeomorphism operator $= Q_1 Q_2^{\otimes 2} [Q_1, Q_2] + [Q_1, Q_2] Q_2^{\otimes 2} - Q_2^{\otimes 2} [Q_1, Q_2]$
\end{enumerate}

"Sier geometry" : take $[E_2, G] = -\text{inv} \circ [G, E_2]$ where inv is the involution of X2X switching factors.

What are the modules $\mathfrak{g}$?

Recall $h: M(X)^* \to Sh(X) \times \otimes \text{ functor}.$

$(A \otimes B \to \Delta \otimes C) \subset P_2 \times ([A, B], C)$.

Let's apply $h$ transversely to the diagonal $\mathcal{F} \to A$, along $\mathcal{F}$.
variable \( \Delta \) transversely is \( \delta \)-functions hence its \( h \) is \( C \):
\[
\begin{align*}
\text{get} & \\
\Delta \xrightarrow{\delta} & \quad B \quad \rightarrow \quad C, \quad \text{i.e.} \\
\Delta \xrightarrow{\delta} & \quad \text{Hom}_D(B, C) \\
\xrightarrow{h} & \quad \text{Hom}_D(h(B), h(C)) \quad \text{which is} \ h \text{ applied to the whole \( D \)-operation} \\
& \quad \text{get more structure via this “partial” application of \( h \).}
\end{align*}
\]

Example \( L \) Lie*-algebra, \( L \) an \( \mathbb{L} \)-module. Then the \( \text{Lie} \) action \( M \rightarrow \mathbb{P}^*_L(L, M, h) \), apply partial \( h \) yields \( h(L) \rightarrow \text{Hom}_D(M, M) = \text{End}_D(M) \). \( h(L) \) is a sheaf of \( \text{Lie} \)-algebras. This gives actions of \( \mathbb{C} \)-sheaf \( h(L) \) on \( M \) (follows from general facts, to be done.)

1. The \( \text{Lie} \) functor \( \Lambda \text{mod} \rightarrow \mathbb{L}(L) \)-mod in \( \mathbb{M}(\mathbb{L}) \) is fully faithful.

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1. The above property of \( h \) makes it an \( \text{augmentation functor} \) for the \( \mathbb{C} \)-structure: \( \text{write} \ h(M) = \mathbb{P}^*_L(M) \).

Now the \( \mathbb{P}^*_L(\{z\}, M) \) have composition \( J \circ I \rightarrow \mathbb{P}^*_L \circ \circ \circ \rightarrow \mathbb{P}^*_L \). Now can extend this to \( J \circ I \) not surjective: For example fibers, replace with \( \mathbb{P}^* \) - exact operations.

1. Right exact. Then \( I \circ J \) (exact - cannot really write \( \text{Hom}(M, N) \) as \( \text{Hom}(\mathbb{C}, N) \).

In the case \( J \circ I \rightarrow \mathbb{L} \)-mod:
\[
\mathbb{P}^*_L(\{z\}, L) \circ \circ \circ \rightarrow \mathbb{P}^*_L(\{z\}, L)
\]
\[
\text{i.e.} \quad \mathbb{P}^*_L(\{z\}, L) \rightarrow \text{Hom}(\circ \circ \circ (M), \mathbb{P}^*_L(\{z\}, L)).
\]

Exercise-Proposition: If \( J \) is nonempty then this map is injective.

Example \( L \) is a \( \mathbb{L} \)-algebra, \( x \in X \), what are \( \mathbb{L} \)-local \( x \)-loans supported at \( x \)?
\[
\begin{align*}
\text{M}(X) & \xrightarrow{\Delta} \text{Vec} \quad \text{So we have vector spaces w/ some extra structure.} \\
\text{h}(L) & \text{ is a \( \mathbb{L} \)-algebra, w/ some topological vector space.} \\
\text{The \( \text{K} \)-theory of \( h(L) \) is always represented for any \( \mathbb{L} \)-module \( L \). Consider all} \ \mathbb{L} \text{-submodules} \quad L' \subset L \quad \text{s.t.} \quad h(L') \text{ is supported at} \ x \ & \quad \Rightarrow \ h(L') \xrightarrow{\Delta} h(L) \quad \rightarrow \text{h}(L \circ L') \xrightarrow{\text{exact}} \ \\
\text{The open sets are then the images of these maps} \quad \text{h}(L') \rightarrow \text{h}(L) \).
\end{align*}
\]
Lemma: If the $L_i$'s are regular $D$-modules then any $*$ operation on $L \in P^*$ induces a continuous operation

$$h: L(\mathfrak{g}) \times: \mathfrak{g} \rightarrow L(\mathfrak{g})$$

take an open in $L(\mathfrak{g})$ killed by sufficiently high powers of $\mathfrak{g}$, hence so will $L(\mathfrak{g})$ in inverse image.

So $h(\mathfrak{g})$ is a topological Lie algebra, and a module over $\mathfrak{g}$.

Consider the action $L \otimes M \rightarrow L \otimes M$, given by $L \otimes h(\mathfrak{g}) \rightarrow M$. Making first variable $h(\mathfrak{g})$, then $L \otimes h(\mathfrak{g}) \rightarrow L \otimes M$ makes first variable continuous.

We can also go backwards: continuous map $h(\mathfrak{g}) \otimes h(\mathfrak{g}) \rightarrow L(\mathfrak{g})$ factors through $h(\mathfrak{g}) \rightarrow h(\mathfrak{g})/h'(\mathfrak{g}) \rightarrow \mathfrak{g}$, where $L(\mathfrak{g})/h'(\mathfrak{g})$ is a Lie algebra over $\mathfrak{g}$ by Sheaves-ness. Set $L \otimes h(\mathfrak{g}) \rightarrow M$, $L \otimes M \rightarrow L \otimes M$ just by lifting back up to $\mathfrak{g}$.

Example: $K-M \otimes \mathfrak{g} = \mathfrak{g} \otimes \mathfrak{g}$, only $h$ set back $\mathfrak{g} \otimes \mathfrak{g}$ - no longer just an infinitesimal vector space. $K-M$ is a $\mathfrak{g}$-module, and $\mathfrak{g}$-modules are vector spaces on which this completed $K-M$ acts - i.e., every element is killed by a form $\cdot \mathfrak{g}$, i.e. category of $\mathfrak{g}$ modules.

Now combine $D$-algebras & Lie* structure

We need to study symmetries in geometry of $\mathfrak{g}$-alg, but sheafing doesn't work, need *-theory instead of group schemes.

Compatibility between $\otimes$ and $P^*$: Fix attention on $M^*(\mathfrak{g})$, which now has tensor structure $\otimes$, $\otimes$ structure:

$$M \otimes M_2 = (M \otimes \mathfrak{g}^{-1}) \otimes_\mathfrak{g} (M_2 \otimes \mathfrak{g}^{-1}) \otimes \mathfrak{g} \mathfrak{g} = (M \otimes \mathfrak{g}^{-1}) \otimes M_2$$

Since we discussed $\otimes$ on left, and now we must shift to right hand.

Compatible $h(\mathfrak{g})$: $h(\mathfrak{g}) \otimes \mathfrak{g} \otimes h(\mathfrak{g}) \otimes \mathfrak{g}$ have canonical map

$$P^* (\mathfrak{g} \otimes \mathfrak{g}) \otimes P^* (\mathfrak{g} \otimes \mathfrak{g}) \rightarrow P^* (\mathfrak{g} \otimes \mathfrak{g}) \otimes P^* (\mathfrak{g} \otimes \mathfrak{g})$$

$$L \otimes \mathfrak{g} \otimes \mathfrak{g}$$

$(L \otimes \mathfrak{g} \otimes \mathfrak{g}) \otimes (M \otimes \mathfrak{g}^{-1}) \rightarrow M \otimes (L \otimes \mathfrak{g}^{-1})$.
What is the map? 
\[ \mathcal{M} \xrightarrow{\varphi} \Delta^{(1)} L, \quad \mathcal{K} \xrightarrow{\psi} \Delta^{(2)} N, \]
\[ \mathcal{M} \otimes \mathcal{K} \xrightarrow{\varphi \otimes \psi} \Delta^{(1)} L \otimes \Delta^{(2)} N \]
Consider hyperplane 
\[ X^m \xrightarrow{\varphi} X^{m+1} \]
pullback our module to this hyperplane — still get something sitting on a diagonal of \( X^m \), which is just what's happening with the case of one module.

So above operations are a generalization of tensoring morphisms...

Note any formulation: Note that \( \otimes \) is not self-dual. In the category, its arrows going wrong ways. Let's make it self-dual by brute force — i.e., cheat with \( \otimes \) on it as the dual. This compound structure with some compatibility. E.g., \( \otimes \) defines \( \otimes \) on dual (\( \otimes \) self-dual) in a compound structure.

Simpler example: \( \otimes \) is \( \otimes \); we can construct compound \( \otimes \) by \( \otimes \) and \( \otimes \).

\[
\begin{align*}
M^{(1)}(k) & \quad I \xrightarrow{1_i} s_j \rightarrow I \xrightarrow{1_s} J \\
\otimes & \quad p^* & \quad \otimes p^* & \quad \leftarrow p^* \\
\text{compatibility} & \quad \text{with } I, J, i, j,
\end{align*}
\]

Assocativity: \( I, J, K \) with \( I, J, K \) to \( I, J, K \)

Two transformations \( (I, J, K) \) to \( I, J, K \)

Different order pullbacks commute...

\( \otimes \) compound tensor category (classical structure)

Matrix algebras — Associates with \( \otimes \), some from following model:

\( V, V' \) vector spaces, \( \langle > \) \( : V \times V \rightarrow \mathbb{C} \)

\( V \otimes V' \) is then an associative algebra:

\( V \otimes V', V' \otimes V, V \otimes V' \) (coordinate-free matrix multiplication)

It acts on \( V \) from left, or \( V' \) from right.

Now assume \( V, V' \in M^{(1)}(k) \), \( \langle > \in \mathfrak{P}^{(1)}(V, V'), \mathfrak{D} \)

Claim: \( V \otimes V' \) is an \( \otimes \) associative algebra that acts on \( V \) from left, \( V' \) from right.

Action on \( V' \): \( \text{id}_V \otimes (\varphi) \quad \langle > \quad \left[ V, V' \right] \rightarrow \mathbb{C} \\
V \xrightarrow{\text{id}_V} V \xrightarrow{\otimes \text{id}_V} V \xrightarrow{\otimes \text{id}_V} V' \xrightarrow{\otimes \text{id}_V} V \)

\( V' \) takes \( \langle > \otimes \text{id}_V \).
Product: \( \text{id}_V \otimes \text{id}_V \quad \xrightarrow{v \otimes v} \quad V \otimes V \)

Endomorphism algebras: \( \text{End}_A(M) \) in a \( \times \) category

\[ M, L \rightarrow \text{Hom}^*(M, L) \quad \text{Assume we have a pairing } \langle \cdot, \cdot \rangle : \text{Hom}(M, L) \times \text{Hom}(L, M) \rightarrow \mathbb{C} \]

Then for any \( I, K \), \( j \cdot \) maps

\[ p^*_j (L, K, X) \rightarrow p^*_j (L, K, M, L) \]

\[ \text{Def } (\mathcal{K}(x)) \text{ is } \text{Hom}^*(M, L) \text{ if for any } I, K \]

\[ p^*_j (L, K, X) \rightarrow p^*_j (L, K, M, L) \]

This is inner hom in our tensor category.

Say \( I \) is a single object:

\[ \text{Hom}(I, \text{Hom}(M, L)) = P^*_x (L, K, M, L) \]

A useful \( \otimes \)-cat: \( \text{Hom}^*(M, L) \) is an object \( X \) equipped with an \( X \otimes M \rightarrow L \) s.t. \( \text{Hom}(K, x) \rightarrow \text{Hom}(K \otimes M, L) \) is iso.

Certainly unique if exists.

- Trying to represent functor \( F \) as \( \text{Hom}(\ldots, X) \): doesn't try

These is like saying a canonical element \( \text{def} = \langle x \rangle \)

corresponds to \( \text{id}_X : \text{Hom}(X, M) \rightarrow \text{Hom}(X, X) \)

\[ \rightarrow \text{id}_X \]

For \( M \) coherent this exists.

Exist: Define inner \( P^*_I \).

\[ p^*_j (L, K, X, M, L) \]

Then \( \text{End}^*(M) \) is an \( \text{Ass}^* \)-algebra acting on \( M \) from right only.

Lemma: In \( \text{End}^*(M) \times \text{Hom}(M, L) \) exists \( T \) \( M \) is coherent.

- Built it locally:

\[ \text{Hom}(M, L) = \text{Sheaf } \text{Hom} (D_x, \text{Hom} (M, L)) \]

\[ = \text{Hom} (M, \text{Hom} (L, D_x)) \]

\[ = \text{Hom} (M, \text{Hom} (L \otimes M, D_x)) \]

Say \( \text{End} \) at sheaf, a diagrams.

If \( M \) is not coherent this last term will be infinite sum.

\[ \text{Hom}(\text{coch}, 2 \cdot C) \text{ not } \geq C \] then \( \text{Hom}(\text{coch}, 2 \cdot C) \text{ is coh.} \)

Example: \( M^0 := \text{Hom}^*(M, \text{coch}) \) usual duality for \( \text{D-modules} \).

\[ D_x^O = C_x \otimes \text{coch } \text{inner } \ast \text{-category} \]

Example: \( V, V', \langle \cdot, \cdot \rangle \rightarrow \text{Hom}(V, V') \rightarrow \text{End}^*V \)

Lemma (exercise): If \( V \) is locally free Fred \( \text{Witt rank} \), \( V = V' \), symmetric

Since \( V \otimes V' \rightarrow \text{End}^*V \).
Now apply \( h \). First assume \( V = \mathbb{F} \) is induced, Frobenius Cartan.

\( \text{Claim } \) \( (\text{End } V) = \text{Diff}(\mathbb{F}, \mathbb{F}) \to \text{End } \mathbb{F} \).

\( L \) a Lie\(^*\) algebra, \( M \otimes L \) \( \longrightarrow \) \( \text{End } \mathbb{F} \) \( \otimes \text{End } L \) \( \longrightarrow \) \( \text{End } \mathbb{F} \otimes L \).

Claim: \( L \)-mod is a tensor category, \( h(L) \) is a tensor functor which is a Hilbert embedding as tensor subcategory.

Proof: Should define \( L \)-action on \( M \otimes N \) when \( M, N \) are \( L \)-modules.

* \( \bullet \) \( M \otimes N \) \( \otimes L \to \text{End } \mathbb{F} \otimes L \to \text{End } \mathbb{F} \).

or explicitly \( [L \otimes M, N] \to \text{End } \mathbb{F} \otimes L \).

\[ A \otimes B \to A \text{ acts } L \text{-action on } B \text{ acts } \]

using tensor of \( L \)-modules.

**Duality**

\[ \text{On } M^d_\mathbb{F}(X) \to M_\mathbb{F}(X) \] (covariant)

\[ M \to M^d = \text{Hom}_\mathbb{F}(M, \omega_X) = \text{Hom}_\mathbb{F}(M, \omega_X \otimes \text{Dx}) \]

(\( \omega_X \otimes \text{Dx} \) has two right dual structure, from \( \text{Dx} \) as right)

from \( \text{Dx} \) as left \( \omega_X \) with \( \omega_X \).-Consider both sides of \( \omega_X \) as \( \text{Dx} \) acts \( M^d_\mathbb{F}(X) \) via \( \omega_X \)

and \( M^d_\mathbb{F}(X) \).

Claim: \( \omega \) lifts canonically to a \( \text{Dx} \times \text{Dx} \) functor

\[ M^d_\mathbb{F}(X) \to M^d \omega_X \]

(Note: tensor of two \( \text{Dx} \)\( \text{Dx} \) is not \( \omega_X \) \( \omega_X \) in general)

- as follow in \( M^d \omega_X \) \( P^\times_\mathbb{F}(E) \cdot M : = \text{Hom}_\mathbb{F}(\omega_X)(M, \omega_X) \)

\[ P^\times_\mathbb{F}(E) \cdot M \]

\[ \text{maps} \]
Recall we have an operation \( \varphi_i \in P^*_k(\mathbb{Z}^i, \mathbb{L}^i, \mathbb{W}_k) \)

\[ \Rightarrow \varphi_i \in P^*_k(\{\mathbb{L}^i, \mathbb{L}^i \}, \mathbb{W}_k) \]

-coproduct tensor at the \( \varepsilon_i \).

If \( \mathfrak{g} = \text{Hom}_k(M, \mathbb{L}^i) \) can compose:

\[ (\otimes \varepsilon_i) (P^*_k(M, \mathbb{L}^i)) \cong P^*_k(\mathbb{L}^i, \mathbb{L}^i, \mathbb{W}_k) \]

\[ \Rightarrow \otimes \varepsilon_i (\mathbb{L}^i, \mathbb{L}^i, \mathbb{W}_k) \]

- now how - this is our \( g \)-module.

We have when of \( D_k \)-schemes, group \( D_k \)-schemes \( g \),
action \( g \otimes \varepsilon_i \rightarrow \varphi_i \) of \( g \) on \( \mathbb{L}^i \). What's inner leading action?

\( \text{Collie}(\mathbb{L}^i) \) is well defined - cotangent fiber of \( \mathbb{L}^i \) at \( \varepsilon_i \), product on \( \mathbb{L}^i \) yields cobracket \( \text{Collie}(\mathbb{L}^i) \rightarrow \text{Collie}(\mathbb{L}^i) \otimes \text{Collie}(\mathbb{L}^i) \)

Assume that \( \text{Collie}(\mathbb{L}^i) \) is constant (e.g. \( \mathbb{L}^i \) locally of the form \( \mathbb{L}^i, \mathbb{W}_k \)).

Then \( \text{Collie}(\mathbb{L}^i) \) is a constant \( D_k \)-module and our duality
(Which is not the abstract inner duality of a \( D_k \)-category -oms falls ! \( \Rightarrow \mathcal{T} \)).

Set \( \text{Lie}(\mathbb{L}^i) = (\text{Collie}(\mathbb{L}^i))^\circ - \)

Which is a Lie algebra!

This \( \text{Lie}(\mathbb{L}^i) \) will act on the sheaf of functions at the same \( \mathbb{L}^i \) acts on... 

\( \mathbf{G} \) acts on \( \mathbb{L}^i \Rightarrow \mathbf{G} \) acts on \( \mathbb{L}^i \)

\( \mathbf{G} \) acts on \( \mathbb{L}^i \)

\( \text{Collie}(\mathbb{L}^i) = \mathcal{L}(\text{Collie}(\mathbb{L}^i) = \text{Lie}(\mathbb{L}^i) \).

- more generally \( \text{Sym} V = \text{Sym} (D_k, \mathbb{L}^i) \) by universal V:

\[
\text{Hom} (\text{Sym} V, A) = \text{Hom} (\text{Sym} (D_k, \mathbb{L}^i), A) = \text{Hom} (\mathbb{L}^i, A)
\]

\[
\text{Hom} (\mathbb{L}^i, A) = \text{Hom} (\text{Collie}(\mathbb{L}^i), A).
\]

\( \text{Lie}(\mathbb{L}^i) = \text{Lie}(\mathbb{L}^i, D_k) \ldots \) when we have central charge can't write our lie algebra as dual to anything... only in some derived cat...

These are the real examples.

\[ \text{Remark: A comm. \( D_k \)-algebra, can consider \( A \)-mod, a tensor category \( \otimes A \):
\]

\( Q_A M; = Q_A M; / I_{Q_A M}; \) being the ideal at \( A \rightarrow R \rightarrow A \)

Claim: \( A \)-mod is a derived functor at:

\[ P^*_A (\mathbb{L}^i, M) = P^*_k(\mathbb{L}^i, \mathbb{W}_k, \mathbb{L}^i, \mathbb{W}_k, \mathbb{L}^i M)
\]

\( \Rightarrow \text{exercise} \)

\( \varphi (\mathbb{L}^i, \mathbb{W}_k, \mathbb{L}^i M) = P^*_k(\mathbb{L}^i, \mathbb{W}_k, \mathbb{L}^i M) \)
Lie algebroids X obj. unity - algebroid is a sheaf L with:

a. L is a (twc) O_X module 

b. L is a sheaf of Lie algebras

c. \( \sigma: L \to \text{Der}_X \text{O}_X \) (Lie alg homomorphism) \( \text{O}_X \)-modules

- \(\text{Lie}_L = \{ \text{smooth } f, g \text{ s.t. } [f, g] = \sigma(f)(g) - \sigma(g)(f) \} \)

- L-module is an \( \text{O}_X \)-module M with an \( L \)-action (as Lie algebra)

\( \text{Lie}_L \text{ men } f, g, h \text{ s.t. } \{ f, g \text{ on } h \} = \sigma(f)(g) - \sigma(g)(f) \text{ on } h \text{ i.e. } \sigma(f)(g) - \sigma(g)(f) \text{ commutes with } f, g \text{ on } h \) .

Examples of sheaf of vector fields (with some finiteness conditions on X) - \( O_X \).

- \( O_X \)-module is precisely a \( O \)-module (left).

1. \( F \) a bundle - pairs \( (x, \xi) \in O_X \times \text{Lifting of F} \)

\( \Rightarrow \text{Lie algebra. Can put extra conditions on } \xi \text{ if } F \text{ is a } G \)-bundle consider lifts \( \xi \) commuting with \( G \)-action \( \Rightarrow E_X \).

- A Lie algebroid is free, if \( \sigma: L \to \text{Der}_X O_X \).

So \( E_X \text{ is free. } O \to O_E \to E_X \to O_X \to 0 \)

In general \( \text{ker } \sigma \) is a Lie \( O_X \)-algebra.

- Connection - an \( O_X \)-linear section of \( \sigma \). It's integrable iff this section is a morphism of Lie algebras.

2. Let \( P \) be a Lie alg. acting on \( X \Rightarrow O_X \otimes P \) is a Lie algebra on \( X \), \( \sigma \) get \( \gamma \) by \( O_X \)-linearly extensions action \( P \to \text{Der}_X O_X \)

Claim \( \exists! \) algebraic structure on \( O_X \otimes P \) with \( O_X \gamma \) acting of \( P \).

- \( E_X \text{ is free } \)

\( \text{Lie algebra structure on } O_X \otimes O_Y \text{ is in fact a sheaf of ideals!} \)

This is Lie algebra version of gro-poids:

- Groupoid - device which gives you a space of orbits - equiv relation

- Y morphism (not rel. subvariety) \( \Rightarrow \) which is symmetric has Y \times Y lifting over diagonal \( X \times X \) "cross" morphism on \( X \times X \).

- Playing with sets, this gives a category - objects points of \( X \), arrows elements of \( Y \) over \( X \times X \), same as category with all arrows isos.

- Morphisms here are arrows of our category.

- so this is a groupoid on \( X \).

- \( \Rightarrow \) gro-poid is just a gro.

\( G \) acts on \( X \Rightarrow \text{groupoid on } X \) projected \( X \times X \text{ action} \)

Groupoid action on \( X \) is local - may restrict to any open set \( U \).
How to get a graded : take unit section of one of the
prosections & take normal bundle... (image of deformed)

Consider relative tangent bundle to presch $P \rightarrow X$.
We have unit section $\mathbb{E} / X \rightarrow Y$. Our graded is $\mathbb{E} \otimes \mathbb{O}[X]$.

4) Poisson structure : $\mathbb{E} / X \rightarrow Y$ canonical Lie algebra on $\mathbb{E}_X$. (Existence defined by this structure.
- The unique graded structure st $\mathbb{O}[X] \rightarrow \mathbb{E}_X$ is an isomorphism of Lie algebras, $\mathbb{O}[X] \otimes \mathbb{E}_X = \mathbb{E}_X$.
- $\mathbb{E}_X$ as abstract Lie alg acts on $\mathbb{E}_X$, may take induced algebra
  $\mathbb{O}[X] \otimes \mathbb{O}[X] \rightarrow \mathbb{O}[X] f \mapsto df$.
- Lemma. Poisson structure $\mathbb{E} / X \rightarrow \mathbb{O}[X]$ is the Lie algebra $\mathbb{E}_X$.

Compound Lie $R$-algebras : $\mathbb{E}$ is a Lie $R$-alg, so there are $D \rightarrow \mathbb{E}$-mod $\mathbb{E}$ with the following structures:
- $\mathbb{E}$-module $\mathbb{E}$
- $\mathbb{E}$-algebra structure $\otimes$ action of $D$ on $\mathbb{E}$ (Lie alg action $\mathbb{E}$-alg action)
- Compatibility $\otimes$-action of $D$ on $\mathbb{E}$ is compatible with a
  unit $\mathbb{E}$-alg structure on $\mathbb{E}$, i.e. $\mathbb{E}$-alg is an isomorphism of $\mathbb{E}$-modules.

$\Rightarrow$ (category CCA) compound Lie $R$-algebras.
$\mathbb{E}$-modules are $D \rightarrow \mathbb{E}$-mod $\mathbb{E}$ with structure of $\mathbb{E}$-module, $\mathbb{E}$-mod
  (as Lie alg) and compatibility $\otimes$ $\mathbb{E}$ $\mathbb{E}$-linear and $\mathbb{E}$-$\mathbb{E}$-compatible with action.

Ex. 1. Tangent $\mathbb{E}$-algebra $\mathbb{E}_x := \mathbb{O}^* \mathbb{E}[X, \mathbb{E}] \subset \mathbb{Hom}(\mathbb{E}_X, \mathbb{E})$.

- polynomial ring in $\mathbb{E}$ (dg, str, $\mathbb{E}$) which are deformations unit variable $\mathbb{E}$. (with correct functor is retractable) $\Rightarrow$
  $\mathbb{Hom}_\mathbb{E}(\mathbb{E}_x, \mathbb{E})$

If $\mathbb{E}$ is smooth then $\mathbb{E}_x$ exists : $\mathbb{Hom}_\mathbb{E}(\mathbb{E}_x, \mathbb{E})$.

Universal of tangent $\mathbb{E}$-algebra : every Lie algebra $\mathbb{E}$ in $\mathbb{E}$-alg (A) has

$\mathbb{E}$ ! $\mathbb{E} \otimes \mathbb{E}$ via action 1 on $\mathbb{E}$.
Tangent algebra to jet algebras $\mathfrak{X} = JB$:

$$A[D^X] \otimes B[A] \to J^2 A$$

(EXERCISE)

$$\Theta_2 = \text{Hom}_B (R^2, A)$$

which is a $D^X - \text{mod}$, $A[D^X] \to \Omega A[1]$.

$$\Theta_2 = \text{Hom}_B (R^2, A) = \text{Hom}_B (R^2, A) = \text{Hom}_B (R^2, A[D^X])$$

But $x^+ \cdot A = \text{ker} A[D^X]$ so may rewrite $\Theta_2$ as

$$\text{Hom}_B (R^2, A[D^X])$$

$$\Rightarrow \Theta_2 = \text{Hom}_B (R^2, A[D^X]) = \text{Hom}_B (R^2, A[D^X])$$

Now $\mathfrak{X}$ should act on $A$:

$$\mathfrak{X} \cdot A = \Theta_2 \cdot A$$

acting in an obvious way on $A$. - inf. acts of $B$ act on $A = \text{acts of } B$.

Now assume $\dim X = 1$. A smooth

Def: $L \subset E(\mathfrak{X})$ is an elliptic algebra if

$L \to E(\mathfrak{X})$ is injective, and the cokernel is a projective $A$-mod of finite rank.

**Compound Poisson (= Caisson) Structure**

A $D^X$-algebra with $\mathfrak{X} \subset E(\mathfrak{X})$ such the adjoint action is an action of $A$ (cos. Lie alg.) on $A$ (cos. Lie alg.).

A smooth have unique extension $A \to \Theta_2\text{ann} \to \text{ann}$ with a canonically $\text{Lie}^\ast A$-algebras,

from which can recover Caisson on $A$.

Def: A caisson is elliptic if $\Theta_2\text{ann}$ is an elliptic algebra.

A caisson structure is sympletic if $\Theta_2 - \Theta_2\text{ann}$ is is 0.

Elliptic is the true generalization of sympletic to infinite dimension... sympletic not that interesting here.

**Ex.** Koszul - Clifford Poisson structure on sym alg:

If $L$ is any Lie alg then Sym $L$ is a caisson algebra - mainly the graded $L$ characterized by property $L \to \text{Sym} L$

is a morphism of Lie algebras.

2. Twisted version:

$$0 \to c_k \to \text{L} \to L \to 0$$

in $c_k - \text{alg}$

$\text{Sym} L$ by ideal gen by $c_k (a = 1)$ - commutative

$\text{Graded}$ algebra with $\text{ass}$ graded $\text{Sym L} \to \text{Sym}^\infty L$. 

Ex. Heisenberg algebra $(R[X], 2y) \delta = \delta(x \cdot \gamma)$. 

Consider $0 \to \omega \to \operatorname{Res} \to \mathcal{D}_{x} \otimes \mathfrak{A} \to 0$ 

Then $(\operatorname{Sym} \mathcal{D}_{x}) \otimes (\mathcal{D}_{x} \otimes \mathfrak{A})$ is elliptic. 

Thus is $\operatorname{Sym} (\mathcal{D}_{x} \otimes \mathfrak{A})$ with twisted center.

$0 \to \omega \\
\mathcal{D}_{x} \otimes \mathcal{D}_{x} \mathfrak{A}_2 \to \mathcal{D}_{x} \mathfrak{A}_2 \to \mathfrak{A}_2$ 

via $x \mapsto \delta x, \quad \delta x \mapsto \delta x$ 

with Cohen cohomology of $\mathcal{D}_{x} \mathfrak{A}$ (cohom order sums)

On a curve $X$ we have the notion of elliptic coisson algebras $A$ 

- Take a smooth $A$-alg with $\mathbb{E} \to A$ 
- elliptic: or injective, cocontract. $\mathcal{E}$ is a proj. $A$-mod of finite rank.

Recall $\mathfrak{A}$ is a projective $\mathbb{E} \otimes \mathfrak{A}$ mod ($\mathfrak{A}$ smooth) — bug compared to $A$.

**Note/Exercise** — we sitter in any $\mathfrak{A}$ alg is always central: 

$\mathfrak{A} \otimes \mathfrak{A} L \mapsto \mathfrak{A} L$ 

Fix a point in $X$ 

- contract variable $x$ 
- $\mathfrak{A} L \otimes \mathfrak{A} L$ 

but there are no maps $\mathfrak{A} L \to \mathfrak{A} L$ other than zero.

Furthermore $\mathfrak{A} L$ will act trivially on any $\mathfrak{A}$-mod (via $\mathfrak{A} L \otimes \mathfrak{A} L$)

- can't feel "level"/central charge classically — only $\mathfrak{A}$ and $\mathfrak{A}$.

$\operatorname{Sym} \mathcal{E}$ — functions on hyperplane ind to $\mathfrak{A}$ ....

No Koszul—Krull (contracted) coissors are elliptic—degree into $0$ ...

Untwisted — $\mathfrak{A}$ free $\mathfrak{A}$-mod of finite rank. Now in general

$\operatorname{Sym}_L \otimes \mathfrak{A} L$ via $\mathfrak{A} L \otimes \mathfrak{A} L$ 

$\operatorname{Sym}_L \otimes \mathfrak{A} L \otimes \mathfrak{A} L$ canonically — $x \otimes \mathfrak{A} L$ gives automorphism

$\operatorname{Sym}_L \otimes \mathfrak{A} L$, use insertion def of $\operatorname{Sym}_L$ — $\mathfrak{A} \otimes \mathfrak{A} L$ 

$x \otimes \mathfrak{A} L \mapsto \mathfrak{A} x 

\operatorname{Hom} \mathfrak{A} \otimes \mathfrak{A} \otimes (\mathfrak{A} L) \to \mathfrak{A} \otimes (\mathfrak{A} L)$ 

with the above freeness condition

$\mathfrak{A} \otimes \mathfrak{A} L \to \mathfrak{A} \otimes \mathfrak{A} \otimes \mathfrak{A} L \to \mathfrak{A} \otimes \mathfrak{A} \otimes \mathfrak{A} L$ 

Image of the monster with quadratic, in particular $\mathfrak{A} L \otimes \mathfrak{A} L \otimes \mathfrak{A}$ is injective: $\mathfrak{A} L$ project degenerates at $0$ to not symmetric.
Assume this is elliptic coker loc. From $A$-mod, take if's quotient mod standard (aug) map ideal in $A$, get $\text{ker} A$. with $L^0 \rightarrow \text{ker} \sigma / \text{coker} \sigma$. But elliptic hypothesis $\Rightarrow \text{coker} / \text{ker} \sigma$ is loc. free $G^*$-mod of $G$. Want $\Rightarrow$ $\text{twisted case}$. twisted again $\text{Sym}^L$ for $H^*_\text{coker}$. visual symplectic bracket on hyperplane, nondegen. $\text{(dim)}$ via splitting of exact sequence.

Int: dim: say we have $\text{ker} A$ and splitting $0 \rightarrow \omega \xrightarrow{\psi} L \rightarrow 0$ get $K$-ic bracket with a correction term.

Let $L \rightarrow \text{Sym}^L \text{loc}^0$. Since $\text{Sym}^L \text{loc}^0 \cong \text{Sym}^L \text{loc}^0 _L L$.

In $L$, we have $\text{ker} \sigma$ and splitting $0 \rightarrow \omega \xrightarrow{\psi} L \rightarrow 0$ successive graded quotient $\text{S}^L \text{loc}^0 \rightarrow \text{Sym}^L \text{loc}^0 = \text{id} \text{loc}^0 \sigma$. so if $\text{ker} \sigma$ graded is $L$, original map is $\sigma$. Our coisson is symplectic/elliptic iff the diff op $c : L \rightarrow L^0$ is iso or elliptic. Here an elliptic differ (map between $A$-mod) is injective with coim $L$, free for gen. $G^*$-mod.

Example: $G(\mathcal{O}) = K \rightarrow \text{Sym}^L \mathcal{F} \xrightarrow{\omega} \mathcal{F} \xrightarrow{\omega} L \rightarrow 0$.

with cocycle $(\mathcal{O} \otimes \mathcal{O}) \nu((\mathcal{O} \otimes \mathcal{O})) \rightarrow \mathcal{O} \otimes \mathcal{O} \otimes \mathcal{O} \otimes \mathcal{O}$.

In local coordinates.

$c : L \rightarrow L^0$. $\omega \otimes D_x \rightarrow \omega \otimes \omega x D_x$

in local coordinate.

Now $\omega \otimes \omega x D_x$ in local coordinate.

Case $\sigma$. $\mathcal{O} \rightarrow \mathcal{O} \otimes \mathcal{O}$, $L \rightarrow \mathcal{O} \otimes \mathcal{O}$ correspond to differ.

d $\mathcal{O} \rightarrow \mathcal{O} \otimes \mathcal{O}$ (these are integral).

The coker $\sigma$ is $\mathcal{O} \otimes$. If $\sigma$ is non-degen, then coker $\sigma = \mathcal{O} \otimes \mathcal{O} \rightarrow \text{elliptic coisson}$. \[\square\]
Local Poisson algebras

\[ \text{dim } x = 1, \ A \ any \ \mathbb{D} \text{-algebra}, \ x_1 \in X, \ A_x = A \otimes \mathbb{C}x_1. \]

\[ J_x : A_x \to X, \ A_x = h(\text{j}_x, x \mathbb{C}x_1, \ A/\mathbb{C}) \] - vert. span of sections at point \( x \).

\[ \text{Spec } A_x = \text{Spec } \mathbb{C} \text{ of horizontal sections of } \text{Spec } \mathbb{C} \text{ over } \text{Spec } \mathbb{C} \text{ formal disk}: \mathbb{C}_x < \mathbb{C}_x \xrightarrow{\text{closed sub}} \mathbb{C}_x. \]

Any point of fiber may be extended via the connection to a horizontal section/ a formal disk.

\[ A(x) = \pi^* \mathbb{A}^x, \ A \text{ varying over all } \mathbb{D} \text{-algebras } A \subset \mathbb{A}. \]

\[ \text{Note } A' / A \to A' \to A_x \]

- (ber at \( x \) is larger).

Consider \( A' \to A \to A / A' \to 0 \) - \( A \text{ on support point and infinitely divisible tensor with } \mathbb{A} / A' \).

\[ \Rightarrow A_x \to A \to A / A' \otimes \mathbb{C}(1) = 0. \]

(\( A' / A \otimes \mathbb{A} / A' \) sum at point \( \Rightarrow \) sum of \( \mathbb{A} / A' \) functions \( \Rightarrow \) \( \mathbb{A} / A' \)).

**Claim:** \( \text{Spec } A(x) = \text{Spec } \text{ over } \mathbb{C}_x \text{ of } \text{Spec } A \text{ over } \mathbb{C}_x \)

- First we have canonical projection \( A(x) \to A_x \), hence \( \text{Spec } A(x) \to \text{Spec } A_x \) restriction of horizontal sections.

- What is a section? horizontal \( \mathbb{A} \)-morphism \( A' \to \mathbb{C}_x \).

Consider \( \mathbb{C}_x < \mathbb{C}_x \) and the prerequisite \( \mathbb{A} \to \mathbb{C}_x \).

- \( A' \) coincides with \( A \) outside \( x \). Now get \( A_x' \to 0 \)

**Determining \( A' \to \mathbb{C}x \) fiber at point.

**Spec** \( A(x) \) := \( \mathbb{D} \text{-algebra} \) (spectrum of projective limit)

Conversely localize \( A' \to \mathbb{D} \to \mathbb{D} \text{-algebra} \) to get \( A \to A_x \) (more完善 localization).

**Proposition:** Assume that \( A \) is a \( \mathbb{D} \)-algebra. Then any Poisson bracket on \( A \) yields a Poisson bracket on \( A(x) \).

**Proof:** Construction: \( \mathbb{A} \to \mathbb{A}_x \to h(\text{j}_x, x \mathbb{A}) \) canonically.

\( \Rightarrow \) morphism \( A_x \to h(\text{j}_x, x \mathbb{A}) \) - dense image (surjective on any quotient ) \( \Rightarrow \mathbb{D} \)-algebra.

\( h(\text{j}_x, x \mathbb{A}) \) has Lie bracket - \( A_x \) is completion of this so bracket extends if it's continuous.

(Continuity: for given \( A' \) and \( f \in h(\text{j}_x, x \mathbb{A}) \), \( f \) is closed for given \( A' \) and \( f \in h(\text{j}_x, x \mathbb{A}) \), \( f \) is.

Check this.

This bracket is Poisson: \( h(\text{j}_x, x \mathbb{A}) \) acts on \( \text{i} \mathbb{A} \)

by derivations. Now \( \text{i} \mathbb{A} \) \( f \in h(\text{j}_x, x \mathbb{A}) \) is adj. \( A^\text{op} \to A_x \).

must be derivation, so our action is direct limit of derivation.
Consider $\mathcal{A}(x) \rightarrow A_x$. Claim $I$ is involutive (closed under the Poisson bracket $\{\cdot, \cdot\}$). Let $h(\mathcal{A})_x := h(\mathcal{A})$ for $\mathcal{A} \otimes \mathfrak{g}$ generates the ideal $I$. For any $\mathcal{A} \rightarrow \mathcal{A} \otimes \mathfrak{g}$, we have $h(\mathcal{A})_x \rightarrow I_x \rightarrow 0$, which comes from $h$ applied to $\mathcal{A} \rightarrow \mathfrak{g}$.

$I$ is a kernel of each $h(A)$. Let $h(A)$ be a Lie algebra of $h(\mathcal{A})$ (right action).

$I$ completion of $h(\mathcal{A})_x$ is involutive.

Now $I/I^2$ is an $A_x$ module. $I/I^2$ is Lie, acts on $A_x$ by Poisson brackets. Let $\pi$ map $I/I^2 \rightarrow \mathfrak{g}$.

$I$ acts on $A_x \otimes \mathfrak{g}$; $I/I^2$ is a Lie $A_x$-algebra (Hamiltonian reduction).

General nonsense - $N$ an $A$-module, may form $h(N)_x := \lim_{N \rightarrow N}$ $h(N/N_x)$, a Lie algebra coinciding with $N$ at $x$.

This is an $A_x$ module: $N/N_x$ is an $A$-module at point $x$.

Lemma Assume $N$ is a coadjoint Lie algebra, $N \subseteq \mathfrak{g}(A)$, and $N$ is $G$-invariant. Then $h(N)_x$ is a Lie $A_x$-algebra.

Action on $A_x$ comes from continuous action of $h(N)_x$ of which this is a completion. \[ \Rightarrow \]

Proposition $I/I^2 = h(A/A_x)$

Prove this. Hint: Compute relative $\mathfrak{g}/A$.

Answer: $\mathfrak{g}/A \cong A/A_x$, $a \rightarrow a + b$ - using infinite duality of $A/A_x$.

The Global Space of Sections

$x$ - compact curve, $A$ a $\mathbb{R}_x$-algebra. $\mathfrak{g}(\mathcal{X}, \mathcal{A}) = \text{Sec}_B$

space of horizontal sections, $B = H^0(\mathcal{X}, \mathcal{A})$ maximal constant 2-form $\omega$ for $x \in \mathcal{X}$, $B_x = A_x/\mathfrak{g}_x. (\mathcal{H}^1(\mathcal{X}_x, A))$

via $\mathfrak{g}_x$, $\mathfrak{g}_x \rightarrow \mathfrak{g}_x. (\mathcal{H}^1(\mathcal{X}_x, A))$

$\mathfrak{g}_x$, write $B = \sigma \mathfrak{g}_x. (\mathcal{H}^1(\mathcal{X}_x, A)). B$ is gen by the image of $\mathfrak{g}_{x}$. $\mathfrak{g}_{x}$ is $\mathcal{H}^1(\mathcal{X}_x, A) \otimes \mathfrak{g}_{x}$.

Consider $\rho : H^1(\mathcal{X}_x, A) \rightarrow \otimes \mathcal{A}_x \subset \otimes \mathcal{A}_x$. A_x
\[
\Gamma^*(\mathcal{X}, \mathcal{A}) = \text{Spec} \bigotimes_j \mathcal{A}_x / \mathcal{M}_x (\mathcal{H}^*(\mathcal{X}, \mathcal{A})) \otimes \mathcal{A}_x
\]

\[
(A \to \mathcal{X} \to \mathcal{Y}, \text{ and pass to } \mathcal{H} \ldots)
\]

Suppose \( \mathcal{A} \) is a coisson, \( \otimes \mathcal{A}_x \) will then be a topological Poisson algebra. Then \( \mathcal{M}_x \) will also be a morphism of Lie algebras - may be considered a Poisson action (Hamiltonian) of \( \mathcal{H}^*(\mathcal{X}, \mathcal{A}) \), a Lie algebra, on the Poisson algebra \( \otimes \mathcal{A}_x \).

Rewrite it as \( \mathcal{H}^*(\mathcal{X}, \mathcal{A}) \to \otimes \mathcal{A}_x \) as a morphism of \( \mathcal{H}^*(\mathcal{X}) \to \mathcal{H}^*(\mathcal{X}) \), and then the space of horizontal sections of \( \mathcal{A}_x \) over \( \mathcal{X} \times \mathcal{Y} \) is the zero fiber of the moment map for this action.

\( \mathcal{H}^*(\mathcal{X}, \mathcal{A}) \to \otimes \mathcal{A}_x \) must expand to \( \text{Sym} \mathcal{H}^* \to \text{Poisson} \).

Hamiltonian reduction - the invariants of this zero fiber -

Example \( \mathcal{X}, \mathcal{Y} \) semi-simple \( \Rightarrow K.M \to g \otimes \mathfrak{d}^\omega \),

\( g = \text{Sym}(\mathfrak{d} \otimes \mathfrak{d}) \), Spec \( \mathfrak{d}^\omega = \text{the space of connections on the trivial } \mathfrak{g} \text{-bundle on the formal punctured disc at } x_0 \),

Spec \( \text{Sym} \otimes g \otimes \mathfrak{d}^\omega \) - dual to \( g = K_x \otimes \mathfrak{d}^\omega \)

\( \mathfrak{d}^\omega - \text{diff on the disc} \)

Horizontal sections will give space of connection on \( \mathcal{X} \times \mathcal{Y} \).

Hamiltonian reduction -

Isomorphism classes of (global) connections on curves over \( \mathcal{X} \times \mathcal{Y} \) to bundles.

Coisson reduction - standard situation \( \mathcal{X} \to \mathcal{A} \)

\( \text{Sym} \mathcal{X} \to \mathcal{A} \), take point of \( \mathcal{O} \) on spectra, reduce out \( \mathcal{O} \).

In other terms - \( \mathcal{X} \to \mathcal{A} \) (image of \( \mathcal{O} \)) - involutive ideal.

\( A / \mathcal{I} \) doesn't inherit bracket - take subspace invariant under \( \mathcal{I} / \mathcal{I}^2 \) - this acts on \( A / \mathcal{I} \) - it is a Lie \( A / \mathcal{I} \)-algebra.

Choosing \( \mathcal{I} = \mathcal{I}^2 \Rightarrow (A / \mathcal{I})^1 / \mathcal{I}^2 = (A / \mathcal{I})^2 = (A / \mathcal{I})^1 \).

In our situation - since \( \mathcal{I} \) is set by \( \mathcal{O} \), in general only need the involutive ideal \( \mathcal{I} \), not \( \mathcal{O} \)-action.

Coisson setting: \( A \) coisson \( A \to \mathcal{I} \) involutive ideal.
\[ 1/9^2 \text{ gets bracket, becomes } \mathcal{L}(\mathcal{A}/\mathcal{B}) \implies \text{ invariants } \]
\[ (\mathcal{A}/\mathcal{B})^{1/9^2} \ldots \text{ If } \mathcal{I} \text{ is gen. by a Lie subalg of } \mathcal{A}^{1/9^2} \]
\[ \text{sufficient to take } \mathcal{I}\text{-invariants} \]
\[ \implies \text{ new coisson algebra.} \]

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\[ x \text{ cannot come, } \mathcal{A} \text{ Q-alg. Spec } \mathcal{H}_0(\mathcal{A}/\mathcal{B}) \text{ horiz. sections of } \mathcal{A}/\mathcal{B}. \]

sufficient. \[ H_0(\mathcal{X}/\mathcal{S}, \mathcal{A}) \implies \mathcal{O}_x \mathcal{S}/\mathcal{S} \text{ -- ideal sm. by the image of } \mathcal{I}\text{-res : } H_0(\mathcal{X}/\mathcal{S}, \mathcal{A}) \to \mathcal{O}_x \mathcal{S} \subset \mathcal{O}_x \mathcal{S} \]

Now \[ H_0(\mathcal{X}/\mathcal{S}, \mathcal{A}) \text{ as large } \ldots \]

Lemma: \[ \text{Let } \mathcal{S} \subseteq \mathcal{A} \text{ be a D-subal that generates } \mathcal{R} \text{ as } \mathcal{D} \text{ alg}. \]

Then \[ \mathcal{I}\text{-res is generated by the image of } H_0(\mathcal{X}/\mathcal{S}, \mathcal{A}) \]

proof suffices to use char as maximal cont. quotient ...

Case (i): \[ \mathcal{A} = \text{Sym } \mathcal{L} \ldots \text{ then } H_0(\mathcal{X}/\mathcal{S}, \mathcal{A}) = \text{Sym} \text{ cont. quotient of } (\mathcal{L} = H_2(\mathcal{X}/\mathcal{L})) \]

(1) General: \[ \text{Sym } \mathcal{L} \to \mathcal{A} \text{ cocompns } \implies \text{ gen. of } \mathcal{R} \text{ from } H_0(\mathcal{X}/\mathcal{S}, \mathcal{A}) \]

Hence must come from \[ \mathcal{L} \]

Corollary: \[ \text{tor sec. at Spec } \mathcal{A} \text{ on } \mathcal{X}/\mathcal{S} = \text{Spec } H_0(\mathcal{X}/\mathcal{S}, \mathcal{A}) = \text{Spec } \mathcal{L}(\mathcal{A}/\mathcal{S}) \text{ over } (\mathcal{R}/\mathcal{A}) \text{ coincides with } \mathcal{A}\text{-bords.} \]

This \[ H_0(\mathcal{X}/\mathcal{S}, \mathcal{A}) = \mathcal{O}_x \mathcal{S}/\mathcal{S} \text{, generated by } \mathcal{I}\text{-res from } H_0(\mathcal{X}/\mathcal{S}, \mathcal{A}) \]

Then again it's sufficient to take ideal coming from \[ \mathcal{A}\text{-sec. (LC ) on } \mathcal{X}/\mathcal{S} \]

Coisson case:\[ H_0(\mathcal{X}/\mathcal{S}, \mathcal{A}) \to H_0(\mathcal{X}/\mathcal{S}, \mathcal{A}) \text{ a subspace of this cont. alg.} \]

May consider reduction with this subspace acting on symmetry 
\[ \implies \left( \mathcal{L}(\mathcal{A}/\mathcal{S}) \right) H_0(\mathcal{X}/\mathcal{S}, \mathcal{A}) \]

- same reduction just from \[ \mathcal{L} \]

Example of (c) \[ \mathcal{A} = \text{Sym } (\mathcal{O} \otimes D_\mathbb{C}^2), \text{ quad.} \text{ coisson algebra.} \]

This is canonically split as algebraic
\[ \text{Spec } \mathcal{L} = \text{Sym } (\mathcal{O} \otimes D_\mathbb{C}^2), \text{ Spec } \mathcal{A} = \mathcal{O} \otimes D_\mathbb{C}^2 \]

Using (c) \[ \otimes \text{ cont. } \text{ Spec } \mathcal{L} = \text{Spec } \mathcal{A} = \mathcal{O} \otimes D_\mathbb{C}^2 \]

connections on trivial G-bundle on Spec \[ \mathcal{O} \mathcal{A} \text{, similarly for } \mathcal{L} \text{ in } \mathcal{L}. \]

\[ h(\mathcal{A} \otimes D_\mathbb{C}) = \mathcal{O} \otimes D_\mathbb{C}, \text{ fiber at } x\text{-completed } g \cdot \mathcal{L} \text{ at } \mathcal{L} \text{.} \]

acts as algebra of gauge transformations on connections
\[ \text{adjoint action of } H(\mathcal{L}) \text{ on } ... \]

Thus for \[ \mathcal{A}(x) \text{.} \]
Global horizontal sections - global connections on trivial bundle
- maximal cotangent of \( \mathfrak{g} \) span by \( \omega \otimes \mathfrak{g} \).
\[ \text{Hom}(\omega \otimes \mathfrak{g}, \mathfrak{g}) = \text{Hom}(\text{max cotangent } \mathfrak{g}) \]
- \( \omega \otimes \mathfrak{g} \) forms - so \( \mathfrak{g} \) acts on \( \omega \otimes \mathfrak{g} \) forms...

Take Hamiltonian reduction - \( \mathfrak{g} \) acts on gauge transformations - as affine scheme quotient.
- At just point: Group of gauge transformations gives isomorphism classes of trivial \( \mathfrak{g} \)-bundle with connection - but this space has no functions but constants \( \Rightarrow \) point space...

DS reduction extracts from this space a nice affine slice...

**General norence on elliptical form**

\[ \ln x = 1, \quad \mathfrak{g} \rightarrow \mathfrak{g} \in C(\mathfrak{g}) \text{ elliptic } \quad \text{assume } \mathfrak{g} \text{ smooth, } \]
\[ \mathfrak{g} \rightarrow \mathfrak{g} \rightarrow \mathfrak{g} \rightarrow \mathfrak{g} \rightarrow 0 \text{ the pre-Lie algebra structure } \]
\[ \mathfrak{g} \text{ pre-Lie algebra} \]

Claim: \( \mathfrak{g} \) is a Lie algebra in the tensor cat of \( \mathfrak{g} \)-modules.

Ex. \( \mathfrak{g} \) comes (as \( R \mathfrak{g} \)) from an elliptic Conner algebra.

Lie-theory case \( \mathfrak{g} = \mathfrak{g} / \mathfrak{z}(\mathfrak{g}) \), so \( \mathfrak{g} \) comes from a Lie algebra - get sheaf of Lie algebras on this, locally free family, whose fiber at any connection is some Lie algebra.

- Flat endomorphism of your bundle (ends preserving connection) - twisted version of \( \mathfrak{g} \).

**Definition of this branch - Preliminary**

Bott-Drinfeld complex of a Lie algebra:\n
Usual situation: A comm alg, \( \mathfrak{g} \) a Lie \( \mathfrak{g} \)-alg.
- A constant, \( \mathfrak{g} \) just a Lie alg, this is a dg alg, \( \mathfrak{g} \) alg.
- A smooth, \( \mathfrak{g} \) vector fields \( \mathfrak{g} \otimes \mathfrak{g} \) => deRham complex of \( \mathfrak{g} \).

General \( \mathcal{C}(\mathfrak{g}, A) = \text{Hom}(\mathfrak{g}, A) \) - can replace \( \mathfrak{g} \)
by any \( \mathfrak{g} \)-mod \( M \). This is a complex with Chevalley differential for Lie alg. calculus.
A $D_x$-alg. smooth $\xrightarrow{\eta} \Theta x$ Lie algebraic over $A$

Claim consider the dual $\mathcal{L}$ (fully enc. reflexive for e.g. proj.):

$0 \xrightarrow{\eta} D_x \rightarrow \mathcal{L}_x \rightarrow \mathcal{P}_x$ - then Hochschild $\mathcal{P}_x$

Is a Lie coalgebra in the tensor category of $\mathcal{A}(\mathcal{M}_x)$-modules

E.g. if $\mathcal{M}_x = 0$, $D_x$ is a Lie alg. over $A$, then $D_x$ strong

$\Rightarrow$ coaug. $\Rightarrow$ Duality: $M^{ad}_A (\mathcal{M}_x) \rightarrow M^{ad}_A (\mathcal{P}_x)$ - inner hom to $\mathcal{P}_x$.

This is an equivalence of categories if we consider proj. $\mathcal{A}(\mathcal{M}_x)$-modules.

In particular, in the elliptic situation: $\text{Coker } \eta \xrightarrow{\mathcal{L}}$.

$\text{Coker } \eta$ is a Lie algebra in the tensor cat. of $\mathcal{A}(\mathcal{M}_x)$-modules.

Elliptic case - coker $\eta$ is e.g. $\mathcal{M}_x$-mod - so no morphisms tend to $\mathcal{L}$ so it is also injective

$0 \rightarrow A \rightarrow B \rightarrow V \rightarrow 0$


$0 \rightarrow V^* \rightarrow A^* \rightarrow B^* \rightarrow 0$ - $V$ dual $A^*$, with dual connection

$L^* = \text{Hom}(\cdot, \mathcal{A}_x)$

$\mathcal{L}^* \rightarrow \text{Hom}(\cdot, \mathcal{A}_x)$ - naive duality

Chevalley - De Rham complex for a Lie algebra - common gen. of Chevalley - De Rham.

In compound setting:

$LC^\bullet (C, D_x)$ - is local CD-R complex $C^\bullet (C, D_x)$.

Terms $P_{\text{coH}}(C, D_x, \text{M})$.

$[C^\bullet (C, D_x)$ is a Lie alg.] Differential is usual formula.

$C^\bullet (C, D_x) \in C^\bullet (C, D_x)$ - forget $A$-structure, $D_x$ plain Lie algebra - define this to be a $CD-R$ complex.

Similar to embedding Lie alg. complex into all vector fields.

$C^\bullet (C, A)$ is a dg. commutative algebra - but differential not $A$-linear.

$\Rightarrow$ like usual De Rham complex.

Local CD-R complex $C^\bullet (C, D_x)$ - replace polynomials $P^n$ by inner homs: terms $P_{\text{coH}}(C, D_x, \text{M})$

Exists when $D_x$ is a prof $C$-rank $\mathcal{A}(D_x)$-mod.
From duality, $\psi_{\infty} (\text{id} \otimes M) = (L^0, L^0) \otimes M$
$A = M: \text{ terms are } \Lambda^0 L^0 = \mathcal{H} (L, \mathcal{A}) \triangleright \Lambda_0 \otimes A$.

Claim: $C^*_R(\mathcal{A}, \mathcal{A})$ is a comm. $R$-algebra in the tensor cat. of $\mathcal{A} \otimes \mathcal{A}$-modules (diff. not $\mathcal{R}$-linear)

$x \mapsto \partial_x$ gives a canonical morphism of $dg$-algebras
$DR (\mathcal{A}) \rightarrow C^*_R (\mathcal{A}_x, \mathcal{A}_x)$. $DR$ - deRham complex.
$0 \rightarrow \mathcal{A}_x \rightarrow \Lambda^0 \mathcal{A}_x$ - just first term of this.
Quotient is $C^*_R (\mathcal{A}_x, \mathcal{A}_x) / \text{ideal (im of } \mathcal{A}_x) = \Lambda^0 \mathcal{A}_x$.
- so latter is $dg$-algebra.
- $\partial_x$ of $0$ coincides with $\mathcal{A}_x$.
Diff is zero $\Rightarrow$ degree $0$ - so $\mathcal{A}_x$ linear differential
$\Rightarrow \mathcal{A}_x \otimes \Lambda^0 \mathcal{A}_x$ or $\Lambda^0 \mathcal{A}_x \otimes \mathcal{A}_x$ - flat structure on $\mathcal{A}_x$
- $\partial_x$ bracket comes from degree zero differential.

**The Quantum Picture**

Quantize coisson algebras... purely in terms of the dual space itself - space of functions solving some differential equations...

Chiral operators $x$ is a curve $\mathcal{M}^0 (x)$
$P^x_{\text{ch}} (\mathcal{M}^0) : = \text{Hom} (\mathcal{M}^0 (x), \mathcal{M})$,
$\mathcal{M}^0 (x) \rightarrow \mathcal{M}^0$ complement of all diagrams $\rightarrow \text{space}$.

$\mathcal{M}^0 (x) = \mathcal{M}^0$ - all paths along diagonal.

Compositions $I \rightarrow I \rightarrow \mathcal{M}^0$,
$\mathcal{M}^0 (x) \rightarrow \mathcal{M}^0 (x) \rightarrow \mathcal{M}^0 (x)$.

$\mathcal{M}^0 (x, x)$ the set of $\mathcal{M}^0 (x)$.

Definition: A chiral algebra is a Lie algebra in $\mathcal{M}^0$ cut $\mathcal{M}^0 = (\mathcal{M}^0 (x))_{x}$.