Real Groups & Geometric Langlands

Explain work in progress with David Nadler
relating geometric Langlands & the rep's of real Lie groups.

Localization of Representations

Lie group $G \subset X$

$\to$ $\mathfrak{g}$ acts by vector fields
$\mathfrak{g} \to \Gamma (\mathfrak{g} X)$

So set $\mathfrak{g}$-module $\mathfrak{g} X \mod \mathfrak{g} \to \mathfrak{g} \to \mathfrak{g} \mod$

$M = M \to \mathfrak{g} X$, just as we have

for $R \to \Gamma (Q)$ (i.e. $X/R$)

$\mathfrak{g} X \mod \to R \mod$. 

Natural place to localize reps of $G$ relative
the algebra $\mathfrak{g} X$ flag variety $G/B$.

Borel-Weil says reps of $G$ live uniquely in this
on $G/B$: has an $G/M$ transformation under $H$
as a character.
Böhm-Hornfest: roughly says
\[ D(G/B) \cong \text{Vog-mod} \otimes \mathbb{C}[k] \]

at least an equivalence for most \([k] = \mathbb{C}\)

An action of \(Z_g \cong \mathbb{C}[k]\)

\[ \text{e.g. } D(G/B) \cong \text{Vog}_0 \text{-mod} : \]
cone acts as a trivial rep.

We'll ignore the distinction between \(G/B \neq G/B\)

reg vs. monodromic / twisted \(D\)-module

& work with regular dominant infi. coweights.

Examples:

- \( D_B(G/B) \): Harish-Chandra bimodules
  (close to \(L^2 G\) Category \(O\) of highest weight modules) \:<br>  reps of \(G\) as real Lie group.

- \( G_R < G \) real form, the derived category
  of Harish-Chandra modules (admissible reps)
  \[ \cong D(D(G_R \setminus G/B)) \quad \text{Kashiwara-Schapira} \]
really bad at real constructible schemes: has
a holomorphic realization by BB as
$D(K_G \setminus G/B)$, related to above by
Matsuki: Mikami-Uzawa-Ukim

... all such carry $\ell$be etale of "finite
etale algebra" $D(K_G \setminus G/B)$ on variety

This a generic analogue of the description of
reps of a group inside functions on that group;
is there a Fourier dual?

Langlands: classified irreducible reps of
reductive real groups; local Langlands
conjecture for $R$.
Works most naturally for several real forms
at once; fix $G$, quasisplit real form
$1 \to G \to G^0 \to \mathbb{R}/2 \to 1$

$S = \{ \mathfrak{o} \subseteq G^0, \mathfrak{g}: \mathfrak{o}^2 = \mathfrak{d}/\mathfrak{s} \}$
$G^0_\mathfrak{o}$ - fixed points
If $\sigma \in S$ is an irreducible representation of $G$, then Langlands parameters:

Complex analytic data for $G^\sigma$.

$1 \rightarrow G^\sigma \rightarrow G^\circ \rightarrow \mathbb{Z}/2 \rightarrow 1$ complex group

$G^\sigma \rightarrow W_\mathbb{R} \rightarrow \mathbb{Z}/2$

$\leftrightarrow$ pairs $(\xi, \lambda)$ with $C^\circ \times C^\circ$, $\lambda \in \mathfrak{o}_G^\ast$.

$d^2 = \xi \epsilon(\lambda)$.

Adams-Barbasch-Vogan: promote Langlands parameterization to a graphical notation on $K_{\mathbb{C}}$.

Let $S(\mathbb{C}) = \bigoplus_{\sigma \in S} D(\mathbb{C}_G^\sigma)$

$K(\mathbb{C}) \hookrightarrow K_{\mathbb{C}}(X^\sigma) \text{ for } X^\sigma = \text{geometric parameter space of }$ $\mathbb{C}^\sigma$ with orbits $\leftrightarrow$ Langlands parameters.

Serre's conjecture: this perfect pairing induces a (Koszul) equivalence of twisted categories; $\mathbb{C}$ in a Langlands description for $\mathbb{C}$ of $\mathbb{C}$. 

Case of G\textsubscript{0} as a real group:
Sommel's conjecture reduces to orbit of
Beilinson-Ginzburg-Sommel giving a (canonical)
equivalence between D(G/\mathbb{R}) \cong D(G/\mathbb{R})^\text{op}

Our aim: express Sommel's conjecture in
a geometric language, relating
the case of Kleinian orbifolds
and deducing spectral data from a
generic local case conjecture.
General scheme: study geometric Langlands on $\mathbb{P}^1$ in detail.

On $\mathbb{P}^1$ the set of $\mathbb{G}_m$ classes of bundles is countable.

- the trivial bundle is open: admits no

- root deformation, has $G$ of order $\geq 2$.

- $\mathbb{P}^1$ of $\mathbb{P}^1$. When stack looks like

  $G(\mathbb{C})$ orbits on $\mathbb{P}^1$ with ordering reversed.

Trivial stack case: look at $Bun = Bun_0(\mathbb{P}^1, 0, 0)$

- $G$-bundles + flags at $0, \infty$.

Open $Bun^0 \subset Bun$ trivial bundle locus

is isomorphic to $G/B \times G/B = G\backslash G/B$

What about real groups?  look at

- real $G$-bundle on (anti)flag of $\mathbb{P}^1$.

Proposition: Trivial bundle locus in

$Bun_0$ is precisely $G/B \times \{0\}$.
Tate co-ideal geometric Langlands

$Bun_o(X, p)$ flag at $p$ does $flag$, on $X, p$

at $p$: comes from $D_K G(K)/I$
affine Hecke category

$K (D (I \backslash G(K)/I)) = \text{constr. fr. } (I \backslash G(K)) = [\mathbb{U}]$

Kazhdan-Lusztig: $K^G_v (T^* \mathfrak{g}^* \times T^* \mathfrak{g}^*)$

Steinberg variety $\{(g, g, n): n \in \mathfrak{g} \otimes \Delta \}$

Bezrukavnikov:

$D (I \backslash G(K)/I) = D (\mathbb{S}^G_v )$

$\mathbb{S}^G_v = \{(B_1, B_2, g) \in g \otimes \mathfrak{g}, \mathfrak{g} \}$
derived stack
Longitudinal view of this:

\[ \text{Ch}(St_{\alpha}) \rightarrow \text{Ch}(\text{Loc}_{\alpha}(X, p)) \]

Local systems with simple pole at \( p \) and flag at \( p \)

Cross with picture of

Type of surface singularity

in 4d gauge theory; no longer strictly canonical but broad.

BZ-Killer:

\[ D(\text{Br}(\mathbb{P}^1, p)) \rightarrow D(St_{\alpha})^* \]

Coady C. 1.5... gettin' linear.

\* = brief/sketch/complexe. For \( \mathbb{P}^2 \) only
How to pass from affine flags to finite flags?

$\mathbb{P}^n_{\mathcal{O}_{\mathbb{C}^n}} \subset \text{affine of } S^t$ in category $\text{Sh}_c(B\mu_0)$.

$S^t$ Twists of $\mathcal{O}$-bundle $\subset B\mu_0$.

$= \text{precisely } S^t$ fixed points.

$\Rightarrow$ can recover $\text{Sh}_c(\mathbb{P}^n_{\mathcal{O}_{\mathbb{C}^n}}) = \text{Sh}_c(B\mu_0)$

via equivariant localization:

Theorem. $D(\mathbb{P}^n_{\mathcal{O}_{\mathbb{C}^n}}) \cong [\{u^\pm 1\} : dg u = 2]

= D(\mathbb{P}^n_{\mathcal{O}_{\mathbb{C}^n}}) S^1 \otimes [\{u^\pm 1\}]_{S^1[0]}$

See for $B\mu_0$: $S^t$ action real?

$D(\mathbb{P}^n_{\mathcal{O}_{\mathbb{C}^n}}) S^1 = D(\frac{1}{x} \in \mathbb{P}^n_{\mathcal{O}_{\mathbb{C}^n}}) = \frac{1}{x} \mathcal{R} C_{S^1} \otimes [\{u^\pm 1\}]_{S^1[0]}$
Miracle $\mathcal{G}_{\mathsf{c}}$ is actually a loop space, in sense of derived algebraic geometry:

$$\mathcal{G}_{\mathsf{c}} = L\mathcal{G}_{\mathsf{c}}^\vee$$

So comes an abstract $S^1$ action

Theorem $\mathcal{O}(LX) S^1 = \mathcal{D}(X)$

for any $X$ — derived interpretation of cyclic homology

Corollary Canonical equivalence of derived categories $\mathcal{D}(\mathcal{G}_{\mathsf{c}}^{U}) = \mathcal{D}(\mathcal{G}_{\mathsf{c}}^{U^2})$ in derived Fourier transform
Real groups: PGL from \((\mathbb{P}^1, 0, \infty)\) to \\
\((\mathbb{R}P^2, 0)\), i.e. introduce \(\mathbb{R}\) structure.

Geometric box change conjugate allows us to pass between coding spaces, predicts

\[ D(D_{\mathfrak{su}_6, \theta}^m) = D(G) \]

**Theorem** Geometric box change \(\rightarrow\) Sergey conjecture:

\[ D(D_{\mathfrak{su}_6, \theta}^m) = \text{Rep}(G) \]

\[ D(D_{\mathfrak{su}_6, \theta}^m) = D(G) \]