TFT & Langlands

Kapustin - Witten:
Geometric Langlands is a manifestation
of an underlying isomorphism of four-dimensional
topological gauge theories $Z_{0,2} \cong Z_{0,2}$
- much richer mathematical structure!

TFTs: Aenomalies, the "coarsest" QFTs,
built up tier by tier.

An n-dim TFT $Z/k$ is an assignment
$M^n \mapsto Z(M^n)$ $k$-n.d. small stably
oriented, diffeo. invariant
- multiplicative: $\phi \mapsto 1$, $11 \mapsto 1$

$Z(M^n) \mapsto \int \exp(-S(\phi))\,D\phi$

Next express locality (cut & paste) of $Z$:

An n-dim TFT $Z$ assigns ( cpt., sm. old. )
$N^{n-1} \mapsto \mathbb{Z}(N^{n-1}) \in \text{Vec}_k$
- functorial
- multiplicativity: $\phi \mapsto C(\phi)^* \mapsto \*_{|\*}$
Also, $M^\partial$ with boundary

$Z : M^\partial \rightarrow Z(M^\partial) \otimes Z(\partial M)$

- $Z(N^{n-1}) = \text{Id} \in \text{Hom}(N, N)$
- functorial under cobordism

\[
\begin{array}{c}
\text{E} \quad \text{Hom}(Z(0), Z(0)) \\
\uparrow \quad <, > \\
\text{Hom}(Z(0)Z(0)) @ \text{Hom}(Z(0)Z(0))
\end{array}
\]

$\Rightarrow Z(\begin{array}{c}
\text{c}
\end{array}) = \text{Tr}(F')$

$Z(N^{n-1}) = \text{dim} Z(N)$

Get map $Z \rightarrow \mathbb{Z}_5$: For any TFT $\mathcal{T}$

$(n-1)$D TFT

$Z_\mathcal{T}(N) = Z(N^{n-1})$

More general dimensional reduction along $\Sigma^k$

$Z_{n-\text{dim}} \rightarrow \mathbb{Z}_{n-\text{dim}}$

$Z_\mathcal{T}(M) = Z(M^{n-\Sigma k})$

- Labeled correpond: $\text{L} \otimes (\text{I})$ (laid by $\psi \in Z(M)$)
Theorem: 2D TFT $\leftrightarrow$ commutative Frobenius algebra

How? $Z : H \rightarrow Z(s')$

$Z(D) = 1 \in H$ \quad $Z(\emptyset) = \text{tr} : H \rightarrow C$

$Z(\emptyset) : H \otimes H \rightarrow H.$

Check $Z(D) = \text{tr}(\mu) : H \otimes H \rightarrow C.$

Non-degenerate. Moreover any $Z^2$ can be decomposed into such building blocks, so recover $Z(\mathbf{3})$ from Frobenius structure.

Commutivity $Z(\mathbf{3}) \leftrightarrow \mathfrak{o}.$

More generally: $Z$ n-dim

$Z(S^{n-1})$ covers multiplication parametrized by little disks.

$Z(S^0) \leftrightarrow \mathfrak{o}$ \quad $Z(S^1) \leftrightarrow \mathfrak{c}$

associate commutative "more commutative"
Moreover $Z(S^{n-1})$ acts on $Z(N^m)$

Generalizations:

- Families version: given a family of manifolds base $B$ assign $Z(N)$
  local system on $B$ (parallel transport over $B$)

- graded TFT $Z(N)_{gr}$

- dg TFT/TFT: assign dg vector space to manifolds up to quasi-isomorphism
  weaken topological invariant to quasi-isomorphism

Given $\mu (\begin{array}{c}
\ldots
\end{array}) \Rightarrow$ chain map of degree zero $Z(N) \to Z(\mathbb{N})$

Given $k$-chain in $\text{Hom}_B (N, N^m)$

$\Rightarrow$ $k$-chain in $\text{Hom}_B (Z(N), Z(N^m))$)

Now $Z(S^n)$ is an $E_n$-algebra: multiplications parametrized by (chains on) little discs.
E_8 \text{ = } \mathfrak{so}_8 \text{ - algebra: } \text{heuristically reduced notion of alg associative algebra}

- in strictify, we'll gloss over the alg/so distinction,

Physics: CY varieties X provide 2 different
2D TFT, the A-model & B-model:
both come from the same underlying
N=2 SUSY \sigma-model: a conformal
field theory ... it depends on complex
structures on our surfaces \Sigma
- fields are maps \Sigma \rightarrow X \text{ extensions on } \Sigma


topological twist: change natural coordinate
deformation of fields to make
change of coordinates exact wrt
different for SUSY \Rightarrow \text{ Calabi-Yau is a TFT},

A-model: \mathcal{Z}(S^1) = \Omega H^*(X) (\text{symplectic})

B-model: \mathcal{Z}(S^1) = \mathcal{D}^{\ast}(\Lambda^1 T X) (\text{holomorphic})
2-functor theories.

$Z : Y^{n-2} \to Z(Y^{2}) \in \text{Cat}^c$

- c-linear (or dg.) category
- 2-functor, monoidal

$Z(N)^n \in \text{Ob } Z(^2N)$ for manifolds $M^n$.

$\circ \otimes \circ \in \text{Func}(Z(y), Z(y')) \otimes \text{co-pseudo}$

natural transformation for $n$-manifolds with arrows:

$\natural$

2D

$A : \xymatrix{ A \ar@{-->}[r]^B & \text{gives vector space}\text{Hom}(A, B) }$

$\Rightarrow c$ composition of hom encoded

in such diagram.

Moore-Segal axioms for 2D TFT

with labelling A of arrows

Equipping category: $Z(S^1)$ linear category with traces $L$ compatibilities.

Note $Z(S^0)$ manifold $Z(S^1)$ braided $Z^n(S^0)$ $K$-2 symmetric.

$Z(Y \times S^1)$ = categorified tree: $K_4(Z(y)) = HH^4(Z(y))$.
Interpretation: \( \text{Shv}(\mathcal{O}) \rightarrow \text{D-branes} \)

Physics: To a CY can associate two categories in a natural way, which are the allowable (topological) boundary conditions for the A- & B-models.

- **A-model**: \( \text{Fuk}(X) \) Fukaya category of \( X \)
  - Objects: Lagrangians \( L \subset X \)
  - Decategorifying (grading, unty category)
  - Morphisms: intersection points & Floer homology in cotangent bundle points: \( \text{Fuk}(X) \) = perturbing big Fukaya 
    - "iso & branes"

- **B-model**: \( D^b(X) \) bounded derived category of coherent sheaves on \( X \)

Both are some kind of object: a dg category, or weaker, an \( A_n \) category, with a trace \( \text{Hom}(A^A) \rightarrow \text{C} \) on \( A \) or \( A^A \) coming from holomorphic 1-form (B-model)

---

- N.C. Seiberg, "..."
Thm (Costello): 2D dyTFT $\leftrightarrow$ NC Calabi-Yaus

(rough paraphrase)

More precisely, given set $\Lambda$ of labels $\Lambda$, let $\Lambda$-labelled $X \to \Lambda$. Let $\Lambda$ be a

\[
\text{NC C}Y \to 2D \text{TCFT which are universally properties...}
\]

So the structure of 2TFT captures a variety in the sense of noncommutative geometry.

- birational type in 3d (Bridgeland)
- up to discrete, grading more generally (And

Thus a 4d (di)TFT yields a functor $C$ surface $\to Z_c$ NC CY

where $Z_c$ is category of boxes $Z(c)$,

with action (for $p \in C$) of symmetric monoidal category $Z(S^2)$.
Electric - Magnetic Duality

\[ G_c \text{ compact Lie group} \]
\[ \rightarrow \text{4d TFT: } N=4 \text{ Supergravity} \]

Yang-Mills with gauge group \( G_c \), in GL twist.

- \( N=4 \text{ D=4 SYM is a gauge theory: fields} \)
  - include coulomb on principal \( G_c \)-bundle +
  - 6 adjoint scalars + fermions: roses from

\( N=1 \text{ D=10 SYM: correct } A + \) adjoint spinors.

require torsional invariance in 6 dimensions
\[ \rightarrow A_0 + \phi^i; i=1,...,16 \text{ (fermions)} \]

Theory has a complex parameter \( T^a \): conserved
of roots of YM action \( \oint F \wedge F \) ( gauge field)
\[ \left( \frac{\theta}{2\pi} \right) \]

& topological (Chern) top \( \theta \) \( F \wedge F \) (theta angle)

Key feature: Theories for gauge \( G \) & \( G^* \)
are isomorphic; generalization of EM
symmetry of Maxwell. 

\[ \text{Exchange } T^a \rightarrow - \text{ part of } \text{SU}(2) \text{ symmetry via } T^a \rightarrow i \tau^a T^a. \]

[Goldstone-Niels.Ode / Manton-Olive]
Topological twist: change action of change of coordinates on $\mathbb{R}^4$ on $\phi_1 \cdot \phi_2$vertex
i.e. $\mathbb{R}^{10} \cong TM \times \mathbb{R}^2$.
So $A, \phi$ complexified coincide.

The resulting theory has two odd generators $Q_1, Q_2$ with $Q^2 = 0 \in \mathcal{O}_{Q_1, Q_2}$,
cohomology of either $Q$ is a TFT

$\Rightarrow \mathbb{P}^1$ - family of twisted TFTs
labelled by $Q = aQ_1 + bQ_2$

... in fact dg TFTs if take $Q$ complex.

Resulting TFT actually depends on a particular
subset of $t, \tau$ called $T \in \mathbb{P}^1$

$$s = \frac{4}{g^2} \int F \wedge F + \{Q_1, -\}$$
Gauge theory = TFTs:
- $Z(M^4) = \#$ sols of field equation (YM equations + modifications = Higgs)
- $Z(N^3) = H^1$ (moduli of sols of "monopole" reps on $N$)

$Z(C^2) = \text{category of branes on moduli of sols on } C^2$

$C \rightarrow 2d$ TFT: $Z_C(x) = Z(C \times x)$

Think of $C$ as very small: gauge fields on $C$ $\leftrightarrow$ reps from $x$ to moduli of sols of field reps on $C$, i.e. $\sigma$-model on $M(C)$ moduli space.

This $Z(C) = \text{category of branes on } M(C)$

- on $NC\text{ CY}$.

What is $M(C)$? Hitchin moduli space
- Hitchin equations on $C$:
- $F - \rho A = 0$ (F = dA)
- $dA = 0 = dA + \rho$

Moduli of Higgs bundles, a hyperkähler manifold
$\leftrightarrow$ in particular CY.
$M_{\text{K}}(C)$ has a P of Kähler structures
such that $\{I, J, K\} \Rightarrow$ a wealth of
$
\tau$-models: here A3 models in each,
1 in flat linear order of both corresponding
to interpolating complex structure

[C symmetry identifies all but $iJ$]

$K: \quad \psi = 0 \Rightarrow J_b \quad \tau = 0 \Rightarrow K_A$

$M_{\text{K}}(C)_T = \text{moduli of stable flat}$
G connections on $C \subset \text{Loc}_G(C)$

$Z_G(C)_{\psi = 0} = D^b(\text{Loc}_G(C))$

$M_{\text{K}}(C)_K = \text{moduli of stable}$
high bundles $\subset \text{T}^* \text{Bun}_G(C)$

$Z_G(C)_{\psi = 0} = \text{"Fuk"}(\text{T}^* \text{Bun}_G(C))$

$S$-duality: $Z_G, \psi = 0 \cong Z_G', \psi = 0$

$\Rightarrow \text{"Fuk"}(\text{T}^* \text{Bun}_G(C)) = D(\text{loc} \text{Loc}_G(C))$
From Fukaya categories to D-modules:
(Kapustin, Willer, Mauler, Zaslow)

KW: $T^* B_{\mathfrak{g}}$ carries canonical symplectic
structure (in fact skew-adjoint) on $B_{\mathfrak{g}}$.
- Calculating that $D_{\mathfrak{g}}$ acts diagonally
on $B_{\mathfrak{g}}$.

Mauler-Zaslow: give a version of this worked
out in spirit of microscopic study of D-modules,

\[ \text{Fuk}(T^* X) \simeq D_{\mathfrak{g}}(X) \text{-mod} \]
't Hooft & Wilson loops

Operators in 't Hooft gauge have two types:

Wilson: given loop L in $M^4$, can calculate holonomy of connections along L and take trace in any representation of $G$: $V \in RepG \Rightarrow tr_L hol_L(A) = W(v)$

In quark theory can insert this into path integral: $\langle W \rangle = \int W(v) e^{-S[\phi]} D\phi$

'Hooft: can introduce magnetic monopole along L & make all calculations in the presence of this monopole: $\int e^{-S[\phi]} D\phi$ ("disorder operator")

What are these labelled by? Canonical singular vortex of rank of YM in $\mathbb{R}^3 \times \mathbb{R}$ in $S^2$
$\Leftrightarrow \text{G}_0$-bundle on $\mathbb{P}^1 \Rightarrow \mathbb{C} \to \mathbb{C}/\mathbb{Z}$

$\Leftrightarrow \text{Rep } \mathbb{G}^\nu$,

From formal pov this is simply adding one allowable labelling of a loop:

- i.e., states in $\mathbb{Z}(S' \times S^2)$

give operators on $\mathbb{Z}(Y)$ for any loop in $Y$.

$S$-duality exchanges Wilson & 't Hooft loops

Topological twist: not all operator distinct for all values of the parameters.

\( 't \text{Hooft Wilson} \)

$F = 0 \quad \checkmark \quad \times$

$F = 00 \quad \times \quad \checkmark$

So $\text{Rep } \mathbb{G}^\nu$ act on $\mathbb{Z}_{\text{rep}}(Y)$ by 't Hooft loop

$\Leftrightarrow \text{Rep } \mathbb{G}^\nu$ act on $\mathbb{Z}_{\text{rep}}(Y)$ by Wilson loops
Down to 2 dimensions: \( Z(S^2 \times S^2) = K_3 Z(S^2) \)

\( \mathcal{Q} = 0 : \ Z(S^2) = D_{\text{mod}}(\text{Bun}_G P^1) = \text{Res}^G G \) (Satake)

\( \mathcal{Q} = \infty : \ Z(G)(S^2) = G_{-\text{mod}}(\text{Loc}_G P^1) = \text{Res}^G G \)

So it looks like Fock spaces...

**Dynamic POV**

describe these correspondences in terms of solutions to Boyer--O'Kane equations on \( G \times I \) with prescribed singularities:

- Boyer--O'Kane \( \leftrightarrow \) holomorphic type on locus
- stays constant away from singularity,
  
  \[ \text{F. geometric analysis to Frobenius characteristic poly} \]
Eigenbranes In $\mathbb{Z}_{6, y_{0}}(L) = \text{Fuk}(\mathcal{P}_{y_{0}})$

have "obvious" objects, given by the locally
on smooth Hitchin fibers (for flat unitary reps).

These correspond to points on $\mathcal{D}_{y_{0}}$ under
5-duality $\rightsquigarrow$ are eigenbranes for the
't Hooft operators: