M compact Riemann manifold
\[ \Delta \] Laplacian (positive definite) if \( m > 0 \).

Consider scalar field theory:
field \( \phi \in C^0(M; \mathbb{R}) \)
action \( S(\phi) = \left( \int_M \phi (\Delta + m^2) \phi \right) \) + \( I(\phi) \)
where \( I(\phi) \) is a local functional:

\[ \sum_{n=0}^{\infty} \int_M \phi (D_1 \phi) (D_2 \phi) \cdots (D_n \phi) \]
where \( D_i \) are differential operators
on \( C^0(M) \).

integrate on expression only depending on Taylor series
of \( \phi \) at a point:

\[ I(\phi) = \sum_{n=0}^{\infty} I_n(\phi) \]

Want to make sense of functional integrals of form:
\[ Z(a) = \int_{\phi \in C^{0}(m)} e^{i \int (\bar{\phi} \partial^2 \phi + \partial \phi I \phi)} \]

with \( a \in C^{0}(m) \) a background field.

(can go from this to typical form with current \( J \phi \) by change of variables but not conversely - this contains more information)

Heat kernel \( K^{0}_{t} \in C^{0}(M^{2}) \) for the

Laplacian - it kernel representing the operator \( e^{-t \Delta} \)

\[ K_{t} = e^{-t m^{2}} K^{0}_{t} \] representing \( e^{-t(\Delta + m^{2})} \)

Propagator \( P = \int_{0}^{\infty} K_{t} \, dt \in \text{Dist}(M^{2}) \)

... kernel representing the inverse operator \( (\Delta + m^{2})^{-1} \).
\[ P(\epsilon, T) = \int_{\mathbb{R}}^{T} k(t) dt, \quad \text{so} \quad P = O(1) \text{.} \]

\text{Feynman graph expression:}

\[ \log Z(a) = \sum_{\text{corrected graphs} R} \frac{b_i(n)}{|A_k(n)|} \cdot \omega_R(a) \]

- \( b_n = \text{Betti number of graph} \)
- \( \omega_R(a) \) is the weight of graph \( R \)
- Put \( p \) on all internal edges, \( a \) on all external edges + interaction \( I \) at vertices (no propagators on external edges)
Simple example: if $I(q) = \int \rho^3$

$$\Omega \sum_{x \in M} \int_0^\infty K_x(x_k) a(x)$$

This is ill-defined! $-\text{dim}(M/2)$

for $t$ small $K_t(x) \sim 1$

$\Rightarrow$ integral diverges

$$\int \int_{x \in M} \int_{y \in M} K_t(x) K_s(y) dxdy ds$$

$$= \int \int_{x \in M} \int_{y \in M} \frac{1}{m(x)} \frac{1}{m(y)} ds$$

Here where external maps get height

Zero... best kind come from

Integrating where measure on sphere

of maps from $s$ to $y$. 


General Definitions

Let $O = O^*(C^0(M))$

$$= \prod_{n=0}^{\infty} \text{Dist}(M^n)^{S_n}$$

$S_n$-invariant distributions

"" $= \text{Sym}^* (C^0(M)^n)$

ie formal power series on the vector space $C^0(M)$.

$O$ is an algebra under direct product of distributions.

If $A \in (C^0(M^n))$ (think of an operator)

$\Rightarrow$ second order diff op $g_A : O \to O$

by $g_A : \text{Dist}(M^n) \to \text{Dist}(M^{n-2})$

= contraction with $A$ in all possible ways.
For any $I \in \mathcal{O}$ define

$$\Gamma^I(A;I) = 2 \lambda \log (e^{2\pi} e^{2\pi/k}) \lambda \in \mathcal{O}[A]$$

$$= \sum_{\gamma \in \mathcal{O}[A]} \frac{k_{\lambda}(\gamma)}{|\lambda \cdot \gamma|} \nu_{\gamma}(\lambda)$$

as before: put $A$ on internal edges, $I$ on vertices $\gamma$ on external edges.

-always well defined for $A \in C^0(M^2)$

What we said before is

$$\kappa \log Z(\lambda) = \lim_{\varepsilon \to 0} \Gamma^I(P;\varepsilon,0), I$$

So above bound is stated for $\varepsilon$ near $\lambda$.
Def: An effective index $I^\text{eff}_{[\xi]}$ at scale $\xi$ tells us what occurs at scale $\xi$.

Def: A system of effective actions is $I^\text{eff}_{[\xi]} \in \mathcal{O}(\xi^2)$ ∀ $\xi > 0$, subject to:

1. $\Gamma(P(\xi, T), I^\text{eff}_{[\xi]}) = I^\text{eff}_{[T]}$
   (renormalized group equations)
  
\[ \begin{align*}
   \text{in} & \quad \left[ \begin{array}{c}
   I^\text{eff}_{[\xi]} \\
   P(\xi, T)
   \end{array} \right] \\
   \text{on} & \quad J(\xi)
\end{align*} \]

2. I some $J(\xi)$ which is bad, $\xi$ dependent. $\lim_{\xi \to 0} \frac{I^\text{eff}_{[\xi]} - J(\xi)}{\xi} = 0$

Wilson: to make QFT should be effective often at every scale... original already correct at scale 0 never is nonsensical!
Theorem: There is a bijection between local functions $I = J^{(1)} + J^{(2)} + \cdots$

$(I^{1})$ & $(I^{2})$ of effective actors

... is any "nonasial" interaction $I$ corresponds to a system of effective actors.

Idea: Given $I$, we construct a canonical series of canonical terms, $I^{(1)}$, $I^{(2)}$, $I^{(n)}$-dependent local functions, $I^{(n)}$:

$\lim_{\epsilon \to 0} \Gamma(P^{(n)}(I^{(n)}), I^{(n)} - I^{(n+1)}) \exists \forall I^{(n)}$

This limit is $I^{\text{eff}}[I]$.

Similar process construct: sub-branch singular terms in $I^{\text{eff}}[I]$ using canonical terms.

$x$: Canonical, there is an inner close which applies to all theories, which we write:
Choice of a way to define "singular part" of certain functionals of $\mathcal{E}$.

(Many the degenerations...)

Philosophy: $I^0, I^1, I^2$ are fundamental, but $I^1, I^2, I^3$ are less natural (depend on the internal class).

How to discuss gauge theory in this language — for every gauge symmetry child from a group...?

Pass to BV formalism, use a quantum master equation at each step.

Yang-Mills in $^{UL}_{GL}$, 1st order formalism:

og Lie algebra with a pairing

\[ \Pi^2 (M) \cong \mathfrak{so} \]

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Action: $\int B^1 F(A) + \int B^2$

On critical points $B = F(A)$.

Second integral enforces $\text{YM}$ equations.

Gauge fix: $\Omega^0(M/\mathbb{R})$.

BV: add antifields $\mathcal{L} \lambda_{\Omega^1} \Omega^1(n) \omega$

(du to force)

2 antifields $\mathcal{L} \lambda_{\Omega^2} \Omega^2(n) \omega$

\[
\begin{array}{cccc}
-1 & l_2^0 & \omega & \\
0 & l_2^1 & l_2^3 & \omega \\
1 & l_3^0 & l_3^1 & l_3^2 & \omega \\
2 & l_4^0 & & \\
\end{array}
\]

3
\( \exists \) has a differential \( Q \) drawn on

Pick a gauge-fixing operator

\[ Q^+ : \mathcal{E} \rightarrow \mathcal{E}, \quad (Q^+)^2 = 0 \]

\( bH = [Q, Q^+] \) is 2nd order elliptic

(e.g. cones from necks or \( m \)).

\( K_t \in \mathcal{E} \in 3 \) heat kernel for \( H \)

\[ p(\mathcal{E}, T) = \int_0^T Q^+ K_t \]

(Hodge Laplacian from \( Q^+ \))

Proceed as before. If \( \Gamma \in \mathcal{C}(\mathcal{E}, \mathcal{E}) \)

Can define

\[ \Gamma(p(\mathcal{E}, T), I) \]

As before, use a bijection given effective actions & real functions \( I \).
Gauge invariance: for each scale \( \Lambda \) have a BV operator \( \Delta_\Lambda : \mathcal{O}(E) \)

\[ \Delta^2 = 0, \quad \Delta \text{ odd order 2}. \]

...version of usual BV operator differenced by the heat kernel,

\[ \mathcal{I}_c^\Lambda \text{ satisfies } \Delta_\Lambda \text{ + gauging} \]

\[ \text{mass eq.} \]

\[ (\xi + \theta \Delta_\Lambda) \mathcal{I}_c^\Lambda |_{\theta = 0} = 0 \]

RG flow takes solution of one QME to another:

**Lemma II**: \( \mathcal{I}_c^\Lambda \) satisfies QME of scale \( \Lambda \)

\[ \Rightarrow \Gamma (P(E,T), \mathcal{I}_c^\Lambda) \text{ satisfies scale } T \]

**QME**

\[ \Rightarrow \text{det of QME with a good } \Delta \text{ is ill-defined}. \]
To find solutions of ODE, first by focussing and then construct a problem in homological style.