Example: comes from joint work with E. Friedel
- would follow easily from the conjectured
equi-valence of categories -
makes infinitesimal progress towards
geometric Langlands.

Counterexample: 1 April was less than two weeks ago.

Background: The Abelian case (GL_r)

\( A = \text{an abelian variety}, \ A' = \text{Pic}^0 A \) dual

\( \Rightarrow \text{universal line bundle} \ P \rightarrow A^n A' \)

Fourier-Mukai gives equivalence of categories:

\( \text{D} - \text{coh} (A) \leftrightarrow \text{D} - \text{coh} (A') \)

\( A^n A' \rightarrow F \rightarrow \text{R}^n \text{P}_x (\Gamma \otimes p^* F) \)

Let \( V = \text{Lie} (A) \).

Fiber over \( V' \):
\( b \)

\[ T^*A = A^* V \]

\[ \Rightarrow D[\text{Coh}(A^* V)] \leftrightarrow D[\text{Coh}(A^* - V)] \]

\[ F \rightarrow M \rightarrow V \].

\( (1) \)

Deformations

\[ 0 \rightarrow \mathcal{O} \rightarrow T^*A \rightarrow D \]

\[ \rightarrow \text{Sym}T \]

RHS deforms to \( A^b \) — moduli space of

flat, holomorphic \( GL_n \) bundles on \( A \)

\[ V \rightarrow A^b \] universal extension of \( A^\nu \)

by a vector space:

\[ A^\nu \rightarrow E \rightarrow H^1(A^\nu, O) \]

\[ = \text{natural class} \ II \in H^1(A^\nu, V^\nu) \]

Have a principal bundle \( Y \) on \( A \times A^b \)

with flat connection along \( A^\nu \).
Theorem (Laurev, Polishchuk-Rothstein)

Even-Meter gives an equivalence of categories
\[ D(\text{D-mod}(A)) \cong \text{D}(\text{D-mod}(A)) \]
\[ (A \rightarrow A^b : \text{use de Rham duality}) \]
\[ A^d \rightarrow A : \text{use derived perfection} \]

(2) Have further deformations with more structure:
\[ I \rightarrow A \text{ ample line bundle} \]

Replace \( D \) by \( D^\ell \) differs on sections
of \( I = I \otimes D \otimes I^\vee \)
- Some symbols as \( D \), e.g. \( D^\ell \mapsto D^\ell = \text{Sym}\).

I defines a (Grothendieck) line bundle \( I^\vee \) on \( A^\vee \)
(inverse curvature form).
\( D^\ell \) : \( I \) twist of \( D \) on \( A^\vee \)

Theorem (Polishchuk-Rothstein) Equivalence of categories
\[ D(\text{D-mod}(A)) \cong \text{D}(\text{D-mod}(A^\vee)) \]
- on level of $O$-modules this is usual $F$-modules.

Observe: $V^o \cong V$ for some choice of $I$

\[ \Rightarrow \quad V = (V^o)^o \quad V^o = T_i A^o \]

So $A^o \cdot V^o \cong T_i A^o$.

Examples of objects & homs:

1. $O$ on $TA \rightarrow O_p$ on $A^o \cdot V^o$

   \[ P = \{ I \cdot V^o \in A^o \cdot V^o \} \]

   ($P$ will serve as base to $A^o$)

   - $P_\alpha = \pi^o(\alpha) \Rightarrow O_{\alpha} \rightarrow A^o \cdot V^o$

     goes to $O(\alpha) \in A^o$ the fibre on $A$.

   - $\text{Hom}_{\pi^o}(O, O) = \Gamma(TA, O) = \Gamma(A, \text{Sym} T)$

     $= \text{Sym} V$

   - $\text{Hom}_{A \cdot V^o}(O_p, O_p) = \Gamma(P, O) = \text{Sym} V.$
- naturally isomorphic as algebras.

- \( \text{Hom}_A (O(U), O(U)) \cong \text{Sym} V \)

\( \Gamma (P, O) = \text{Sym} V \) no changes.

- \( \text{Ext}_A^1 (O, O) = H^1 (TA, O) \)
  \[ = H^1 (A, \text{Sym} 1) = \text{Sym} V \otimes H^0 (A) \]

- \( \text{Ext}_A^1 (Q_p, O_p) \)
  \[ = \Gamma (A \otimes V, Q_p) \otimes \Lambda^1 (\text{non-flat}) \]
  \[ = \text{Sym} V \otimes \Lambda^1 V \text{ isomorphic algebras.} \]

Note \( \text{Ext}^1 \) classifies first order deformations.
\( \tilde{V} \) normalizations of line bundle \( P \).

Other side \( \tilde{V} = H^1 (A, O) \) infinitesimal
defs of line bundle \( O \) on \( A \) going
defections on \( TA \).
If we didn’t know $F_{\lambda}$, what if the equivalence could we construct?

If $M$ is an $O$-module on $\Lambda^w V$ supported on $P$ (i.e., an $Op$-module)

$M \rightarrow \Gamma(P, M) \rightarrow \tilde{M}$ Sym $V$-module

Can localize $M$ on $\Gamma(A)$, (i.e., $M = \tilde{M}$)

almost tautological that $M$ is $F_{\lambda}$ transform of $\tilde{M}$

$\Rightarrow$ Baby version of classical B-D constr.

To construct an equivalent abelian category

O-modules on $\text{Col}(A)$ $\rightarrow$ O-modules

(“globally presented by $\Gamma(A)$” supported on $P$)

Cute but not very useful:

Embeds in these abelian categories are not
the Thom groups in $D(\text{Col}(A))$

BUT...
Theorem: There is an equivalence of triangulated categories

\[
\text{full subcat of } \\
\text{full subcat of } \\
\text{full subcat of } \\
\text{full subcat of } \\
\text{full subcat of } \\
\text{full subcat of }
\]

\[
\text{Defn (A^{\text{ppr}})} \\
\text{objects where } \\
\text{objects where } \\
\text{objects where } \\
\text{objects where } \\
\text{objects where } \\
\text{objects where }
\]

\[
\text{D of final subt of } P
\]

\[\varepsilon: \text{need to know deformations are unobstructed}
\]

\[= \text{vanishing of higher Whitehead brackets}
\]

\[\text{and need more info to pin down the obstruction.}
\]

\[\text{Ext}^n \text{p}^k \text{diam first order}
\]

(1) \[D \text{ on } A, \quad 0 \text{ on } A^b
\]

\[
\text{Ext}^n_{D^{\text{op}}/A} (D, D) = H^n (A, D) = \text{Sym}_V V \otimes A^b
\]
\[ \text{Ext}^*_A(\mathcal{O}_P, \mathcal{O}_P) = \text{Sym}^* V \otimes \Lambda V \]

Note: something changes a bit with deformations:

- \( \mathcal{O}_P \) is no longer canonically isomorphic to \( P \), so self-ext is non-canonically \( \text{Sym}^* V \)

\[ \mathcal{O}_P \leftrightarrow D \otimes \mathcal{O}(1) \quad \text{as } D\text{-module.} \]

\[ \text{Hom}_D(D \otimes \mathcal{O}(1), D \otimes \mathcal{O}(1)) \cong \text{Hom}_D(\mathcal{O}(1), D \otimes \mathcal{O}(1)) \]

\[ = D^h \text{ differs on } \mathcal{O}(1) \]

(isomorphic to \( D \) after choice of connect on \( \mathcal{O}(1) \).)

(2) Further deformation:

\[ \text{Proj } H^0(A, D^2) \cong C, \text{ degree zero} \]

\[ = \text{Ext}^1(D^2, D^2) \]

\( A^b \) site: neighborhood of \( P \) in \( A^b \)

looks like \( T^*P \), & deforms to \( D^2 \)

\( \leftrightarrow \) grading \( T^*P \).
Prop. (case 1) This requires \(FM\) in
a formal neighborhood of \(P\) in \(\mathbb{A}^b\)

\[\text{Note: } \text{Ext} = \text{Sym} \, \text{V} \circ \text{V}^* \circ \Delta^{(1)}(P)\]

**Blanket theorem (w/ E. Frank)**

All this holds for any reductive \(G\)

\(P \rightarrow \text{subvariety of Borel-Dual of } G\)

On \(\mathfrak{g}_P\) look at \(\mathfrak{d}^\mathbb{R}\)

\[\text{Ext}_{\mathfrak{g}_P} (D, D) \rightarrow \Omega^*_{\mathcal{L}_G}(P)\]

algebraic differential forms on \(G\)

\(L\) deformations are unobstructed to all orders

\(\Rightarrow\) extend the B-D construction of \(D\)-modules
from \(P\) to a formal neighborhood

(construct \(FM\) over the formal neighborhood)
Part 2

Study of the FM transform
between coherent sheaves on $T^*B_{m_0}$
& on $T^*B_{m_0}$. 

(Chernyuk's conjecture holds for all G.
"Global" issues - gerbes, action of coadj 
exts for groups other than GLn -
see Donagi-Pantev).

Recall $T^*B_{m_0}$, $T^*B_{m_0}$,

$\mathcal{Z} \rightarrow \mathcal{F}$

$\mathcal{U} = \mathcal{G}(\Sigma, K^{OD})$

$\Sigma =$ curve, $K =$ canonical bundle
$d =$ exponents $+ 1 = 1, n \text{ for GL}_n.$
Generic fibers are (almost) all abelian varieties. $GL_n$: Jacobians of spectral curves.

$p : \mathcal{U} \to \mathcal{M} \times (\mathbb{P}^1)^{n-1}$

Spectral curve given by $\{ \mathbb{C}^n \times \mathbb{C}^n \times \ldots \times \mathbb{C}^n = 0 \}$

$\mathcal{U} \in \text{Tot}(\mathcal{K})$.

Structure sheaf $\mathcal{O}^n$ of spectral curve, as algebra over $\Sigma$, is $\text{Sym} T$ principal ideal generated by above equation.

$\text{Sym} T \otimes \mathcal{O}^n \to \text{Sym} T \to \mathcal{O}^n \to 0$

map given by characteristic polynomial $p$ above.

For $GL_n$ can fix a base point in each fiber by choosing the trivial line bundle on each curve.

(Fiber over $p$ = something containing Pic of spectral curve).

$\Rightarrow$ associated vector bundle on $\Sigma$ is $\mathcal{T} \mathcal{O}^n = \mathcal{E}$.
Can see \( E : T^* G = \bigoplus_{i=0}^d T^{i+1} \) (col 1 #1)

Higgs field on \( E \) by \( \theta = \left[ \begin{array}{c} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_n \end{array} \right] : E \to E \otimes k \)

This defines the Hitchin section

of \( T^* \text{Bun}_G \to \mathfrak{g} \).

Sometimes not convenient to use

\( E = K^{d_1} \otimes \cdots \otimes K^{d_n} \), so \( \text{det} = 1 \)

(i.e. \( S^2 \text{Bun} \) bundle)

...may need to choose \( K^i \) to define this.

We know part of the Pansu kernel

for FM transform ... a line bundle over

the locus where \( S \) is smooth.

Can extend this to a slightly bigger set \( S \).

For \( G_n \) - let \( R = T^* \text{Bun}_G \)

be the open subset of pointwise regular

Higgs field. \( \mathcal{L} \) is its complement

\( \mathcal{L} \) extends to a line bundle on
Can write a formula in terms of data of spectral curve.

\[ \text{GL}_n : \text{regular Higgs field} \leftrightarrow \text{Higgs sheaf is a line bundle} \]

For general \( G \), know how to extend to a slightly smaller subset.

"Geometric" construction of \( P \):

* \( P \) is \( \mathcal{O} \) on \( T^* B_{\mathcal{G}} \times \sigma \mathfrak{g} \)
  * Hitchin section.

Recall function on \( \mathfrak{g} \) and by Hamiltonian flow on \( T^* B_{\mathcal{G}} \) (linear flows along fibers).

Restrict to linear function, a copy of \( \mathfrak{h} \).

(\text{Hitch}) \( \text{Then } \mathbb{R}^N \times \mathcal{O} = \text{vector bundle canonically isomorphic to } \mathfrak{h}^* \) as \( \mathfrak{g} \)

\( \mathbb{R} \) was a fiber bundle with fibers abelian varieties.
\( H^* \) fibers target to \( b \).

\( \mathbb{R}^* \times \mathbb{G} \) gives infinitesimal deformation of the bundles.

Exponentiation gives a class in \( H^*(T^* \mathbb{G}_x, \mathbb{G}) \)

Prop. If you move the Hitchin section or by \( \exp(h) \), then

\[ \Rightarrow \text{ p becomes} \]

\[ \mathcal{O}(\exp h) \in \text{Pic}(T^* \mathbb{G}_x \times \exp h, \mathcal{O}(h)) \]

What is meaning of transformed Hitchin section \( \exp h \cdot \sigma(h) \subset T^* \mathbb{G}_x \) ?

\( \sigma(h) \) itself represents all spectral curves + method the bundles on them.

\[ h \cdot H^* = \oplus H^*(\mathcal{E}, T^* \mathcal{E}) \]

defines an infinitesimal line bundle on \( T^* \mathcal{E} \).
Deformed Hitchin section gives all spectral curves & restriction of this line bundle.

(Corresponding to the section of the Hitchin fibration).

Pretty good indication that this story defers.

due NC case --- $\sigma(\lambda) \mapsto \mathcal{O}_\lambda \subset \mathcal{M}$

transformed sections --- local flat limit of $\mathcal{M}_\lambda$.

(Corresponding)

Can define $\mathcal{P}$ on $T^*\text{Hom} \times T^*\text{Hom} \setminus S \times S$

E.g.: for a closed stack whose support does not meet $S$ can define $\mathcal{P}$ on $\mathcal{M}$.

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Fig.: Cohomology sheaf of $\sigma(\lambda) \to \mathcal{O}$

$\mathcal{O}_\mathcal{P} \mapsto \mathcal{O}(\log(-\lambda))$ line bundle.

Check to get zero self-cuts.
\[ \text{Ext}^k_{\mathcal{B}_q} (O_0, \mathcal{O}_a) = \mathcal{C} \oplus \Lambda^k \mathcal{H} \]

\[ \text{H}^* (\mathcal{B}_q, \text{Sym} \mathcal{T}) \]

Corollary: \[ \mathcal{H} \]

Reason of Frenkel-Teleman

Good check! this detects information about all of \( T^* \mathcal{B}_q \).

(this follows from Fisler-Grajnowski-Teleman)

Would like to claim fully faithful functor from

\[ \text{D-Coh} \] (with support in \( R \)) to

\[ \text{D-Coh} (T^* \mathcal{B}_q) \]

(only follows for a neighborhood of \( \sigma \), not checked on all of \( R \))

Eg. \( \mathcal{O}_a \) on \( \mathcal{B}_q \) = del of codimension

\[ \rightarrow \text{given } \mathcal{O}_a \text{ on } T^* \mathcal{B}_q \]
Claim: on $R$ the transfer of $O(R)$ is $O(R)$.  

Check: $\text{Ext}(O, O(R)) = H^*(T^*B_{\mathcal{R}}, O(R)) = \text{sections of}$

$\mathcal{O} \otimes \text{bundle on Hilb space}$

$\text{Ext}(\mathbb{G}_m \otimes B_{\mathcal{R}}, O(R)) = \text{free rank 1 module}$

on $\text{Hilb}$

Try $\text{Ext}(O(R), O) = H^*(T^*B_{\mathcal{R}}, O(R))$

$= H^*(B_{\mathcal{R}}, O(R) \otimes \text{sym}^n)$

$0$

BUT $
\text{Ext}(O(R), \mathbb{G}_m \otimes \mathcal{O}) = \text{rank 1 free module over } C[x,y, ...]
$

in degree $d$...

Counterexample to geometric Langlands;
appears to deform to Deligne-Serre...
... Some duality fails on $B_{fg}$, so
it's of infinite type....

$\Rightarrow$ as't give local definition of
category of coherent sheaves, need
constraints on behavior at infinity
... or for Atiyah-Bott states.

\begin{enumerate}
\item \textbf{Evidence for fully faithful functor}
\item $D\left(\text{Coh } \left(E^\infty_{F^*}\right)\right) \to D\left(B_{fg}, D\right)$
\item is strong, but not equivalence...?
\end{enumerate}

- problem: $B_{fg}$ not proper, so base
change doesn't work.

\begin{align*}
\text{Open } \begin{array}{c}
\circ \text{ a bundle } E \text{ with hol. conn. } D \\
\circ \text{ a full flag } 0 \subsetneq E^1 \subsetneq \cdots \subsetneq E^k = E \\
\circ \text{ s.t. } DE^k = E^{k+1} \circ D \\
\circ \begin{array}{c}
\text{or } \begin{array}{c}
V : E^k/E^k \to E^{k+1}/E^k \circ D
\end{array}
\end{array}
\end{array}
\end{align*}
... associated graded of an oper is a Koszul complex \( \mathcal{O} \) in the Hitchin section.

Want: a local version of \( \mathcal{O} \),

of which \( \mathcal{O}_p \) are a leaf.

Rephrase the definition:

an oper is actually a D-module

graded by \( \mathbb{D}_0 \mathcal{K}^d \) of rank \( n \),

by a priori flat ideal.

Recall: a spectral curve was a graded

of \( \text{Sym} T \) of rank \( n \).

Idea: defining an oper is on NC spectral curve.

Want: line bundle on this coming from

a line bundle on \( \text{Spec} \mathcal{D} \)

... i.e. a locally free \( \mathcal{D} \)-module of rank 1.

... classified by \( H^1(\Sigma, GL_r(\mathcal{D})) \).
initial variables: \( H^i(\Sigma, \mathcal{O}, D) \)
\[ = H^i(\Sigma, D) \]
\[ \cong \bigoplus H^i(\Sigma, T^{\text{can}}) \text{ (numerically)} \]

Unfortunately these don't integrate to honest sheaves.

But classes from \( H^i(\Sigma, T) \) do integrate to sheaves of the cure.

Result of a dot of \( D \) on \( \Sigma \) will be a \( D_{\text{reg}} \)\( \neq \)\( \Sigma \)
of \( D_{\text{reg}} \) on \( B_{\text{reg}} \) from \( L \)
\[ \text{to moment of subspace of } \Sigma \]