Kevin Costello - Chern-Simons theory

What algebraic structures are free of models?

$X$: topological space

$C^\infty(X, \mathbb{Q})$ (rational calculus) is a differential graded commutative algebra for right choice of notion of calculus.

Rational homotopy theory (Quillen, Sullivan).

This dg-commutative algebra is a complete invariant of the rational homotopy theory of $X$ (for $X \otimes \mathbb{Q}$).

--- invert all the primes in a space!

Sullivan's localization of $X$, $X \otimes \mathbb{Q}$.

Let $M$ be a manifold (not).

Q: What does this mean for rational homotopy theory?

--- What structure does $Sp^*(M, \mathbb{R})$ have reflecting the fact that $M$ is a manifold?
1. Commutative dga

2. $T: \text{Comm} \to R$

$\text{Vec}(R)$-module $\text{C} = \Omega^ullet(\text{diag}_{\text{Com}})$

\begin{align*}
\text{Alg} & \to \text{Q-module} \\
\text{Com} & \to \text{Q-module}
\end{align*}

$O^w(\mu) \to \text{Alg}$

$C^w(\mu) \to \text{Q-module}$
Private: algebraic structure $A$ on complex $(V,d) \to$ homotopy $A$-structure on $H^*(V)$

So $H^*(M,R)$ is a homotopy commutative object

(covering real bundles), & on hands

Nondegenerate part on $H^*(M,R) \Rightarrow$

Strength to a homotopy Frobenius object

(Add "Massey products")

Theorem (Hopkins, Lurie) The homotopy Frobenius structure just encodes the fact that the rational homotopy type has Poincaré duality

-- it gets just a Poincaré dual $s$.

What is homotopy commutative?

If $A$ is a differential graded commutative algebra & $a_*$ is a lip algebra
A  group is a dg Lie algebra.

So commutative algebras → functor
from Lie algebras to Lie algebras.

Homotopy commutative algebras:
functors dg Lie algebras → homotopy Lie algebras
of → A ○ A

Homotopy Lie: \( V \) is an \( n \)-algebra complex
+ maps \( l_n: V^\otimes^n \to V \quad n \geq 2 \)
which are antisymmetric (\( \delta \))
of degree \( 2-n \) (\( l_2 \) is the bracket)
satisfying some equations:

eg Jacobi:
\[
\begin{align*}
l_2(x, l_3(y, z)) + l_2(z, l_3(x, y)) + l_2(y, l_3(x, z)) & = [l_3(x, y), z] \\
& = d l_3(x, y, z) - l_3(d x, y, z) - l_3(x, d y, z) + l_3(x, y, d z)
\end{align*}
\]
In pictures: \( l_2(x_1) \)

\[ l_k(x_1, \ldots, x_n) : \]

\[ j(x) : \]

\[ d\left( \vec{x} \right) = \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i \]

\[ d\left( \vec{x} \right) = \sum_{\text{all partitions of } n} \sum_{k} \left( \sum_{\text{all } k} \right) \]

**Definition:** An even invariant pairing on a Lie algebra \( g \) is a nondegenerate pairing \( \langle \cdot, \cdot \rangle_2 \) is totally antisymmetric.

An invariant pairing on a Handky Lie algebra is a nondegenerate pairing \( \langle \cdot, \cdot \rangle_2 \) is totally antisymmetric.
can also talk about odd paraos in the sense
where \( \langle x, y \rangle \neq 0 \) or if \( p(x) \cdot p(y) = 1 \). (1)

A commutative Frobenius algebra, with an odd grading,
y lie with an even grading:

\[ \Rightarrow \text{Adg is lie with odd paraos} \]

Homotopy Frobenius algebra with odd grading:

\[ \text{fundamental lie algebra} \rightarrow \text{homotopy lie algebra} \]

eg. M odd-dimensional manifold, \( \text{odd Lie with even paraos} \)
\( \Rightarrow \text{odd paraos} \)
\( \Rightarrow \mathbb{H}^*(M) \circ \text{adj} \text{ is a homotopy Lie algebra} \)

\[ \text{compatible with an odd grading.} \]

Theorem: odd dimensional manifolds have zero Euler characteristic

\[ \text{.... really a homotopy trivializer!} \]
Let $A$ be a finite dimensional algebra over an odd ring, $2 = 0$.

**Lemma** For $a \otimes \epsilon \in A \otimes \mathbb{Z}$ define an operator $T_{a \otimes \epsilon}^\mathbb{Z} : A \otimes \mathbb{Z} \rightarrow A \otimes \mathbb{Z}$

Then $T_{a \otimes \epsilon}^\mathbb{Z} + 3 = 0$

Equivalently let $D \in A \otimes A$, $C \otimes \mathbb{Z}$ be the nodes of the process. Then $C \otimes \mathbb{Z} \in (A \otimes \mathbb{Z})^2$

$& \left[ C \otimes \mathbb{Z} \right] = 0$

Why? $D$ is antisymmetric: $\Sigma D_{\sigma, \sigma} = 0$

$D = \Sigma D_{\sigma, \sigma}, \quad C = \Sigma C_{\sigma, \sigma}$

$\Rightarrow \Sigma \left[ C_{\sigma, \sigma} C_{\sigma, \sigma}' \right] = \Sigma D_{\sigma, \sigma} \left[ C_{\sigma, \sigma} C_{\sigma, \sigma}' \right] = 0$

If $A = H^*(M)$, $M$ odd dimension,
Def. Let $V$ be a bosonic Lie algebra
with a minimal paring. $V$ is a minimal
if $\forall x \in V$, $Tr_{\mathbb{C}} \sum_{x} x_{i} \cdot x_{j} = 0$

Theorem. Let $M$ be an odd-dimensional manifold,
y a Lie algebra with even points.
Then $\mathfrak{h}^*(M) \otimes \mathbb{C}$ is a minimal
bosonic Lie algebra, up to trivial
homotopy.

Is $\mathfrak{h}^*(M) \otimes \mathbb{C}$ minimal?
for $x \in \mathfrak{h}^*(M) \otimes \mathbb{C}$, $x_{i} \cdot x_{j} = [x_{i}, x_{j}]$

is not true class
that of traces as integers.
uniformity is a regularization of this integral.

\( C \in \mathfrak{g}^{m^2} \) move to \( \mathfrak{g}^{m^2} \).

\( D \) current on \( M \) given by \( f \) function on \( \mathfrak{g}^{m^2} \).

\[ D @ \mathfrak{g}^{m^2} \] needs to restrict \( D \) to be diagonal in \( M \), which is infinite.

\[ \Omega \in H^1(M) \] may have homotopy lift

operators \( f \), \( C \in (H^1(M) \otimes \mathfrak{g})^{m^2} \)

unstable \( \Omega = 0 \) \( \theta = 0 \) \( \Psi = 0 \) \( \cdots \)

Homotopy mistake: \( \Omega \) has homotopy \( \Gamma \)

with \( d \Theta = 0 \)

\[ \Sigma \] with \( d \Sigma = \Psi + \Omega \)
A k be \( \mathcal{O} \) with

\[
d^* \omega = \sum_{n=1}^{\infty} \omega_n + \cdots
\]

Compressy structure on \( \Omega^*(M) \otimes \mathcal{O} \):

Let \( \Omega^*(M) = \text{harmonic forms, for a metric on } M \).

Problem: \( \Omega^*(M) \otimes \mathcal{O} \) is not a Lie subalgebra:

\[
\left[ d \omega, \alpha \otimes \gamma \right] = d(\gamma \wedge \alpha) \otimes [x,y]
\]

but \( d \gamma \text{ harmonic } \not\rightarrow \gamma \wedge \alpha \text{ harmonic } .
\]

Write down a one parameter family of Lie subalgebra on \( \Omega^*(M) \otimes \mathcal{O} \) s.t.

\( t \to 0 \) get usual, \( t \to 0^+ \): \( \mathcal{O} \otimes \mathcal{O} \) is a subalgebra.
\[ q^+(xy) = \gamma \to e^{-t\Delta} \]

At \( t=0 \), \( e^{t\Delta} \) might go to infinity so they're a subspace.

**Problem: Jacob \( \) not satisfied**

\[ \mathcal{B}(xy) = \gamma \to e^{-t\Delta} \to \gamma \to e^{-t\Delta} \to \gamma \to e^{-t\Delta} \to \gamma \to e^{-t\Delta} \]

Can't compare the \( e^{t\Delta} \) because of back.

\( e^{-t\Delta} \): Brann with \( \) the \( f \) ... so we " " to enrich!

Let \( h(t) = \int_0^t \Delta \to e^{-t\Delta} \)

\[ [\mathcal{B}, h(t)] = \int_0^t \mathcal{B} \to e^{-t\Delta} \int_0^t \Delta \to e^{-t\Delta} \]

\[ = -(e^{-t\Delta} - 1) : \text{boundary when} \frac{1}{2} e^{-t\Delta} \]
From now on write $H$ for $A=L^k$. So we can now add constants.

\[ l_3^+ (x, y, z) = x + e^{y} + e^{-y} + e^z + e^{-z} \]

\[ l_3^+ (x, y, z) = 0 \]

Similarly $l_4^+ = \sum_{\text{trees}} \ldots$.

At $\infty$, there is a singularity.

\[ \Rightarrow \text{monday 10 alg stats on} \]

\[ \text{calculus 10:05 am in \#70005} \]
Define $\mathcal{D} : \mathcal{O} \rightarrow \mathbb{R}$ by $\mathcal{D}(a) = \sum_{n} <a^n, a>$

Then $\mathcal{D}(a)$ is the tree level version

$$\lim_{t \to 0} \log \left( \int_{x \in \text{Im } d^x} \exp \left( \frac{1}{2t} <x, [x] > + I_{CS} (x, [x] / \mathbf{g}) \right) \right)$$

$$= \mathcal{D}(a)$$

$I_{CS} (x) : = <x, [x] >$

$d : \text{Im } d^x \rightarrow \text{Im } d$ is an isomorphism

$d^{-1} = \int_{0}^{\infty} \exp(-t H) dt$,

Version for universal $g$: cobords

In graph calculus,

1. graph complexes build symmetric monoidal category with cobords a tree
we begin with trivial things

$$H = \int d^4 x \ e^{-iHt}$$

has kernel given by

$$\text{Propagator} \quad p(\theta, \Phi) = \int_0^\infty d^4 K \ e^{iK \theta}$$

What is $\theta$? $\Phi$?

$$d(\theta, \Phi) = \vec{k}_0 - \vec{k}_\perp$$

but $\vec{k}_\perp$ is self-interacting.

of the diagonal: nearly true.

Even worst: $\Phi$ infinite.

$k_0 = 0$ solves CS action.

satisfies the quantum master equation.
So if $J_{35}$ satisfies QME, fix
an harmonic form $\Omega_H > e^{-\varphi_H} = 0$

since the cube is $\varphi_H = 0$

\[ T_{ikl}(x_1, x_2) = \sum_{\text{terms}} \frac{g_{ijk} \cdot \alpha_{ijl}}{\text{weight}} \]

$g_{ijk}$ is a function

$d\varphi$: only points that don't map to $P(0,0)$

$d\varphi$: cost ends have an edge of $k_{35}$ or are wis to

Quanta under curve: $k_{35}$ favors $x_{35}$
QME: $F$ Roton or diagonal $\lambda_{\text{diag}} = 0$

---

Sage: The Lie algebra is unimodular.

Not well defined, but can pretend to be unimodular up to boundary.

$$d_{\mathfrak{g}} \Phi_{\mathfrak{k}} = \sum_{\mathfrak{g} \neq k} \psi_{\mathfrak{g} \to \mathfrak{k}} + \sum_{\mathfrak{g} \neq k} \psi_{\mathfrak{k} \to \mathfrak{g}}$$

$$\psi_{\mathfrak{k} \to \mathfrak{g}}$$

which is the quantum master equation for $\mathfrak{g}$, or in $\mathfrak{k}$

- the BV algebra is the algebra crossing

for $\mathfrak{g}$ to $\mathfrak{k}$, or $\mathfrak{k} \to \mathfrak{g}$

with BV structure since this is an odd symplectic form.
The group has an odd symmetry

\[ \text{h} \circ g \text{ is odd symmetry} \]

\[ \Rightarrow \text{h} \circ g (x) \text{ is odd symmetry} \]

\[ \Rightarrow \text{h} \circ g (x) \text{ is a BV algebra.} \]

\[ f \in O(\text{h} \circ g (x)) \]

\[ \Rightarrow \text{h} \circ g = \{ h, g \} \text{ Hamilton vector field} \]

\[ \Delta f = \text{Div}_X f \text{ BV algebra.} \]

\[ \text{diagonal for lattice measure.} \]

\[ \mathfrak{g} = \mathfrak{e} \mathfrak{g} \mathfrak{k}_d (\mathfrak{g} \oplus \mathfrak{a})/\mathfrak{a} \]

\[ \text{satisfies } \frac{1}{\sqrt{2}} \mathfrak{g} + k \Delta \mathfrak{g} = 0 \]

\[ \Leftrightarrow \text{homotopy minimal stratify on Lie algebra} \]

Here the BV operator is one you can write for any odd symplectic vector field.
M odd, symplectic manifold, $\psi$

$\Rightarrow$ odd element of $\Lambda^*TM$

$\Leftrightarrow$ odd element $\forall \omega \in \Lambda^{2n}TM$, the symbol of a differential operator

A BV operator on $M$ is an operator $\Delta$ with $\Delta^2 = 0$ i.e. $\text{symbol}(\Delta) = \psi$.

\[ \Delta \text{ with } \Delta^2 = 0 \Rightarrow \text{symbol}(\Delta) = \psi, \]

i.e. a differential $\Delta = d + d_1 + \sum k_i D_i$

with $D \text{ mod } k = d$, $\circ(\psi) = k_\psi$. 