Oxford Hecke Algebras

The Bushba & Divisors (in disc)

Local description of Div \( X \) : any \( \sigma \) admits a meromorphic section \( s \):

-Away from finitely many zeros & poles \( \{ x_i \} \)
-This trivializes \( \sigma \)

\[
\text{Div} \ s = \sum_i \text{ord}_{x_i} s = \sum_{x \in X} \text{ord}_x s \in \mathbb{Z} \otimes \mathbb{Z}
\]
-Only finitely many nonzero orders

Origin of divisor: can trivialize \( \sigma \) in small disc around each \( x \) \( \Rightarrow \) \( s \) gives nonzero Laurent series in each trivialization \( e_{x_i} \in \mathbb{C}[[z^{-1}]] \)
-Almost everywhere \( e_{x} \in \mathbb{C}[z^{-1}]^* \)

\( \{ \mathfrak{L}, s \}, \text{local trivial} \) \( \leftrightarrow \quad \mathbb{T} K_{x_i} \quad \text{or} \quad \mathbb{T}' K_{x_i} = \mathbb{A} \)

Divisor classes by forgetting local trivializations:
-Change of trivialization \( e_{x} \in \mathbb{C}^* \quad \text{and Taylor series} \)

\( K_{x_i}/\mathcal{O}_{x_i} = \mathbb{Z} \quad \text{remember only order of pole} \)

\( \{ \mathfrak{L}, s \} \in \mathbb{T}' K_{x_i}/\mathcal{O}_{x_i} = \mathbb{T}' \mathbb{Z} = \text{Div} \ X \)
Charged section $s$: ratio of any two
$s/s' \in \mathcal{O}(X)^*$ nonzero rational function
$\text{Div } s/s'$ is a principal divisor

$\Rightarrow \text{Pic } X = \{ [I] \} = \text{Div } X = \text{Pic } \mathcal{O}(X)/\mathcal{O}^*$

$= \text{GL}_r(\mathcal{O}(X)) - \text{GL}_r(\mathcal{A}_X)/\text{GL}_r(\mathcal{O}_X)$

General group $G$: seek symmetry of $\mathcal{B}_G$
$\text{P } G$-bundle, trivialize away from finitely many $x \in X$ & near each $x$; $\Rightarrow$ transition functions $\star$ in base group

- topologically $\text{LG} = \{ s' \to G \}$
- algebraically: Laurent series in $G$
$G(x) = G(\mathcal{O}(x))$
- invertible matrices of Laurent series

Change of local trivialization $\Leftrightarrow$
hol. map from disc to $G$ : $L \in G \in LG$
or Taylor series in $G$ $G(O_+) \subset G(k)$
Charge & meromorphic trivializers $G(X, \{x\})$

or in extreme setting $G(C(X))$

meromorphic action

So $Bun_0 X = G(C(X)) \backslash \Pi_1 G(C_x)/G(C_x)$

If $G$ is semisimple, (e.g. $SL_n$, not $GL_n$)

or we work analytically, only need one part:

$Bun_0 X = G(X, \{x\}) \backslash LG/L + G$

The local data at $x$ is

Affine $\mathcal{E}_r = LG/L + G = G(K_x)/G(L)$

Grossmann = $G-$bundle + trivialization on $X - x$

-- generates $\mathcal{Z} = \mathcal{O}_{G_0}$.

However, $\mathcal{Z}$ is a group & acts on Pic $X$:

$L \mapsto L(x)$: add for divisor of $L$

-- no sections allowed to be pole at $x$.

Abel-Jacobi map: $X \rightarrow Pic^0 X$

all line points generate Pic $= \prod \mathcal{Z}$. 
Hecke algebra $G$ finite group $G \times G$ \\
\[ \Rightarrow \text{ functions in } G \text{ here} \]
\[ \text{convolution } f \ast g = \mu_m \left( \pi_i f \pi_j g \right) \]
\[ \text{group algebra } CG. \]

$K \subset G$ subgroup \$\Rightarrow_{
\text{Hecke algebra } KG \subset KG \text{ invariant subspace of } G
\]
\[ CG_{K \subset K} = CG \left[ K \backslash G / K \right] \subset CG \]

Why subalgebra? Given by same diagram

\[ K \backslash G / K \leftarrow K \backslash G / K \text{ (for } K \right) \]

\[ G = \{(g,1) \sim (g,k), (1,g) \}
\]
\[ = \{(g,k) \sim (g,k), (g,k) \}
\]

What's its meaning? Q: What does $K$-invariant? 
\[ V_{g,K} = CG_{K \subset K} \text{ C} \text{ GL } K \text{ (for } K \text{ irreducible)} \]
\[ \text{induced representation, generated by } \]

\[ \text{fixed } K \text{-representations} \]

\[ \Rightarrow \text{ Hom}_G \left( V_{g,K}, W \right) = W^K \text{ } K \text{-invariants} \]
\[ \text{(determined by image of } 1, \text{ must be } K \text{-invariant)} \]
\[ \text{End}_G(V_{\mathfrak{g}, k}) = H^{0}_{\mathfrak{g}}(V_{\mathfrak{g}, k}, V_{\mathfrak{g}, k}) = (V_{\mathfrak{g}, k})^* = \mathbb{C}[G \backslash G/k] = \mathbb{P}_{\mathfrak{g}, k}, \]

\[ \mathbb{P}_{\mathfrak{g}, k} \text{ acts on } K \text{-invariants } W^k \]

Geometrically: \( X \times G \) G acting, take quotient by \( K \), and what remains of \( G \) symmetry:

\[ X \times G \quad \Rightarrow \quad X^G \quad \Rightarrow \quad X/K \]

- Get fiber bundle over \( X/K \) with fiber \( G/K \)& structure group \( K \) acting on \( X \)
- \( \mathbb{P}_{\mathfrak{g}, k} \) acts on \( \mathbb{C}[X/K] = \mathbb{C}[x]^k \)
- Also get localization \( \mathbb{C}[x_0, \ldots, x_n]/(x_{ij} x_{ik}) \) from \( X \times G \quad \Rightarrow \quad X/K \times X/K \) \( \mathbb{P}_{\mathfrak{g}, k} \)

Some picture holds for any theory of descent
- \( \text{D}(X, D) \) \( D \)-modules.
Symmetry of \( \text{Bun}_G X = \prod_{x \in X} G(x)/G(\mathbb{Q}) \):

at each \( x \in X \) base action of
\[ G(\mathbb{Q}) \backslash G(X)/G(\mathbb{Q}) = L_G \backslash G / L_G \]

Hecke correspondence: \( x \in X \)
\[ \text{Hecke}_x = \{ P_1, P_2 | P_1 \mid_{x^{-1}} \cong P_2 \mid_{x^{-1}} \} \]
\( \text{Bun}_G \rightarrow \text{Bun}_G^{\text{Hecke}} \)

- fiber over \( x \) is trivial:
\[ \{ P \text{ + trivialization on } X \times x \} = L G / L G_1 \cong \text{Gr} \]

- Hecke \( x \) is a \( \text{Gr} \) bundle over \( \text{Bun}_G \) with structure group \( L G_1 \).

Corollary: \( D \left( L G / L G_1, D \right) = D_{G_0} \left( \text{Gr}, D \right) \)

is morfed (associative) \( \star \) acts on \( D \left( \text{Bun}_G, D \right) \)

- remit of loop group symmetries
of bundles with trivialization.
$\mathcal{G}_n$: instead of $\mathbb{Z}$ co-variants find particular closed strata: elementary modifications

e.g. $0 \leq k < n \{ V + \text{ root } k \text{-subspace } W \subseteq V \}$

$v \in \mathcal{B}_n \quad \mathcal{B}_n$

\[
\text{elementary transform of } V:\ 
\text{sections are sections of } V \text{ with value in } V \text{ at } x.
\]

... closed $L_G$ orbit $\mathcal{G}_n \subset \mathcal{G}_n$

... these general Hecke correspondences for $\mathcal{G}_n$.

Note: $H^*(\mathcal{G}_n) \cong \Lambda^k \mathbb{C}$

representation of $\mathcal{G}_n$...
The affine Grassmannian
\[ \text{Gr}_n : \mathcal{L}G/\mathcal{L}G_+ = \text{Gr}_n(\mathbb{C}^n)/\text{Gr}_n(\mathbb{C}^1) \]

\[ = \left\{ \text{lattices } V \subset \mathbb{C}^n : \text{preserved by } z \right\} \]

Can likewise model with any kind of loops,
Hilbert space Grassmannian, etc.

- union of finite dimensional orbits of \( \mathcal{L}G_+ \): 
  \[ \cup \text{ lattices further & further from } H_+. \]

Orbits (double cosets) \( \mathcal{L}G \setminus \mathcal{L}G/\mathcal{L}G_+ : \)
- Laurent matrices up to invertible
- Todd row & column operations
- \[ \rightarrow \begin{pmatrix} z^k & 0 \\ 0 & z^k \end{pmatrix}, \quad k_1 \leq k_2 \leq \ldots \leq k_n \]
- normal form.

General G is T - weighted torus :
- \( \chi \cdot A \) - cocharacter lattice = \( \text{Hom}(\mathbb{C}^* T) \)
- Interpret as loop in T, get special 
  - double cosets \[ [\lambda] \in \text{Gr}. \]
L(G) orbit $\leftrightarrow$ W [V] Weyl group orbit.

$\Lambda = \text{Hom}(T^\vee, \mathbb{C}^*)$ - characters of dual torus $(= \Lambda^\vee \otimes \mathbb{C}^*)$

$\Lambda / \mathbb{W} \leftrightarrow$ equivalent characters of $T^\vee$.

$\rightarrow$ irreducible representations of group $G^\vee$ with $T^\vee$ as its maximal torus.

Laumon dual group: dualize root data of $G$.

\[ \Pi_1(G^\vee) = \mathbb{Z}(G)^\vee \]

Simply connected form $\leftrightarrow$ adjoint form.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$G^\vee$</th>
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<tbody>
<tr>
<td>GL_n</td>
<td>GL_n</td>
</tr>
<tr>
<td>SL_n</td>
<td>PGl_n</td>
</tr>
<tr>
<td>Sp_n</td>
<td>SO_{2n+1}</td>
</tr>
<tr>
<td>Spin_{2n+1}</td>
<td>Sp_{n+1}</td>
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<tr>
<td>Spin_{2n}</td>
<td>SO_{2n}/\mathbb{Z}^2</td>
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Let's look at the classical analog of D-modules on \( LG \setminus G \): functions on \( G \) constant along \( LG \) orbits (\( L \)-equivariantly supported) - in fact this is the \( K \)-group \( K(LG, G) \).

To define convolution duality: consider \( L G \setminus LG \setminus LG \) by replacing \( C([a]) \mapsto \frac{1}{\pi} \frac{1}{a} \).

This doesn't affect the constant variation question, but now have a nice locally compact field \& can integrate!

\[ \Rightarrow \text{ spherical Hecke algebra} \]

\[ \mathbb{C} [ L G \setminus LG \setminus LG \setminus ] \]

Satake Theorem: The spherical Hecke algebra is isomorphic to the representation ring \( \text{Rep}\ G' (\cong \mathbb{C}[t^+/w]) \), in particular it's commutative.
Honest definition of $G$: Tannakian theory

Given a group $H$ we have a category $\text{Rep} \, H$ of (finite) representations on $\mathbb{C}$-vector spaces.

Structure:
- Linear category
- Monoidal category: $V, W \mapsto V \otimes W$
  (associative)
- Concatenation: $V \otimes W \cong W \otimes V$
- Unit: $\text{id}_V$

- Fiber functor: $\text{Rep} \, H \to \text{Vec}$
  faithful & full
  $(H \text{-mod} \subset \text{C-mod})$

Def: A Tannakian category is a category which obeys:

\[ C, \otimes, F : C \to \text{Vec}, \text{etc.} \]

Tannakian reconstruction: Given $C \to \text{group } H$

$H = \text{Aut} (F)$, natural isomorphisms $C \cong \text{Vec}_F$

ie automorphisms of "underlying vector spaces" of every $V \otimes F$ compatibly.

Theorem: $C = \text{Rep} \, H$, ie $\text{Tannakian categories } \leftrightarrow \text{groups}$
eg \( \mathbb{C} = \mathbb{Z} \)-graded vector spaces, \( \otimes \)

\[ H = \mathbb{C}^* \] multiplicative group

\( \mathbb{C} \) = flat vector bundles on \( X \),

\( F \) = fiber of \( \mathbb{C} \times X \)

\( H = \pi_1(X,*) \) (or pre-sheaf completion...)

Origin of \( G^v \):

Geometric Satake Theorem (Mirkovic-Vilonen, Ginzburg, Lusztig, Drinfeld, Groth)

\( LG \)-equivariant D-modules on \( LG/LG \)

are a Tannakian category; \( G^v \) Tannakian group

is a reductive group with dual vector data.

Example: \( GL_1 : G^v = \mathbb{Z} \),

\( D\)-module = \( \mathbb{Z} \)-graded vec, Tannakian group =

\( \mathbb{C}^* = GL_1 \)

Consequences \( D(\text{Bun}_G, D) \) comes

lots of commuting operators:

\[ x \in X \mapsto \text{Rep } \mathbb{C}^* \to H_x,v : D(\text{Bun}_G, D) \]

\( \Rightarrow \) try to simultaneously diagonalize

\( \Rightarrow \) find analog of characters on \( \text{Pic} \).
Spectral decomposition (Fourier transform)

Want an equivalence between "functions" on $B_{\mathbb{Q}}$ & "functions" on some other space, so that Hecke operators $H_{\lambda,\nu}$ become multiplication operators.

i.e. find space $Z$ & bundles $W_{\lambda,\nu}$ on $Z$

s.t. $D(B_{\mathbb{Q}},D) \cong D(Z,\mathcal{O})$

$H_{\lambda,\nu} \otimes W_{\lambda,\nu}$

Universal guess for such a $Z$ is

"spectrum of Hecke algebra $\otimes_{\mathbb{R}^d}^\text{max} \hspace{1cm}$"

= machinery that turn representations of $G$ into flat vector bundles on $X$

(i.e. fully varying vector bundles for $G \times X$)

$\mathfrak{g} \sigma^\nu X \moduli$ of flat $G$ bundles:

$V \rightarrow V_{\lambda,\nu}$ associated vector bundle

$W_{\lambda,\nu} \mid_L := V_{\lambda,\nu} \mid_L$

So our optimistic hope is that functions on $B_{\mathbb{Q}}$ have multiplicity one (no degeneracy) over spectra.
Geometric Langlands Conjecture: There is an equivalence of categories

\[ \text{D}(\text{Bun}_G, D) \rightarrow \text{D}(\text{Con}_G, \mathcal{O}) \]

Intertwining Hecke algebras \( \otimes \text{Rep}_G \)

I really need more rest because Tom is singular, & Ben is big... needs some modification “along the Ganges”.

Hecke eigenstates: \( L \in \text{Con}_G \)

\( L \) skyscraper, \( \text{Aut}_L \) corresponds! Derake

\[ H_{x,v} \text{Aut}_L = L_v |_x \otimes \text{Aut}_L \]

“Eigenvalue” for Hecke operators with eigenvalues determined by the \( G \) conjugacy \( L \).

— Analog of character shift in Delin setting.
Geometric Satake

Idea of proof: Lie algebra is simply connected

$\Rightarrow$ only trivial flat connections on $G$

$\Rightarrow$ only extensions of $G$-bundles

Fiber functor: global cohomology

Mirko: Vitali's canonical $\Lambda$-grading

on the cohomology, labelled by fixed points

$\Lambda \in \mathbb{C}r$: "proceed cell decomposition"

with one cell through each $\mathcal{D}_{\lambda}$

$\Rightarrow$ find dual torus $T^\vee = \text{torus}^\vee$

Most interesting aspect: commutativity

Why is this convolution algebra commutative?

Drinfeld's answer: operator product expansion

or fusion picture, coming from conformal field theory.

Example: $G = \text{GL}_n$, $\mathbb{C}r = \mathbb{Z}$. Why is $\mathbb{Z}$ abelian?

Addition of divisions: $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

Change of division: $\frac{1}{(x-1)^3} \Rightarrow \frac{1}{x}$
Commutativity comes from colliding points in different orders.

Recall $C_{\alpha} = LG/L\alpha = G(K_{x})/G(\alpha)$

$= G$-bundles with trivialization on $X \times x$.

As $x$ varies get $G$-bundle $\mathcal{E}_{x} \rightarrow \mathcal{C}_{x} \times X$

$\xrightarrow{i_{x}} \mathcal{C}_{x} \times X$

Let $\mathcal{C}_{x \times x} \rightarrow \{ G$-bundles on $X \times x$ trivial on $X \times \{y, y\}\}$

$\xrightarrow{i_{x \times x}} X \times X \rightarrow \{x, y\}$

Claim 1. $\mathcal{C}_{x \times x}$ is a fibration (flat) over $X \times X$.

2. $\mathcal{C}_{x \times x} \mid_{x \times y} \cong \mathcal{C}_{x} \times \mathcal{C}_{y}$ two copies

$\mathcal{C}_{x \times x} \mid_{x \times y} \cong \mathcal{C}_{x}$ one copy!

Argue of addition of divisors as points collide.
Fusion of sheaves: $F, G$ D-modules on $X \times X$
put $F \boxtimes G$ on $X \times X \times X$
$c_{X \times X}$ now collide!
take limit as $G$ approaches diagonal

$F \boxtimes G$

Claim: This agrees with $\cdot F \boxtimes G$

$\Rightarrow$ convolution is commutative.

Rational-Drinfeld: geometric theory of
vertex algebras based on this picture.

- a vertex algebra is a vector bundle $V$ on $X$ & a vector bundle $V^\ast$ on $X^\ast$
  (quasi-coherent sheaf flat along $\Delta$)
  which glues $V \boxtimes V^\ast$ off the diagonal
to $V$ on the diagonal (plus unit section)

-gluing low along $\Delta$ gives OPE of sections
-Linearizing $G$-algebra viewpoint gives
Kac-Moody vertex algebra - version of
    enveloping algebra of loop algebra $\mathfrak{g} = \mathfrak{g}(\mathfrak{g})$