Motivation: X smooth projective / Fq, F local system on X
Global epsilon factor (constant in $F_{	ext{arch}}$)

\[ E(x, F) = \det (-\text{Fr}_* + ; \text{RP}(x, F)) \]

- monomial of degree = Euler characteristic (up)

\[ E - \text{factorization} \quad E(x, F) = \prod_{x \in X} E_x(F) \quad \text{local fields} \]

- density only on base, mean x --- need to choose
- global to meromorphic form on curve (or additive character of field)

Geometric meaning? rewrite in terms of graded lines
\[ \det \text{RP}(x, F) \]
\[ E(x, F) = \text{Tr}_x (\text{Fr}_* +, \det \text{RP}(x, F)) \]
- supertrace in correct degree, don't need - sign can

Q: what is local factorization on level of superrings ($\mathbb{Z}$-graded)?

\[ E_x(\langle \rangle) \quad \mathbb{Z} - \text{graded superlie with} \]
- Frob action depending on 1-form V.

- want \[ \det \text{RP}(x, F) = \bigotimes_{x \in X} E_x(F) \]

- for almost all points local factor is trivialized
- connicilly... part of our hypotheses (like local geometry is trivial).

What is a superline? \[ \otimes \] connectivity constant
- corrected by signs --- eg want \[ \det (C_{\theta} \otimes C_{\theta}) \]

\[ \otimes \det C_{\theta} \otimes \det C'_{\theta} \]

Independent of ordering, so needed
- if want to tensor over all points on a curve
Today explain why such a sheafification is reasonable---
in model topological str. 

From now on \( X \) = compact real analytic manifold 
\( F \) = constructible complex 
real analytic --- used only to have canonical class of stratifications (subanalytic). 

\( \Gamma_0(X,F) \) complex with \( \dim \) cohomology = 
get graded super lie det \( \Gamma_0(X,F) \) 
(degree = Euler class). would like det \( \Gamma_0 = \bigotimes E_{x_i} \).

How to compute this.

Basic example of computation.
\( X = \mathbb{S}^2 \), \( U = d \theta \) differential form
constructible sheaf \( F \) on \( \mathbb{S}^2 \); order \( 1 \) at 
points where \( F \) jumps in orientation given by \( U \).
\[ \{ x_i^+ \} \text{ jump points}, \quad I_x = (x_{i-1}, x_i) \]
\[ I^+ = (x_{i-1}, x_i^+) \] (again orientation from \( U \)).

\[ \det \Gamma_0(X,F) : \]
\[ 0 \to j_! j^* F \to F \to \bigotimes F_{x_i} \to 0 \]
\( j_i \) : inclusion of complements of points.

sum of constant sheaves induced by \( j_i \). 

\( x_i \leftarrow \det \Gamma_0(X,F) = \left( \bigotimes \det F_{x_i} \right) \left( \bigotimes \det R \Gamma_1(I_x, F_{x}) \right) \)
also have \( 0 \to F_1 \to F_\alpha \to F \to 0 \)

so \( (*) : \bigotimes t \text{ del R}_{\alpha} (F_x^+, F_{\alpha}^+) = \bigotimes \mathbb{E} \quad \forall \mathbb{E} \in X \)

\( (E = C \text{ where small is smooth}) \)

\[ E \chi(u) (F) \delta \theta \quad \text{on } F_{\alpha} \]

... behaves microlocally: depends only on microlocal behavior in \( x \) in \( C \theta \) radius, locally.

General situation: To each \( F \) assign singular support

\( SS(F) = T_x X \quad \text{critical Lagrangian subvariety} \)

Characteristic cycle \( CC(F) \) \& \( z \)-cycle support

on \( SS(F) \) of \( \dim X = \dim X \quad \text{sum of components} \) of \( SS(F) \)

- Case of local system: \( SS(F) \) is \( T_x X \)

\( CC(F) = \partial \cdot T_x X \)

- Case of skyscraper: \( SS(F) = T_x^X \), \( CC(F) = \partial t \cdot T_x^X \)

Dubson-Kashiwara formula:

\[ \chi (X, F) = \langle \langle CC(F) , T_x X \rangle , X \rangle \quad 0 \text{-section} \]

... Euler char. given by intersection number.

For \( F \) local system, just got rank times self intersection of 0-section \( \Rightarrow \) of diago in \( X \times X \).

... essentially obvious in D-module theory, not obvious directly!

New attentive: gradient specie's form Picard group!
... $\mathcal{Z}$ is a class of objects here in this Picard groupoid, so want to animate $\mathcal{R}(\mathcal{F})$...

... lift $F$ to $\mathcal{Z}$ valued in Picard groupoid of some group $\mathcal{G}$. ...

Generalize further to make obvious.

Consider $K$-theory spectrum of coefficients

... consider constructible sheaves with coeff in any associative or dg algebra $R$,

if $R$ is perfect

$\Rightarrow \mathcal{R}(\mathcal{F}, F)$ perfect complex $\Rightarrow \mathcal{R}(\mathcal{F}, F)$ \text{Ker} of $K(R)$ homotopy, and $K$-theory spectrum ... defined only up to homotopy.

On level of $T(\mathcal{O})$ get Euler characteristic

$\text{def } R$'s cohomology ... little deeper:

(\text{def ... must assume } R \text{ commutative})

Any spectrum $K$ defines Picard groupoid

$\mathcal{T}_K$ Poincaré groupoid is a Picard groupoid

-- $K$ is infinite loop space $\Rightarrow$ get addition, comm & assoc. constraints come from delooping (calculus time)

$\mathcal{T}_K$ remembers $T_0, T_1$ ...

$\mathcal{T}_K$ comes from intermediate spectra.

Picard groupoids $\mathcal{S}$ spectra with two adjoint homotopy groups.
K(R) -> spectrum \rightarrow \Omega K(R). Picard group

\text{Picard group of graded } \mathbb{R} - \text{super-algebras}

So for any complex set a lift of its boundary line to homotopy point of K(R).

So now \( K = K(R) \)

\[ K(X) = K(\mathbb{P}(X), R) \]

K-theory of complex \( X \).

There is a morphism (cohomology)

\[ K(X) \rightarrow K \]

\[ \mathbb{R}^n \]

\[ [F] \rightarrow [R], \mathbb{R}[X,F] \]

1. For topological space \( X \), spectrum \( K \) can take cohomology of \( X \) with \( K \) coefficients \( \mathbb{C}(X,K) \), which is itself a spectrum, e.g., for EM space \( H(R) \) this is \( \mathbb{C}(X,R) \) via (cohomology).

2. Generally \( K = (K_i, S^k_i \rightarrow K_{i+1}) \)

\[ \Rightarrow \mathbb{C}(X,K) = (X \wedge K_i, \ldots) \]

3. Functional on \( X,K \).

Claim \( \mathbb{R}^n \) can be decomposed as

\[ K(X) \rightarrow C(X,K) \rightarrow K \]

\[ \mathbb{R}^n \]
Definition of $E$: use fact that $K(X)$ is additive with partitions of $X$, $X = \bigsqcup X_x$, with $X_x$ subanalytic and locally closed.

$\Rightarrow$ get map

$$K(X) \xrightarrow{(i_a!)} \prod K(X_x)$$

for each $X_x$. Consider extensions by zero for each $X_x$.

Quillen $\Rightarrow (i_a!) \text{ is a hodge equivalence.}$

Suppose also each $X_x \subset Y_x$ contractible subset (just note strata small enough).

In a few words get:

$$K(X) \xrightarrow{(i_a!)} \prod K(X_x) \xrightarrow{R_{\text{loc}}} \prod K$$

we choose $X_x$, $Y_x$ but these data are directed & have eventual monomorphisms.

... invalid here for local additive construction?

Restrict to finite, false there locally, put it on my point of contractible sets $Y_x$ is, & then add up:

$$\sum x_x [R_{\text{loc}}(X_x, F|_{x_x})]$$
Microlocalization of $\mathcal{E}$

Localization "singulartoday" $\mathcal{E}$:

- see as the cotangent with coefficients in an algebraic complex $\mathcal{E}$

$\mathcal{E}$ is $\mathcal{E}$

$U \rightarrow$ relative algebra $\operatorname{Core}(\mathcal{E}, \mathcal{E}(U))$

get presheaf of sheaves $K^!(U)$.

This is flabby, so $K^!(U) \rightarrow R\Gamma(U, K^!)$

$U = X$, so $\mathcal{E}(X, K) \rightarrow R\Gamma(X, K^!)$

$p: T^*X \rightarrow X$

$R\Gamma(T^*X, p^*K^!)$

Let $S \subset T^*X$ be closed & conical, $V = T^*X \setminus S$

Inside of category of complexes have dy subcategory of

$\mathcal{P}(X) \supset \mathcal{P}(X)_S$. Thick subcategory $\mathcal{P}$

quotient is the category of microlocal sheaves on $V$.

$K(X)_S \rightarrow K(X)$

$K(X)_S \rightarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow$ *

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$K^!(X)_S \rightarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow$ *

Claim: composition is canonically homotopic to zero.

$K^!(X)_S \rightarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow$ *

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$K^!(X)_S \rightarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow$ *
Take \( \varphi \) \( \mapsto \) \( K(X)_S \rightarrow K(X) \rightarrow K^+(U) \)
\[ \downarrow E \downarrow E^+ \]
\[ R\Gamma_\mathcal{S}(T^*X, p^*K^{-1}) \rightarrow R\Gamma_\mathcal{S}(T^*X, p^*K^{-1}) \rightarrow R\Gamma_\mathcal{S}(V, p^*K^{-1}) \]

Take \( \varphi \) \( \mapsto \) \( K(X)_S \rightarrow R\Gamma_\mathcal{S}(V, p^*K^{-1}) \) locally \( h \)
zeros \( \rightarrow \) get commutable diagram as above,

defining minimalization of \( E, E^+ \).

\( E_S \) : minimal of characteristic cycle:

Replace \( K \) by \( \mathbb{Z} \) then \( R\Gamma_\mathcal{S}(T^*X, p^*K^{-1}) \)
becomes just cycles \( \oplus \) supported on \( S \), get usual characteristic cycle. \( -p^*\mathbb{Z}! \) will
take care of all combinator problems.

\( E \)-factorization: Suppose \( U \subset X \) \& \( V \) 1-form on \( U \).
\( \) with \( h(U) = V \) doesn't intersect our singular support
\( S = SS(S) \) \( \) \( \) \( V = T^*X - S ^ \).

\[ K(X)_S \rightarrow K(X) \]
\[ \downarrow E \]
\[ C(X,k) \rightarrow (\omega \in C^\infty(X \times U_k) \rightarrow C^\infty(X, k)) \]
\[ = K^0(U) \]

\( \omega \) defines conical boundary to \( 2\pi \) of coisotropic
\( K(X)_S \rightarrow \) \( K^0(U) \); use \( \omega \) to pull back to
\( R\Gamma(V, p^*K^{-1}) \rightarrow R\Gamma(V, K^{-1}) \)
So just as before we can formally write

\[
\begin{array}{c}
K(X)_s \rightarrow K(X) \\
\downarrow \varepsilon_u \\
\bigcup (X \cup U, k) \rightarrow (X, k) \rightarrow K'(U)
\end{array}
\]

A commutative diagram, so we get

\[
[\text{RF}(X, F)] = \tau \cdot \varepsilon_u (\{ F \})
\]

\[
\in \bigcup (X \cup U, k)
\]

If \( X \cup U = \{ x_k \} \) for many points, then

\[
\bigcup (X \cup U, k) = \prod K_k, \quad & \text{here is just some sum}
\]

so for each \( x_k \) get \( E \)-factor \( E(F)_V, x_k \)

\[
\cdots \text{can now pass to etf lines or Euler chowrings get usual } E\text{-factors.}
\]